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SOME FIXED POINT RESULTS FOR GENERALIZED RATIONAL TYPE CONTRACTION MAPPINGS IN PARTIALLY ORDERED B-METRIC SPACES

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Abstract. The purpose of this paper is to state some fixed point results for generalized rational type contraction mappings in partially ordered b-metric spaces that generalize the main results of [7] and [22]. Also, some examples are given to illustrate the results.

Keywords: fixed point; b-metric space; rational contraction mapping; nonlinear integral equation

1. Introduction and preliminaries

In 1977, Jaggi [16] proved the following theorem for a contractive condition of rational type.

Theorem 1.1. [16, Theorem 1] *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a mapping such that*

$$d(Tx, Ty) \leqslant \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, $x \neq y$ and some $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$. Then T has a unique fixed point in X.

In 2010, Harjani *et al.* [12] proved a version of Theorem 1.1 in partially ordered metric spaces. In 2011, Luong *et al.* [22] proved the following theorem for generalized weak contractions satisfying rational expressions in partially ordered metric spaces, which is a generalization of the result of [12].

Denote by Ψ the family of all lower semi-continuous functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that $\psi(t) = 0$ if and only if $t = 0$.

Theorem 1.2. [22, Theorem 2.1] . *Let $(X, \preceq, d)$ be a complete, partially ordered metric space and $T : X \longrightarrow X$ be a mapping such that*

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1. *T is a non-decreasing mapping.*

2. *There exists $\psi \in \Psi$ such that $d(Tx, Ty) \le M(x, y) - \psi\Big(M(x, y)\Big)$ for all $x, y \in X$ with $x \succ y$, where*
$$M(x, y) = \max\Big\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\Big\}.$$

3. *T is continuous or X has the property: if $\{x_n\}$ is a non-decreasing sequence in X such that $\lim\limits_{n\to\infty} x_n = x$, then $x = \sup x_n$.*

4. *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*

Then T has a fixed point.

In 2013, Chandok *et al.* [7] established some common fixed point results for weak contractive conditions satisfying rational type expressions in partially ordered metric spaces, which are generalizations of the the main results in [22].

Definition 1.1. [15] Let $(X, \preceq)$ be a partially ordered set and $T, f : X \longrightarrow X$ be two mappings. T is called *monotone f-nondecreasing* if for all $x, y \in X$, $fx \preceq fy$ implies $Tx \preceq Ty$.

Definition 1.2. [18] Let (X, d) be a metric space and $T, f : X \longrightarrow X$ be two mappings. The pair (T, f) is called *compatible* if $\lim\limits_{n\to\infty} d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} fx_n = t$ for some $t \in X$.

Definition 1.3. [19] Let (X, d) be a metric space and $T, f : X \longrightarrow X$ be two mappings. The pair (T, f) is called *weakly compatible* if they commute at their coincidence points, that is, $Tfx = fTx$ for all $x \in X$ with $Tx = fx$.

The main results of [7] were stated as follows.

Theorem 1.3. [7, Theorem 2.1] *Let $(X, \preceq, d)$ be a partially ordered metric space and $T, f : X \longrightarrow X$ be two mappings such that*

1. *$TX \subset fX$ and fX is a complete subspace of X.*

2. *T is a monotone f-nondecreasing mapping.*

3. *There exists $\psi \in \Psi$ such that*
$$d(Tx, Ty) \le M(x, y) - \psi(M(x, y))$$
for all $x, y \in X$ with $fx \ne fy$, fx and fy being comparable, where
$$M(x, y) = \max\left\{\frac{d(fx, Tx)d(fy, Ty)}{d(fx, fy)}, d(fx, fy)\right\}.$$

4. *T and f are continuous mappings.*

5. *T and f are compatible.*

6. *There exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$.*

Then T and f have a coincidence point.

