NEW INEQUALITIES OF WIRTINGER TYPE FOR CONVEX AND MN-CONVEX FUNCTIONS

Tatjana Z. Mirković

Abstract. In this paper, we obtain some inequalities of Wirtinger type by using some classical inequalities and means for convex functions and establish some applications to special means for positive real numbers.

Keywords. Inequalities; inequalities of Wirtinger type; convex functions.

1. Introduction

Let f be a periodic function with period 2π and let $f' \in L^2$. Then, if $\int_0^{2\pi} f(x)\,dx = 0$, the following inequality holds

$$
(1.1) \qquad \int_0^{2\pi} f^2(x)\,dx \leqslant \int_0^{2\pi} f'^2(x)\,dx,
$$

with equality if and only if $f(x) = A\cos x + B\sin x$, where A and B are constants.

Inequality (1.1) is known in the literature as Witinger's inequality. The proof of W. Wirtinger was first published in 1916 in the book (see [1]) by W. Blaschke. There are many studies which generalize and extend Wirtinger's inequality in the literature, (see [2], [3]). However, Inequality (1.1) was known before this, though with other conditions on the function f. For example, in 1905, E.Almansi proved that

$$
(1.2) \qquad \int_a^b f^2(x)\,dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2\,dx,
$$

under the condition that f and f' are continuous on the interval (a, b), that $f(a) = f(b)$ and that $\int\limits_a^b f(x)\,dx = 0$.

Theorem 1.1. *(Hölder inequality) Let $f(x)$ and $g(x)$ be positive continuous functions on $[a, b]$. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$(1.3) \qquad \int\limits_a^b f(x)\,g(x)dx \leqslant \left(\int\limits_a^b f(x)^p\,dx\right)^{\frac{1}{p}} \left(\int\limits_a^b g(x)^q\,dx\right)^{\frac{1}{q}}.$$

Theorem 1.2. *(Reverse Hölder inequality) For two positive functions f and g satisfying $0 < m \leqslant \frac{f^p}{g^q} \leqslant M < \infty$, on the set X, and for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we get*

$$\left(\int\limits_X f^p\,d\mu\right)^{\frac{1}{p}} \left(\int\limits_X g^q\,d\mu\right)^{\frac{1}{q}} \leqslant \left(\frac{M}{m}\right)^{\frac{1}{pq}} \int\limits_X fg\,d\mu.$$

Definition 1.1. *A function $I \subseteq R \to R$ is said to be convex (concave) if whenever $x, y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds:*

$$(1.4) \qquad f(tx + (1-t)\,y) \leqslant (\geqslant)\,tf(x) + (1-t)\,f(y).$$

Anderson mentioned mean function in [4] as follows:

Definition 1.2. *A function $M : (0, \infty) \to (0, \infty)$ is called a mean function if*
   (a) Symmetry: $M(x, y) = M(y, x)$;
   (b) Reflexivity: $M(x, x) = x$;
   (c) Monotonicity: $\min\{x, y\} \leqslant M(x, y) \leqslant \max\{x, y\}$;
   (d) Homogeneity: $M(\lambda x, \lambda y) = \lambda M(x, y)$, for any positive scalar λ.

Definition 1.3. *Let $I \to (0, \infty)$ be continuous, where I is a subinterval of $(0, \infty)$. Let M and N be any two mean functions. We say f is MN-convex (concave) if $f(M(x, y)) \leqslant (\geqslant) N(f(x), f(y))$ for all $x, y \in I$.*

Taking into account Definition 1.3, MN-convex function will be defined by the formulas:

   1. f is AA-convex iff (1.4) holds;

   2. f is AG-convex iff

$$f(t\alpha + (1-t)) \leqslant [f(\alpha)]^t\,[f(\beta)]^t, \quad 0 \leqslant t \leqslant 1;$$

3. f is AH-convex iff

$$f\left(\left(1-t\right)\alpha+t\beta\right)\leqslant\frac{f\left(\alpha\right)f\left(\beta\right)}{tf\left(\alpha\right)+\left(1-t\right)f\left(\beta\right)},\quad 0\leqslant t\leqslant 1;$$

