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EXISTENCE OF GLOBAL SOLUTIONS FOR A SYSTEM OF REACTION-DIFFUSION EQUATIONS HAVING A FULL MATRIX

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Abstract. In this paper we generalize a result obtained in [8] concerning uniform boundedness and so the global existence of solutions for reaction-diffusion systems with a general full matrix of diffusion coefficients. Our techniques are based on invariant regions and Lyapunov functional methods.

1. introduction

We are interested in global existence in time of solutions to the reaction- diffusion systems of the form

$$\frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f(u,v) \quad \text{in} \quad]0,+\infty[\times\Omega \tag{1.1}$$

$$\frac{\partial v}{\partial t} - c\Delta u - d\Delta v = g(u,v) \quad \text{in} \quad]0,+\infty[\times\Omega \tag{1.2}$$

with the following boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \qquad \text{in} \qquad]0,+\infty[\times\partial\Omega \tag{1.3}$$

and the initial data

$$u(0,x) = u_0, \quad v(0,x) = v_0 \quad \text{in } \Omega, \tag{1.4}$$

where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\frac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$, Δ denotes the Laplacian operator

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with respect to the x variable, a, b, c, d, are positive constants satisfying the condition $(b+c)^2 < 4ad$ which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion is positive definite. The eigenvalues λ_1 and $\lambda_2(\lambda_1 < \lambda_2)$ of the matrix of diffusion are positive. If we assume that $a < d$), then we have

$$\lambda_1 < a < d < \lambda_2.$$

The initial data are assumed to be in the following region

$$\Sigma = \left\{(u_0, v_0) \in \mathbb{R}^2 \text{ such that } \frac{a-\lambda_2}{c} v_0 \leq u_0 \leq \frac{a-\lambda_1}{c} v_0\right\}.$$

We suppose that the reaction terms f and g are continuously differentiable on Σ, $(f(r,s), g(r,s))$ is in Σ, for all (r,s) in $\partial\Sigma$ (we say that $(f; g)$ points into Σ on $\partial\Sigma$), i.e.,

$$\frac{a-\lambda_2}{c} g\left(\frac{a-\lambda_2}{c}s, s\right) \leq f\left(\frac{a-\lambda_2}{c}s, s\right) \text{ and } f\left(\frac{a-\lambda_1}{c}s, s\right) \leq \frac{a-\lambda_1}{c} g\left(\frac{a-\lambda_1}{c}s, s\right), \tag{1.5}$$

for all $s \geq 0$.

See [5, 12], for more details.

Assume further that

$$\sup\left(\left|\left(-f + \frac{a-\lambda_1}{c} g\right)(r,s)\right|, \left|\left(f - \frac{a-\lambda_2}{c} g\right)(r,s)\right|\right) \leq C(|r| + s + 1)^m, \forall\, r, s \in \Sigma, \tag{1.6}$$

where C is a positive constant and $m \geq 1$.

We suppose that one of the following conditions is satisfied:

1-There exist $p \geq 2$; $c(p) > 0$ and positive numbers $(B_i(p))_{0 \leq i \leq p}$ such that

$$(B_{i-1}(p) - B_i(p)) f(r,s) + \left(\frac{a-\lambda_1}{c} B_i(p) - \frac{a-\lambda_2}{c} B_{i-1}(p)\right) g(r,s) \leq C(p)(r+s+1), \tag{1.7}$$

where

$$(a+d)^2 B_i^2(p) \leq 4.(a.d - bc) B_{i-1}(p) B_{i+1}(p). \tag{1.8}$$

2-There exist $c(1) > 0$ and $B_i(1)$, $0 \leq i \leq 1$ such that

$$(B_0(1) - B_1(1)) f(r,s) + \left(\frac{a-\lambda_1}{c} B_1(1) - \frac{a-\lambda_2}{c} B_0(1)\right) g(r,s) \leq C(1)(r+s+1). \tag{1.9}$$

N. Alikakos [1] established global existence and L^∞-bounds of solutions for positive initial data for $b = c = 0$, $f(u,v) = -g(u,v) = -uv^\beta$ and $1 < \beta < (n+2)/n$. In

[6] K.Masuda showed that solutions to this system exist globally for every $\beta > 1$. A. Haraux and A. Youkana [3] generalized the method of K. Masuda with the reactions term

$$f(u, v) = -g(u, v) = -u\varphi(v),$$

where φ is a nonnegative function satisfying the following condition

$$\lim_{v \to +\infty} \frac{\log(1 + \varphi(v))}{v} = 0.$$

The components $u(t, x)$ and $v(t, x)$ represent either chemical concentrations or biological population densities and the system (1.1)–(1.4) is a mathematical model describing various chemical and biological phenomena (see E. L. Cussler [2], J.Savchik [11]).