Theorem 1.4. [7, Theorem 2.2] *Let $(X, \preceq, d)$ be a partially ordered metric space and $T, f : X \longrightarrow X$ be two mappings such that*

1. *$TX \subset fX$ and fX is a complete subspace of X.*

2. *T is a monotone f-nondecreasing mapping.*

3. *There exists $\psi \in \Psi$ such that $d(Tx, Ty) \leq M(x, y) - \psi(M(x, y))$ for all $x, y \in X$ with $fx \neq fy$, fx and fy being comparable, where*

$$M(x,y) = \max\left\{\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}, d(fx,fy)\right\}.$$

4. *If $\{fx_n\}$ is a non-decreasing sequence in X such that $\lim_{n\to\infty} fx_n = fx$, then $fx = \sup fx_n$.*

5. *T and f are weakly compatible.*

6. *There exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$.*

Then T and f have a common fixed point. Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have only one common fixed point.

There have been many generalizations of a metric space and many fixed point theorems on generalized metric spaces have been stated [3, 6, 17, 25]. The notion of a b-metric space was introduced by Bakhtin in [4] and then extensively used by Czerwik in [8, 9] as follows.

Definition 1.4. [9] Let X be a non-empty set and $d : X \times X \longrightarrow [0, \infty)$ be a function such that for all $x, y, z \in X$ and some $s \geq 1$.

1. $d(x, y) = 0$ if and only if $x = y$.

2. $d(x, y) = d(y, x)$.

3. $d(x, y) \leq s\big(d(x, z) + d(z, y)\big)$.

Then d is called a b-*metric* on X and (X, d, s) is called a b-*metric space*.

The first important difference between a metric and a b-metric is that the b-metric need not be a continuous function in its two variables, see [21, Example 13]. In recent years, many fixed point theorems on b-metric spaces were stated, the readers may refer to [1, 2, 5, 10, 13, 14, 24, 26, 27, 28, 29] and references therein.

The purpose of this paper is to state some fixed point results for generalized rational type contraction mappings in partially ordered b-metric spaces that generalize the main results of [7] and [22]. Also, some examples are given to illustrate the results.

First, we recall some notions and lemmas which will be useful in what follows.

Definition 1.5. [9] Let (X, d, s) be a b-metric space.

1. A sequence $\{x_n\}$ is called *convergent* to x, written $\lim\limits_{n\to\infty} x_n = x$, if $\lim\limits_{n\to\infty} d(x_n, x) = 0$.
2. A sequence $\{x_n\}$ is called *Cauchy* in X if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
3. (X, d, s) is called *complete* if every Cauchy sequence is a convergent sequence.

Definition 1.6. [20] A function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is called an *altering distance function* if

1. φ is continuous and non-decreasing.
2. $\varphi(t) = 0$ if and only if $t = 0$.

Lemma 1.1. [11] *Let X be a nonempty set and $f : X \longrightarrow X$ a mapping. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \longrightarrow X$ is one-to-one.*

2. Main results

The following result is a sufficient condition for the existence of the fixed point for a generalized rational type contraction mapping in partially ordered b-metric spaces.

Theorem 2.1. *Let $(X, d, s, \preceq)$ be a complete, partially ordered b-metric space and $T : X \longrightarrow X$ be a mapping such that*

1. *T is a non-decreasing mapping.*
2. *There exist $\psi \in \Psi$ and an altering distance function φ such that*

$$\varphi\Big(sd(Tx, Ty)\Big) \le \varphi\Big(M(x, y)\Big) - \psi\Big(M(x, y)\Big) \tag{2.1}$$

for all $x, y \in X$ with $x \succ y$, where

$$M(x, y) = \max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\}.$$

3. *T is continuous.*

4. *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $x_0 \preceq Tx_0$, we construct a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for $n \geq 0$. Since T is a non-decreasing mapping, by induction, we can show that

$$x_0 \preceq x_1 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots$$

If there exists $n \geq 0$ such that $x_n = x_{n+1}$, then $x_n = Tx_n$, that is, x_n is a fixed point of T. So, we suppose $x_n \neq x_{n+1}$ for all $n \geq 0$. Since $x_n \succ x_{n-1}$ for all $n \geq 1$, from (2.1), we have

$$\begin{aligned} \varphi\Big(d(x_{n+1}, x_n)\Big) &= \varphi\Big(d(Tx_n, Tx_{n-1})\Big) \\ &\leq \varphi\Big(sd(Tx_n, Tx_{n-1})\Big) \\ &\leq \varphi\Big(M(x_n, x_{n-1})\Big) - \psi(M(x_n, x_{n-1})) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\left\{\frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1})}, d(x_n, x_{n-1})\right\} \\ &= \max\Big\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\Big\}. \end{aligned} \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\begin{aligned} \varphi\Big(d(x_{n+1}, x_n)\Big) \leq\ & \varphi\Big(\max\Big\{d(x_{n+1}, x_n), d(x_{n-1}, x_n)\Big\}\Big) \\ & -\psi\Big(\max\Big\{d(x_{n+1}, x_n), d(x_{n-1}, x_n)\Big\}\Big). \end{aligned} \tag{2.4}$$