4. f is GA-convex iff

$$f\left(\alpha^{t}\beta^{1-t}\right)\leqslant tf\left(\alpha\right)+\left(1-t\right)f\left(\beta\right),\quad 0\leqslant t\leqslant 1;$$

5. f is GG-convex iff

$$f\left(\alpha^{t}\beta^{1-t}\right)\leqslant\left[f\left(\alpha\right)\right]^{t}\left[f\left(\beta\right)\right]^{1-t},\quad 0\leqslant t\leqslant 1;$$

6. f is GH-convex iff

$$f\left(\alpha^{1-t}\beta^{t}\right)\leqslant\frac{f\left(\alpha\right)f\left(\beta\right)}{tf\left(\alpha\right)+\left(1-t\right)f\left(\beta\right)},\quad 0\leqslant t\leqslant 1;$$

7. f is HA-convex iff

$$f\left(\frac{\alpha\beta}{\left(1-t\right)\alpha+t\beta}\right)\leqslant tf\left(\alpha\right)+\left(1-t\right)f\left(\beta\right),\quad 0\leqslant t\leqslant 1;$$

8. f is HG-convex iff

$$f\left(\frac{\alpha\beta}{\left(1-t\right)\alpha+t\beta}\right)\leqslant\left[f\left(\alpha\right)\right]^{t}\left[f\left(\beta\right)\right]^{1-t},\quad 0\leqslant t\leqslant 1;$$

9. f is HH-convex iff

$$f\left(\frac{\alpha\beta}{\left(1-t\right)\alpha+t\beta}\right)\leqslant\frac{f\left(\alpha\right)f\left(\beta\right)}{\left(1-t\right)f\left(\alpha\right)+tf\left(\beta\right)},\quad 0\leqslant t\leqslant 1.$$

The main aim of this paper is to prove some new Wirtinger-type integral inequalities for convex and MN-convex functions.

2.   Main Results

Theorem 2.1. *Let f and f' be continuous functions on the interval $\left(a,b\right)$, with $a<b$, $f\left(a\right)=f\left(b\right)$ and $\int_{a}^{b}f\left(x\right)dx=0$. If $\left(f'\right)^{2}$ is convex on $\left[a,b\right]$, then*

$$(2.1)\qquad\int_{a}^{b}\left[f\left(x\right)\right]^{2}dx\leqslant\frac{\left(b-a\right)^{3}}{\left(2\pi\right)^{2}}\frac{\left[f'\left(a\right)\right]^{2}+\left[f'\left(b\right)\right]^{2}}{2}$$

Proof: Since $(f')^2$ is a convex function on $[a, b]$, therefore for $t \in [0, 1]$ we have

$$\frac{1}{b-a} \int_a^b [f'(x)]^2 \, dx = \int_0^1 [f'(ta + (1-t) b)]^2 \, dt$$

$$\leqslant \int_0^1 \left[t[f'(a)]^2 + (1-t) [f'(b)]^2 \right] dt = \frac{[f'(a)]^2 + [f'(b)]^2}{2}.$$

Now multiplying both sides of the above inequality by $\frac{(b-a)^3}{(2\pi)^2}$ and with (1.2), we get the desired inequality in (2.1). $\square$

Theorem 2.2. *Let f and f' be continuous functions on the interval (a, b), with $a < b$, $f(a) = f(b)$ and $\int_a^b f(x) \, dx = 0$. If f' is convex on $[a, b]$, then*

$$(2.2) \quad \int_a^b [f(x)]^2 \, dx \leqslant \frac{(b-a)^3}{(2\pi)^2} \left\{ \frac{(f'(a))^2 + (f'(a))(f'(a)) + (f'(b))^2}{3} \right\}.$$

Proof: We have

$$\left(\frac{b-a}{2\pi} \right)^2 \int_a^b [f'(x)]^2 \, dx = \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [f'(ta + (1-t) b)]^2 \, dt$$

$$\leqslant \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [t f'(a) + (1-t) f'(b)]^2 \, dt$$

$$= \frac{(b-a)^3}{(2\pi)^2} \left\{ \frac{(f'(a))^2 + (f'(a))(f'(a)) + (f'(b))^2}{3} \right\}.$$