The present investigation is a continuation of results obtained in [8, 9]. In this study, we will treat the case of a general full matrix of diffusion coefficients.

2. Existence of local solutions

The usual norms in spaces $L^p(\Omega), L^\infty(\Omega)$ and $C(\overline{\Omega})$ are respectively denoted by

$$\|u\|_p^p = \frac{1}{|\Omega|} \int_\Omega |u(x)|^p \, dx$$

$$\|u\|_\infty = ess \sup_{x \in \Omega} |u(x)|$$

$$\|u\|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |u(x)|$$

For any initial data in $C(\overline{\Omega})$ or $L^p(\Omega)$, $p \in [1, \infty[$ local existence and uniqueness of solutions to the initial value problem (1.1)-(1.4) follow from the basic existence theory for abstract semilinear differential equations (see D. Henry [4] and A. Pazy [10]). The solutions are classical on $]0, T^*[$, where T^* denotes the eventual blowing-up time in $L^\infty(\Omega)$.

Furthermore, if $T^* < +\infty$, then

$$\lim_{t \uparrow T^*} (\|u(t)\|_\infty + \|v(t)\|_\infty) = +\infty.$$

Therefore, if there exists a positive constant C such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq C, \forall t \in]0, T^*[,$$

then $T^* = +\infty$.

3. Existence of global solutions

Multiplying equation (1.2) one time through by $\frac{a-\lambda_1}{c}$ and subtracting (1.1) and another time by $-\frac{a-\lambda_2}{c}$ and adding (1.1), we get

$$\frac{\partial w}{\partial t} - \lambda_1 \Delta w = F(w,z) \quad \text{in }]0,T^*[\times\Omega, \tag{3.1}$$

$$\frac{\partial z}{\partial t} - \lambda_2 \Delta z = G(w,z) \text{ in }]0,T^*[\times\Omega, \tag{3.2}$$

with the boundary conditions

$$\frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0 \text{ in }]0,T^*[\times\partial\Omega, \tag{3.3}$$

and the initial data

$$w(0,x) = w_0(x), z(0,x) = z_0(x) \text{ in } \Omega, \tag{3.4}$$

where

$$\begin{aligned} w(t,x) &= -u(t,x) + \frac{a-\lambda_1}{c}v(t,x), \\ z(t,x) &= u(t,x) - \frac{a-\lambda_2}{c}v(t,x), \end{aligned} \tag{3.5}$$

for any (t,x) in$]0,T^*[\times\Omega$ and

$$\begin{aligned} F(w,z) &= (-f + \frac{a-\lambda_1}{c}g)(u,v), \\ G(w,z) &= (f - \frac{a-\lambda_2}{c}g)(u,v), \text{ for all } (u,v)\in\Sigma. \end{aligned} \tag{3.6}$$

To prove that the solutions of (1.1)-(1.4) are global, comes back in even to prove it for problem (3.1)-(3.4). To this subject, it is well known that, it is sufficient to derive a uniform estimate of the quantity

$$\sup(\|(F(w,z)\|_q, \|(F(w,z)\|_q),$$

on $]0,T^*[$ for some $q > \frac{n}{2}$.

Now, we present the main result

Theorem 3.1. *Let $(w(t,.), z(t,.))$ be a solution of (3.1)-(3.4). If one of the conditions (1.7) or (1.9) has been satisfied, there would exist an integer $p \geq 1$ and a continuous function $c_p : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\sup(\|(w(t,.)\|_p, \|z(t,.)\|_p) \leq c_p(t), t < T^*.$$

Proof. We consider the functional

$$L_P(t) = \int_\Omega (\sum_{i=0}^{i=p} C_p^i B_i(p) w^i z^{p-i}) dx = \int_\Omega (\sum_{i=0}^{i=p} \alpha_i(p) w^i z^{p-i}) dx, \tag{3.7}$$

where

$$\alpha_i(p) = C_p^i B_i(p), i = 0, \dots, p.$$

Differentiating L_P with respect to t, we obtain

$$L_P'(t) = \int_\Omega (\sum_{i=1}^{i=p} i\alpha_i(p) w^{i-1} z^{p-i}) \frac{\partial w}{\partial t} dx + \int_\Omega (\sum_{i=0}^{i=p-1} (p-i)\alpha_i(p) w^i z^{p-i-1}) \frac{\partial z}{\partial t} dx.$$