If there exists $n \geq 1$ such that $d(x_{n+1}, x_n) > d(x_n, x_{n-1})$, then from (2.4), we have

$$\begin{aligned} \varphi\Big(d(x_{n+1}, x_n)\Big) &\leq \varphi\Big(d(x_{n+1}, x_n)\Big) - \psi(d(x_{n+1}, x_n)) \\ &< \varphi\Big(d(x_{n+1}, x_n)\Big). \end{aligned} \tag{2.5}$$

It is a contradiction. Hence, $d(x_{n+1}, x_n) \leq d(x_{n-1}, x_n)$ for all $n \geq 1$. Then, (2.4) becomes

$$\begin{aligned} \varphi\Big(d(x_{n+1}, x_n)\Big) &\leq \varphi\Big(d(x_n, x_{n-1})\Big) - \psi(d(x_n, x_{n-1})) \\ &< \varphi\Big(d(x_n, x_{n-1})\Big) \end{aligned} \tag{2.6}$$

for all $n \geq 1$ and $\{d(x_{n+1}, x_n)\}$ is a non-increasing sequence of non-negative real numbers. Hence, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_{n+1}, x_n) = r$. Now, we shall show that $r = 0$. Suppose to the contrary that $r > 0$. Taking the upper limit as $n \to \infty$ in (2.6) and using the property of φ and ψ, we get

$$\varphi(r) \leq \varphi(r) - \liminf_{n\to\infty} \psi(d(x_n, x_{n-1})) \leq \varphi(r) - \psi(r) < \varphi(r). \tag{2.7}$$

It is a contradiction. Therefore, $r = 0$, that is,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.8}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. If otherwise, then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that for every $k \in \mathbb{N}$, we have

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \quad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.9}$$

So, we have

$$\begin{aligned} \varepsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq sd(x_{m(k)}, x_{n(k)-1}) + sd(x_{n(k)-1}, x_{n(k)}) \\ &\leq s^2 d(x_{m(k)}, x_{m(k)-1}) + s^2 d(x_{m(k)-1}, x_{n(k)-1}) + sd(x_{n(k)-1}, x_{n(k)}). \end{aligned} \tag{2.10}$$

Taking the upper limit as $k \to \infty$ in (2.10) and using (2.8), we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}). \tag{2.11}$$

We also have

$$\begin{aligned} d(x_{m(k)-1}, x_{n(k)-1}) &\leq sd(x_{m(k)-1}, x_{m(k)}) + sd(x_{m(k)}, x_{n(k)-1}) \\ &< sd(x_{m(k)-1}, x_{m(k)}) + s\varepsilon. \end{aligned} \tag{2.12}$$

Taking the upper limit as $k \to \infty$ in (2.12) and using (2.8), we get

$$\limsup_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) \leq s\varepsilon. \tag{2.13}$$

Therefore, from (2.11) and (2.13), we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) \leq s\varepsilon. \tag{2.14}$$

Similarly, we also have

$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) \leq s\varepsilon. \tag{2.15}$$

Since $x_{n(k)-1} \succ x_{m(k)-1}$, from (2.1) we have

$$\begin{aligned} &\varphi\Big(sd(x_{n(k)}, x_{m(k)})\Big) \\ &= \varphi\Big(sd(Tx_{n(k)-1}, Tx_{m(k)-1})\Big) \\ &\leq \varphi\Big(M(x_{n(k)-1}, x_{m(k)-1})\Big) - \psi(M(x_{n(k)-1}, x_{m(k)-1})), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} (2.17) \qquad & M(x_{n(k)-1}, x_{m(k)-1}) \\ &= \max\Big\{\frac{d(x_{n(k)-1}, Tx_{n(k)-1})d(x_{m(k)-1}, Tx_{m(k)-1})}{d(x_{n(k)-1}, x_{m(k)-1})}, d(x_{n(k)-1}, x_{m(k)-1})\Big\} \\ &= \max\Big\{\frac{d(x_{n(k)-1}, x_{n(k)})d(x_{m(k)-1}, x_{m(k)})}{d(x_{n(k)-1}, x_{m(k)-1})}, d(x_{n(k)-1}, x_{m(k)-1})\Big\}. \end{aligned}$$