By applying (1.2), we get (2.2). $\square$

Theorem 2.3. *Let f and f' be continuous functions on the interval (a, b) with $a < b$, $f(a) = f(b)$ and $\int_a^b f(x) \, dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are convex on $[a, b]$, then the following inequality holds*

$$\int_a^b [f(x)]^2 \, dx \leqslant \alpha (b-a)^3 \frac{[f'(a)]^{\frac{1}{\alpha}} + [f'(b)]^{\frac{1}{\alpha}}}{8\pi^2} + \beta (b-a)^3 \frac{[f'(a)]^{\frac{1}{\beta}} + [f'(b)]^{\frac{1}{\beta}}}{8\pi^2},$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

Proof: By using the well-known inequality $cd \leqslant \alpha c^{\frac{1}{\alpha}} + \beta d^{\frac{1}{\beta}}$ $(\alpha, \beta, c, d > 0$ and $\alpha + \beta = 1)$, the convexity of $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$, we get

$$
\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 \, dx = \frac{(b-a)^3}{(2\pi)^2} \int_0^1 f'(ta + (1-t)b)\, f'(ta + (1-t)b)\, dt
$$

$$
\leqslant \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 [f'(ta + (1-t)b)]^{\frac{1}{\alpha}} \, dt + \beta \int_0^1 [f'(ta + (1-t)b)]^{\frac{1}{\beta}} \, dt \right\}
$$

$$
\leqslant \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 [tf'(a) + (1-t)f'(b)]^{\frac{1}{\alpha}} \, dt + \beta \int_0^1 [tf'(a) + (1-t)f'(b)]^{\frac{1}{\beta}} \, dt \right\}
$$

$$
\leqslant \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 \left[t\,(f'(a))^{\frac{1}{\alpha}} + (1-t)\,(f'(b))^{\frac{1}{\alpha}} \right] dt \right.
$$

$$
\left. + \beta \int_0^1 \left[t\,(f'(a))^{\frac{1}{\beta}} + (1-t)\,(f'(b))^{\frac{1}{\alpha}} \right] dt \right\}
$$

$$
= \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \frac{(f'(a))^{\frac{1}{\alpha}} + (f'(b))^{\frac{1}{\alpha}}}{2} + \beta \frac{(f'(a))^{\frac{1}{\beta}} + (f'(b))^{\frac{1}{\beta}}}{2} \right\}.
$$

Combining with (1.2), we get the required inequality. □

Theorem 2.4. *Let f and f' be continuous on the interval (a,b), with $a < b$, $f(a) = f(b)$, $\int_a^b f(x)\, dx = 0$ and $f > 0$. Let $0 < m \leqslant \frac{|f|^p}{|f|^q} \leqslant M < \infty$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f|^p$, $|f|^q$ are concave on $[a,b]$ then*

$$
(2.4) \qquad \left(\frac{m}{M}\right)^{\frac{1}{pq}} [f(a) + f(b)]^2 \leqslant \frac{b-a}{\pi^2} \int_a^b [f'(x)]^2 \, dx.
$$

Proof: Making changes to the variable, using the reverse Hölder inequality and inequality $|u + v|^r \leqslant 2^{r-1}(|u|^r + |v|^r)$, $u, v \in \mathbb{R}$, we have

$$
\frac{1}{b-a} \int_a^b [f(x)]^2 \, dx = \int_0^1 [f(ta + (1-t)b)]^2 \, dt
$$

$$
\geqslant \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\int_0^1 |f(ta + (1-t)b)|^p \, dt \right)^{\frac{1}{p}} \left(\int_0^1 |f(ta + (1-t)b)|^q \, dt \right)^{\frac{1}{q}}
$$

$$\geqslant \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\int\limits_0^1 [t|f(a)|^p + (1-t)|f(b)|^p]\, dt\right)^{\frac{1}{p}}$$