Consequently,

$$L_P'(t) = \int_\Omega (\sum_{i=1}^{i=p} i\alpha_i(p) w^{i-1} z^{p-i}) \frac{\partial w}{\partial t} dx + \int_\Omega (\sum_{i=1}^{i=p} (p-i+1)\alpha_{i-1}(p) w^{i-1} z^{p-i}) \frac{\partial z}{\partial t} dx.$$

using (3.1) and (3.2), we get

$$\begin{aligned} L_P'(t) = & \int_\Omega (\sum_{i=1}^{i=p} i\alpha_i(p) w^{i-1} z^{p-i})(F(w,z) + \lambda_1 \Delta w) dx + \\ & + \int_\Omega (\sum_{i=1}^{i=p} (p-i+1)\alpha_{i-1}(p) w^{i-1} z^{p-i})(G(w,z) + \lambda_2 \Delta z \ ) dx. \end{aligned}$$

which implies

$$
\begin{aligned}
L_P^{'}(t) = & \int_\Omega \sum_{i=1}^{i=p} i\alpha_i(p)F(w,z)\, w^{i-1}z^{p-i}dx+ \\
& \int_\Omega (\sum_{i=1}^{i=p}(p-i+1)\alpha_{i-1}(p)G(w,z)\, w^{i-1}z^{p-i}dx+ \\
& + \int_\Omega \sum_{i=1}^{i=p}(\lambda_1 i\alpha_i(p)\Delta w)w^{i-1}z^{p-i})dx+ \\
& + \int_\Omega \sum_{i=1}^{i=p}(\lambda_2(p-i+1)\alpha_{i-1}(p)\Delta z)w^{i-1}z^{p-i}dx.
\end{aligned} \tag{3.8}
$$

We distinguish two cases:

1-when $p = 1$, we obtain from (3.8)

$$
\begin{aligned}
L_1^{'}(t) &= \int_\Omega (\lambda_1\alpha_1(1)\Delta w + \lambda_2\alpha_0(1)\Delta z)dx + \\
& \int_\Omega (\alpha_1(1)F(w,z) + \alpha_0(1)G(w,z))dx,
\end{aligned}
$$

By applying Green's formula, we have

$$
L_1^{'}(t) = \int_\Omega (B_1(1)F(w,z) + B_0(1))G(w,z))dx,
$$

Using conditions (1.9),(3.5) and (3.6), we deduce

$$
\begin{aligned}
L_1^{'}(t) &\leq c_0^{'} \int_\Omega (w+z+1)dx = \\
& c_0^{'} \int_\Omega (w+z)dx + c_0^{'} mes(\Omega),
\end{aligned}
$$

It follows that

$$
L_1^{'}(t) \leq c_1 L_1(t) + c_2,\, t < T^*, \tag{3.9}
$$

where

$$
\begin{aligned}
c_1 &= c_0^{'} \max(B_1(1), B_0(1)), \\
c_2 &= c_0^{'} mes(\Omega).
\end{aligned}
$$

By a simple integration of (3.9) for all $t < T^*$, we have

$$L_1(t) \leq (L_1(0) + \frac{c_2}{c_1}) \exp(c_1 t) - \frac{c_2}{c_1}),$$

from (3.7), we obtain

$$\begin{aligned} L_1(t) \quad \geq \quad & \min(\alpha_1(1), \alpha_0(1)) \int_\Omega (w + z) dx \geq \\ & \min(\alpha_1(1), \alpha_0(1)) \sup \|w(t,.), z(t,.)\|_1 , \end{aligned}$$

Then, we get

$$\sup \|w(t,.), z(t,.)\|_1 \leq c_1(t), \text{for } t \leq T^*,$$

where

$$c_1(t) = \frac{1}{\min(\alpha_1(1), \alpha_0(1))} \times \left\{ (L_1(0) + \frac{c_2}{c_1}) \exp(c_1 t) - \frac{c_2}{c_1}) \right\},$$