Taking the upper limit and the lower limit as $k \to \infty$ in (2.17) and using (2.8), (2.14), (2.15), we get

$$(2.18) \qquad \frac{\varepsilon}{s^2} \le \limsup_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) \le s\varepsilon$$

and

$$(2.19) \qquad \frac{\varepsilon}{s^2} \le \liminf_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) \le s\varepsilon.$$

Taking the upper limit as $k \to \infty$ in (2.16) and using (2.9), (2.18), (2.19), we obtain

$$\begin{aligned} (2.20) \qquad & \varphi(s\varepsilon) \\ \le\ & \varphi\Big(s \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)})\Big) \\ \le\ & \varphi\Big(\limsup_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1})\Big) - \psi\Big(\liminf_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1})\Big) \\ \le\ & \varphi\Big(s\varepsilon\Big) - \psi\Big(\liminf_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1})\Big) \\ <\ & \varphi(s\varepsilon). \end{aligned}$$

It is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in X. Since (X, d, s) is a complete b-metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since T is a continuous mapping, then $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T(\lim_{n\to\infty} x_{n-1}) = Tx$. It implies that x is a fixed point of T. □

The next result is another one for the existence of the fixed point for a generalized rational type contraction mapping in partially ordered b-metric spaces.

Theorem 2.2. *Let $(X, d, s, \preceq)$ be a complete, partially ordered b-metric space where d is continuous in each variable and $T : X \longrightarrow X$ be a mapping such that*

1. *T is a non-decreasing mapping.*
2. *There exist $\psi \in \Psi$ and an altering distance function φ such that*

$$(2.21) \qquad \varphi\Big(sd(Tx, Ty)\Big) \le \varphi\Big(M(x, y)\Big) - \psi\Big(M(x, y)\Big)$$

for all $x, y \in X$ with $x \succ y$, where

$$M(x, y) = \max\Big\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\Big\}.$$

3. *If* $\{x_n\}$ *is a non-decreasing sequence in* X *such that* $\lim\limits_{n\to\infty} x_n = x$, *then* $x = \sup x_n$.

4. *There exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$.

Then T *has a fixed point.*

Proof. Following the proof of Theorem 2.1, we have $\{x_n\}$ is a non-decreasing sequence. If there exists $n \geq 0$ such that $x_n = x_{n+1}$, then $x_n = Tx_n$, that is, x_n is a fixed point of T. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Also, from the proof of Theorem 2.1, we have $\lim\limits_{n\to\infty} x_n = x$. By the assumption (3), we have $x = \sup x_n$. Particularly, $x_n \prec x$ for all $n \geq 0$. Since T is a non-decreasing mapping, we have $Tx_n \preceq Tx$ for all $n \geq 0$, that is, $x_{n+1} \preceq Tx$ for all $n \geq 0$. Moreover, as $x_n \prec x_{n+1} \preceq Tx$ for all $n \geq 0$ and $x = \sup x_n$, we obtain

$$x \preceq Tx. \tag{2.22}$$

Consider the sequence $\{y_n\}$ in X that is constructed by $y_0 = x, y_{n+1} = Ty_n$ for $n \geq 0$. Since $x \preceq Tx$, we have $y_0 \preceq Ty_0$. By using the similar argument as in the proof of Theorem 2.1, we obtain that $\{y_n\}$ is a non-decreasing sequence and $\lim\limits_{n\to\infty} y_n = y$ for some $y \in X$. By the assumption (3), we have $\sup y_n = y$. Now suppose that $x \neq y$. Since $x_n \prec x = y_0 \preceq Tx = Ty_0 \preceq y_n \preceq y$ for all $n \geq 0$, from (2.21) we have