$$\cdot \left(\int\limits_0^1 [t|f(a)|^q + (1-t)|f(b)|^q]\, dt\right)^{\frac{1}{q}}$$

$$= \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\frac{|f(a)|^p + |f(b)|^p}{2}\right)^{\frac{1}{p}} \left(\frac{|f(a)|^q + |f(b)|^q}{2}\right)^{\frac{1}{q}}$$

$$\geqslant \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\frac{|f(a) + f(b)|^p}{2^p}\right)^{\frac{1}{p}} \left(\frac{|f(a) + f(b)|^q}{2^q}\right)^{\frac{1}{q}}$$

$$= \left(\frac{m}{M}\right)^{\frac{1}{pq}} \frac{(f(a) + f(b))^2}{4}. \quad \square$$

By (1.2), we get the inequality (2.4). $\square$

Theorem 2.5. *Let f and f' be continuous on the interval (a,b), with $a < b$, $f(a) = f(b)$ and $\int_a^b f(x)\, dx = 0$. Then:*

1. *If $|f'|$ is AG convex, then*

$$\int\limits_a^b [f(x)]^2\, dx \leqslant \frac{(b-a)^3}{8\pi^2} \frac{[f'(a) f'(b)]^2 - 1}{\ln[f'(a) f'(b)]};$$

2. *If $|f'|$ is AH convex, then*

$$\int\limits_a^b [f(x)]^2\, dx \leqslant \frac{(b-a)^3}{(2\pi)^2} f'(a) f'(b);$$

3. *If $|f'|$ is GA convex, then*

$$\int\limits_a^b [f(x)]^2\, dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \left\{ \left[-a + \frac{2a}{\ln \frac{a}{b}} + 2\frac{b-a}{\ln^2 \frac{a}{b}}\right] [f'(a)]^2 \right.$$

$$+ \left[-2(a+b)\frac{1}{\ln \frac{a}{b}} - 4\frac{b-a}{\ln^2 \frac{a}{b}}\right] f'(a) f'(b)$$

$$+ \left.\left[b + \frac{2b}{\ln \frac{a}{b}} + \frac{2(b-a)}{\ln^2 \frac{a}{b}}\right] [f'(b)]^2 \right\};$$

4. If $|f'|$ is GG convex, then

$$\int_a^b [f(x)]^2 \, dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \ln \frac{a}{b} \frac{b\,[f'(b)]^2 - a\,[f'(a)]^2}{\ln a\,[f'(a)]^2 - \ln b\,[f'(b)]^2};$$

5. If $|f'|$ is HA convex, then

$$\int_a^b [f(x)]^2 \, dx \leqslant \frac{1}{(2\pi)^2} \left\{ \left[a(b-a)\left(b + a(b-a)^2\right) - a^2 b \ln \frac{b}{a} \right] [f'(a)]^2 \right.$$

$$+ \left[ab(b-a)\left(1 - (b-a)^2\right) - ab(a+b) \ln \frac{b}{a} \right] f'(a) f'(b)$$

$$\left. + \left[b(b-a)\left(a + b(b-a)^2\right) - ab^2 \ln \frac{b}{a} \right] [f'(b)]^2 \right\}.$$

Proof.

1. From (1.2) and by using the AG convexity of $|f'|$ we have

$$\int_a^b [f(x)]^2 \, dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 \, dx = \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [f'(ta + (1-t)b)]^2 \, dt$$

$$\leqslant \frac{(b-a)^3}{(2\pi)^2} \int_0^1 \left[(f'(a))^t (f'(b))^t \right]^2 \, dt = \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [(f'(a))(f'(b))]^{2t} \, dt$$

$$= \frac{(b-a)^3}{8\pi^2} \frac{[f'(a) f'(b)]^2 - 1}{\ln[f'(a) f'(b)]};$$

2. Since $|f'|$ is an AH-convex function, we can write

$$\int_a^b [f(x)]^2 \, dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 \, dx = \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [f'((1-t)a + tb)]^2 \, dt$$

$$\leqslant \frac{(b-a)^3}{(2\pi)^2} \int_0^1 \left[\frac{f'(a) f'(b)}{t f'(a) + (1-t) f'(b)} \right]^2 \, dt = \frac{(b-a)^3}{(2\pi)^2} f'(a) f'(b);$$