2-If $p \geq 2$

We set

$$\begin{aligned} T = & \int_\Omega \sum_{i=1}^{i=p} (\lambda_1 i \alpha_i(p) \Delta w) w^{i-1} z^{p-i} dx + \\ & \int_\Omega \sum_{i=1}^{i=p} (\lambda_2 (p - i + 1) \alpha_{i-1}(p) \Delta z) w^{i-1} z^{p-i}) dx. \end{aligned} \tag{3.10}$$

The inequality (3.10) can be written as

$$\begin{aligned} T = & \sum_{i=1}^{i=p} \int_\Omega \Delta(\lambda_1 i \alpha_i(p) w) w^{i-1} z^{p-i} dx + \\ & \sum_{i=1}^{i=p} \int_\Omega \Delta(\lambda_2 (p - i + 1) \alpha_{i-1}(p) z) w^{i-1} z^{p-i} dx. \end{aligned}$$

By a simple use of Green's formula, we obtain

$$T = -\sum_{i=1}^{i=p} \int_\Omega (\nabla(\lambda_1 i \alpha_i(p) w) \nabla(w^{i-1} z^{p-i}) dx - \sum_{i=1}^{i=p} \int_\Omega \nabla(\lambda_2 (p - i + 1) \alpha_{i-1}(p) z) \nabla(w^{i-1} z^{p-i}) dx.$$

Which implies

$$\begin{aligned} T &= -\int_\Omega \sum_{i=2}^{i=p} \lambda_1 i(i-1)\alpha_i(p) w^{i-2} z^{p-i} \nabla^2 w dx + \\ & \int_\Omega \sum_{i=1}^{i=p-1} \lambda_1 i(p-i\ ) w^{i-1} z^{p-i-1} \nabla w \nabla z dx + \\ & \int_\Omega \sum_{i=2}^{i=p} \lambda_2 (i-1)(p-i+1)\alpha_{i-1}(p) w^{i-2} z^{p-i} \nabla w \nabla z dx + \\ & \int_\Omega \sum_{i=1}^{i=p-1} \lambda_2 (p-i+1)(p-i)\alpha_{i-1}(p) w^{i-1} z^{p-i-1} \nabla^2 z dx, \end{aligned}$$

By a simple computation and from (3.8), it follows that

$$\begin{aligned} L_P'(t) = & \int_\Omega \sum_{i=1}^{i=p} (iC_p^i B_i(p)) F(w,z)) w^{i-1} z^{p-i} dx + \\ & \int_\Omega (\sum_{i=1}^{i=p} (p-i+1) C_p^{i-1} B_{i-1}(p) G(w,z)) w^{i-1} z^{p-i}) dx - \\ & \int_\Omega (\sum_{i=1}^{i=p-1} \left\{ \begin{array}{c} \lambda_1 i(i+1) C_p^{i+1} B_{i+1}(p) \nabla^2 w + (a+d) i(p-i\ ) C_p^i B_i(p) \nabla w \nabla z + \\ \lambda_2 (p-i\ )(p-i+1) C_p^{i-1} B_{i-1}(p) \nabla^2 z \end{array} \right\} w^{i-1} z^{p-i-1}) dx, \end{aligned}$$

Using the fact that

$$\begin{aligned} iC_p^i &= (p-i+1) C_p^{i-1} = p C_{p-1}^{i-1}, \\ i(i+1) C_p^{i+1} &= i(p-i) C_p^i = (p-i)(p-i+1) C_p^{i-1} = p(p-1) C_{p-2}^{i-1}, \end{aligned}$$

we conclude

$$\begin{aligned} L_P'(t) = & \int_\Omega (\sum_{i=1}^{i=p} (p C_{p-1}^{i-1} \left[B_i(p) F(w,z) + B_{i-1}(p) G(w,z) \right] w^{i-1} z^{p-i}) dx - \\ & p(p-1) \int_\Omega (\sum_{i=1}^{i=p-1} C_{p-2}^{i-1} \times \left[\lambda_1 B_{i+1}(p) \nabla^2 w + (a+d) B_i(p) \nabla w \nabla z + \lambda_2 B_{i-1}(p) \nabla^2 z \right] w^{i-1} z^{p-i-1}) dx. \end{aligned}$$

From (1.8), it follows that the quadratic forms

$$\lambda_1 B_{i+1}(p) \nabla^2 w + (a+d) B_i(p) \nabla w \nabla z + \lambda_2 B_{i-1}(p) \nabla^2 z,$$

are positive since

$$((a+d)B_i(p))^2 - 4\lambda_1 B_{i+1}\lambda_2 B_{i-1} = (a+d)^2(B_i(p))^2 - 4(ad-bc)B_{i+1}B_{i-1} \le 0,$$