$$\begin{aligned} \varphi\Big(d(y_{n+1}, x_{n+1})\Big) &= \varphi\Big(d(Ty_n, Tx_n)\Big) \\ &\leq \varphi\Big(sd(Ty_n, Tx_n)\Big) \\ &\leq \varphi\Big(M(y_n, x_n)\Big) - \psi(M(y_n, x_n)) \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} M(y_n, x_n) &= \max\Big\{\frac{d(y_n, Ty_n)d(x_n, Tx_n)}{d(y_n, x_n)}, d(y_n, x_n)\Big\} \\ &= \max\Big\{\frac{d(y_n, y_{n+1})d(x_n, x_{n+1})}{d(y_n, x_n)}, d(y_n, x_n)\Big\}. \end{aligned} \tag{2.24}$$

Taking the limit as $n \to \infty$ in (2.24) and using the continuity in each variable of d, we obtain $\lim\limits_{n\to\infty} M(y_n, x_n) = \max\{0, d(y, x)\} = d(y, x)$. Then, taking the limit as $n \to \infty$ in (2.23) and using the property of φ and ψ, we have

$$\varphi\Big(d(y, x)\Big) \leq \varphi\Big(d(y, x)\Big) - \psi\Big(d(y, x)\Big) < \varphi\Big(d(y, x)\Big).$$

It is a contradiction. Therefore, $x = y$. Since $Tx \preceq y$, we have $Tx \preceq x$. Then, from (2.22), we have $Tx = x$, that is, x is a fixed point of T. □

In what follows, we shall prove the uniqueness of the fixed point in Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *Assume that*

1. *Either the assumptions of Theorem 2.1 or the assumptions of Theorem 2.2 hold.*
2. *For each $x, y \in X$, there exists $z \in X$ that is comparable to x and y.*

Then T has a unique fixed point.

Proof. From Theorem 2.1 and Theorem 2.2, we conclude that T has a fixed point. Suppose that $x, y \in X$ are two fixed points of T. By the assumption (2), there exists $z \in X$ such that z is comparable to x and y. We define the sequence $\{z_n\}$ by $z_0 = z, \;\; z_{n+1} = Tz_n$ for $n \geq 0$. Since z is comparable to x, we may assume that $z \preceq x$. Since T is a non-decreasing, by induction, we can show that $z_n \preceq x$ for all $n \geq 0$. Suppose that there exists $n_0 \geq 0$ such that $z_{n_0} = x$, then $z_n = Tz_{n-1} = Tx = x$ for all $n \geq n_0$. Hence, $\lim\limits_{n\to\infty} z_n = x$. On the other hand, if $z_n \neq x$ for all $n \geq 0$, we have

$$\begin{aligned}
\varphi\Big(d(x, z_n)\Big) &= \varphi\Big(d(Tx, Tz_{n-1})\Big) \\
&\leq \varphi\Big(sd(Tx, Tz_{n-1})\Big) \\
&\leq \varphi\Big(M(x, z_{n-1})\Big) - \psi(M(x, z_{n-1}))
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
M(x, z_{n-1}) &= \max\Big\{\frac{d(x, Tx)d(z_{n-1}, Tz_{n-1})}{d(x, z_{n-1})}, d(x, z_{n-1})\Big\} \\
&= \max\Big\{\frac{d(x, x)d(z_{n-1}, z_n)}{d(x, z_{n-1})}, d(x, z_{n-1})\Big\} \\
&= d(x, z_{n-1}).
\end{aligned} \tag{2.26}$$

From (2.25) and (2.26), we have

$$\begin{aligned}
\varphi\Big(d(x, z_n)\Big) &\leq \varphi\Big(d(x, z_{n-1})\Big) - \psi(d(x, z_{n-1})) \\
&< \varphi\Big(d(x, z_{n-1})\Big)
\end{aligned} \tag{2.27}$$

for all $n \geq 1$. Then from the property of φ, we conclude that $d(x, z_n) \leq d(x, z_{n-1})$ for all $n \geq 1$, that is, $\{d(x, z_n)\}$ is a non-increasing sequence of positive real numbers. Therefore, there exists $\alpha \geq 0$ such that $\lim\limits_{n\to\infty} d(x, z_n) = \alpha$. We shall show that $\alpha = 0$. Suppose to the contrary that $\alpha > 0$. Taking the upper limit as $n \to \infty$ in (2.27) and using the property of φ and ψ, we have

$$\begin{aligned}
\varphi(\alpha) &= \varphi\Big(\lim_{n\to\infty} d(x, z_n)\Big) \\
&\leq \varphi\Big(\lim_{n\to\infty} d(x, z_{n-1})\Big) - \liminf_{n\to\infty} \psi\Big(d(x, z_{n-1})\Big) \\
&\leq \varphi(\alpha) - \psi(\alpha) < \varphi(\alpha).
\end{aligned}$$