3. Taking into account that $|f'|$ is GA- convex, we have

$$\int_a^b [f(x)]^2\,dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2\,dx = \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b} \int_1^0 [f'(a^t b^{1-t})]^2\, a^t b^{1-t}dt$$

$$\leqslant \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b} \int_1^0 [tf'(a) + (1-t)f'(b)]^2\, a^t b^{1-t}dt$$

$$= b\left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b}\left\{ \left[(f'(a))^2 - 2f'(a)f'(b) + (f'(b))^2\right]\int_1^0 t^2\left(\frac{a}{b}\right)^t dt\right.$$

$$\left. + \left[2f'(a)f'(b) - 2(f'(b))^2\right]\int_1^0 t\left(\frac{a}{b}\right)^t dt + (f'(b))^2\int_1^0 \left(\frac{a}{b}\right)^t dt\right\}$$

$$= \left(\frac{b-a}{2\pi}\right)^2 \left\{ \left[-a + \frac{2a}{\ln\frac{a}{b}} + 2\frac{b-a}{\ln^2\frac{a}{b}}\right][f'(a)]^2\right.$$

$$+ \left[-2(a+b)\frac{1}{\ln\frac{a}{b}} - 4\frac{b-a}{\ln^2\frac{a}{b}}\right]f'(a)f'(b)$$

$$\left. + \left[b + \frac{2b}{\ln\frac{a}{b}} + \frac{2(b-a)}{\ln^2\frac{a}{b}}\right][f'(b)]^2\right\};$$

4. Since $|f'|$ is GG-convex, we have

$$\int_a^b [f(x)]^2\,dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2\,dx$$

$$= \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b} \int_1^0 [f'(a^t b^{1-t})]^2\, a^t b^{1-t}dt$$

$$\leqslant \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b} \int_1^0 \left\{[f'(a)]^t [f'(b)]^{1-t}\right\}^2 a^t b^{1-t}dt$$

$$= b\left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b} [f'(b)]^2 \int_1^0 \left\{\frac{a[f'(a)]^2}{b[f'(b)]^2}\right\}^t dt$$

$$= \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b} \frac{b[f'(b)]^2 - a(f'(a))^2}{\ln a[f'(a)]^2 - \ln b[f'(b)]^2};$$

5. Since $|f'|$ is HA-convex, we have

$$
\int\limits_a^b \left[f\left(x\right)\right]^2 dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int\limits_a^b \left[f'\left(x\right)\right]^2 dx
$$

$$
= ab\frac{(b-a)^3}{(2\pi)^2} \int\limits_0^1 \left[f'\left(\frac{ab}{(1-t)\,a+tb}\right)\right]^2 \frac{1}{\left[(1-t)\,a+tb\right]^2} dt
$$

$$
\leqslant ab\frac{(b-a)^3}{(2\pi)^2} \int\limits_0^1 \left[\left(\frac{t f'\left(a\right)+(1-t)\,f'\left(b\right)}{(1-t)\,a+tb}\right)\right]^2 dt
$$

$$
= \frac{1}{(2\pi)^2}\left\{\left[a\left(b-a\right)\left(b+a\left(b-a\right)^2\right)-a^2b\ln\frac{b}{a}\right]\left[f'\left(a\right)\right]^2\right.
$$

$$
+\left[ab\left(b-a\right)\left(1-\left(b-a\right)^2\right)-ab\left(a+b\right)\ln\frac{b}{a}\right]f'\left(a\right)f'\left(b\right)
$$

$$
\left.+\left[b\left(b-a\right)\left(a+b\left(b-a\right)^2\right)-ab^2\ln\frac{b}{a}\right]\left[f'\left(b\right)\right]^2\right\}. \quad \square
$$

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Tatjana Z. Mirković
College of Applied Professional Studies
Filipa Filipovića bb
17000 Vranje, Serbia
tmirkovic75@gmail.com