Consequently,

$$L_P'(t) \le p(\int_\Omega \sum_{i=1}^{i=p} C_{p-1}^{i-1}\left[B_i(p))F(w,z) + B_{i-1}(p)G(w,z)\right] w^{i-1}z^{p-i}dx.$$

Using conditions (1.7),(3.5) and (3.6), we get for an appropriate constant $c_0(p)$

$$L_P'(t) \le c_0(p)\int_\Omega (\sum_{i=1}^{i=p} C_{p-1}^{i-1}(w+z+1)w^{i-1}z^{p-i})dx,$$

By a simple computation, we conclude

$$L_P'(t) \le c_0(p)(\int_\Omega \sum_{i=0}^{i=p} C_p^i w^i.z^{p-i}dx + \int_\Omega (\sum_{i=0}^{i=p-1} C_{p-1}^i w^i.z^{p-i-1}dx), \tag{3.11}$$

Using the fact that

$$\sum_{i=0}^{i=p-1} C_{p-1}^i w^i.z^{p-i-1} = (w+z)^{p-1},$$

the inequality (3.11) can be written as

$$L_P'(t) \le c_1(p)L_p(t) + c_0(p)\int_\Omega (w+z)^{p-1}dx,$$

Applying Hölder's inequality to the second term in the right hand side of the above inequality, we obtain

$$L_P'(t) \le c_1(p)L_p(t) + c_0(p)(mes(\Omega))^{\frac{1}{p}}(\int_\Omega (w+z)^p dx)^{\frac{p-1}{p}},$$

Since the following inequality holds,

$$(w+z)^p = \sum_{i=0}^{i=p} C_p^i w^i z^{p-i} \le \frac{\sup_{0\le i\le p} C_p^i}{\min_{0\le i\le p}\alpha_i(p)} \sum_{i=0}^{i=p} \alpha_i(p) w^i z^{p-i}$$

We conclude that the functional L_p satisfies the following differential inequality

$$L_P'(t) \le c_1(p)L_p(t) + c_2(p)(L_p(t))^{\frac{p-1}{p}}, \forall t < T^*,$$

where

$$c_2(p) = c_0(p)(mes(\Omega))^{\frac{1}{p}}\Big(\frac{\sup_{1\le i\le p} C_p^i}{\min_{1\le i\le p}\alpha_i(p)}\Big)^{\frac{p-1}{p}},$$

while putting

$$q(t) = (L_p(t))^{\frac{1}{p}}.$$

One gets

$$pq'(t) \le c_1(p)q(t) + c_2(p),$$

which gives us, by a simple integration

$$(L_p(t))^{\frac{1}{p}} \le \left[(L_p(0))^{\frac{1}{p}} + \frac{c_2(p)}{c_1(p)}\right]\exp\Big(\frac{c_1(p)}{p}t\Big) - \frac{c_2(p)}{c_1(p)}, \tag{3.12}$$

By using the inequality

$$L_p(t) = \int_\Omega \Big(\sum_{i=0}^{i=p}\alpha_i(p)w^i z^{p-i}\Big)dx \ge \int_\Omega (\alpha_p(p)w^p + \alpha_0(p)z^p)dx,$$

we have

$$(L_p(t))^{\frac{1}{p}} \ge \min(\alpha_p(p),\alpha_0(p))^{\frac{1}{p}} \times \sup\Big(\Big(\int_\Omega w^p dx\Big)^{\frac{1}{p}}, \Big(\int_\Omega z^p dx\Big)^{\frac{1}{p}}\Big), \tag{3.13}$$

and therefore, for all $t < T^*$,

$$\sup(\|(w(t,.)\|_p, \|, z(t,.)\|_p) \le \frac{(L_p(t))^{\frac{1}{p}}}{\min(\alpha_p(p),\alpha_0(p))^{\frac{1}{p}}}, \tag{3.14}$$

from (3.12),(3.13) and (3.14), we obtain

$$\sup(\|(w(t,.)\|_p, \|z(t,.)\|_p) \le c_p(t), \forall t < T^*, \tag{3.15}$$

where

$$c_p(t) = \frac{\left[L_p(0) + \frac{c_2(p)}{c_1(p)}\right]\exp\big(\frac{c_1(p)}{p}t\big) - \frac{c_2(p)}{c_1(p)}}{\min(\alpha_p(p),\alpha_0(p))^{\frac{1}{p}}}.$$

The proof of Lemma is complete. □

Theorem 3.2. *Let $(w(t,.), z(t,.))$ be a solution of the problem (3.1)-(3.4). We assume that the condition (1.6) holds and one of the conditions (1.7) or (1.9) are satisfied. In addition if $\frac{p}{m} > \frac{n}{2}$, then the solution $(w(t,.), z(t,.))$ exists globally in time.*

Proof. From (1.6), we have

$$\begin{aligned}\sup(\|F(w,z)\|^{\frac{p}{m}}_{\frac{p}{m}}, \|G(w,z)\|^{\frac{p}{m}}_{\frac{p}{m}}) &= \sup(\left\|(-f+\frac{a-\lambda_1}{c}g)(u,v)\right\|^{\frac{p}{m}}_{\frac{p}{m}}, \left\|(f-\frac{a-\lambda_2}{c}g)(u,v)\right\|^{\frac{p}{m}}_{\frac{p}{m}}) \\ &\leq C^{\frac{p}{m}}\int_\Omega (|u|+v+1)^p dx \\ &\leq C'\int_\Omega (w+z+1)^p dx \forall r, \quad s \geq 0\end{aligned}$$

using the following formula

$$\begin{aligned}\int_\Omega (w+z+1)^p dx &= \int_\Omega (\sum_{i=0}^{k=p} C_p^i (w+z)^i dx = \int_\Omega ((w+z)^p+1)dx + \\ &+ \sum_{i=1}^{i=p-1} C_p^i \int_\Omega (w+z)^i dx,\end{aligned} \tag{3.16}$$

An application of Hölder's inequality from (3.16) gives

$$\begin{aligned}\int_\Omega (w+z+1)^p dx &\leq mes(\Omega) + \int_\Omega (w+z)^p dx + \\ &+ \sum_{i=1}^{i=p-1} C_p^i (mes(\Omega))^{\frac{p-i}{p}} (\int_\Omega (w+z)^p dx)^{\frac{i}{p}},\end{aligned} \tag{3.17}$$

using (3.15) we get

$$(\int_\Omega (w+z)^p dx)^{\frac{1}{p}} = \|w(t,.)+z(t,.)\|_p \leq \|w(t,.)\|_p + \|z(t,.)\|_p \leq 2c_p(t),$$

and the inequality (3.17) can be written as follows

$$\begin{aligned}\int_\Omega (w+z+1)^p dx &\leq mes(\Omega) + 2^p (c_p(t))^p + \\ \sum_{i=1}^{i=p-1} C_p^i 2^i (c_p(t))^i (mes(\Omega))^{\frac{p-i}{p}} &\leq \sum_{i=0}^{i=p} C_p^i 2^i (c_p(t))^i (mes(\Omega))^{\frac{p-i}{p}}.\end{aligned}$$

□

Therefore

$$\sup(\|F(w,z)\|_{\frac{p}{m}}^{\frac{p}{m}}, \|G(w,z)\|_{\frac{p}{m}}^{\frac{p}{m}}) \leq C'_m \int_\Omega (w+z+1)^p dx$$
$$\leq C'_m \sum_{i=0}^{i=p} C_p^i 2^i (c_p(t))^i (mes(\Omega))^{\frac{p-i}{p}},$$

which gives that

$$\sup(\left\|(-f+\frac{a-\lambda_1}{c}g)(r,s)\right\|_{\frac{p}{m}}^{\frac{p}{m}}, \left\|(f-\frac{a-\lambda_2}{c}g)(r,s)\right\|_{\frac{p}{m}}^{\frac{p}{m}}) \leq C_{p,m}(t), \forall t < T^*,\ \frac{p}{m} > \frac{n}{2},$$

where

$$C_{p,m}(t) = C'_m(\sum_{i=0}^{i=p} C_p^i 2^i (c_p(t))^i (mes(\Omega))^{\frac{p-i}{p}}).$$

If we take $q = \frac{p}{m}$, we obtain a uniform estimate of $\sup(\|(F(w,z)\|_q, \|(F(w,z)\|_q)$, for some $q > \frac{n}{2}$.

This ends the proof of Theorem.

Remark 3.1. It is clear that condition (1.5) implies the positivity of the solution of the system (3.1)-(3.4) on its interval of existence. See [12], for more details.

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