It is a contradiction. Hence, $\alpha = 0$, that is, $\lim\limits_{n\to\infty} d(x, z_n) = 0$. Thus, $\lim\limits_{n\to\infty} z_n = x$. Therefore, in both cases, we have

$$\lim_{n\to\infty} z_n = x. \tag{2.28}$$

Similarly, we may show that

$$\lim_{n\to\infty} z_n = y. \tag{2.29}$$

From (2.28) and (2.29), we get $x = y$. Therefore, T has a unique fixed point. □

Remark 2.1. *Since a b-metric space is a metric space when $s = 1$, so our results can be viewed as a generalization of corresponding results in [22].*

By using Theorem 2.1, Theorem 2.2 and Theorem 2.3, we obtain the following corollaries as a generalization of [7, Theorem 2.1] and [7, Theorem 2.2] in partially ordered b-metric spaces.

Corollary 2.1. *Let $(X,d,s,\preceq)$ be a partially ordered b-metric space and $T, f : X \longrightarrow X$ be two mappings such that*

1. *$TX \subset fX$ and fX is a complete subspace of X.*
2. *T is a monotone f-nondecreasing mapping.*
3. *There exist $\psi \in \Psi$ and an altering distance function φ such that*

$$\varphi\Big(sd(Tx,Ty)\Big) \le \varphi\Big(M_f(x,y)\Big) - \psi\Big(M_f(x,y)\Big) \tag{2.30}$$

for all $x, y \in X$ with $fx \succ fy$, where

$$M_f(x,y) = \max\Big\{\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}, d(fx,fy)\Big\}.$$

4. *T and f are continuous mappings.*
5. *T and f are compatible.*
6. *There exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$.*

Then T and f have a coincidence point.

Proof. By Lemma 1.1, there exists $E \subset X$ such that $fE = fX$ and $f : E \longrightarrow X$ is one-to-one. Now, define a map $h : fE \longrightarrow fE$ by $h(fx) = Tx$. Since f is one-to-one on E, h is well-defined. Then, $fE = fX$ is complete and (2.30) becomes

$$\varphi\Big(sd\Big(h(fx),h(fy)\Big)\Big) \quad\le\quad \varphi\Big(M_f(x,y)\Big) - \psi\Big(M_f(x,y)\Big)$$

for all $x, y \in X$ with $fx \succ fy$, where

$$M_f(x,y) \quad=\quad \max\Big\{\frac{d(fx,h(fx))d(fy,h(fy))}{d(fx,fy)}, d(fx,fy)\Big\}.$$

Let $x_0 \in E$ such that $fx_0 \preceq Tx_0 = h(fx_0)$. Choose $x_1 \in E$ such that $fx_1 = Tx_0 = h(fx_0) = hy_0$. By continuing this process, we obtain a sequence $\{fx_n\}$ in fE such that $fx_{n+1} = Tx_n = h(fx_n)$ for $n \geq 0$. By using the similar argument as in the proof of Theorem 2.1, we obtain that $\{fx_n\}$ is a Cauchy sequence in fE. Since fE is complete, there exists $u \in fE$ such that $\lim_{n\to\infty} fx_n = u \in fX$. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_{n-1} = u.$$

Since the pair (T, f) is compatible, we have

$$\lim_{n\to\infty} d(f(Tx_n), T(fx_n)) = 0. \tag{2.31}$$

Also, we have

$$d(Tu, fu) \leq sd(Tu, T(fx_n)) + s^2 d(T(fx_n), f(Tx_n)) + s^2 d(f(Tx_n), fu). \tag{2.32}$$

Taking the limit as $n \to \infty$ in (2.32) and using the continuity of T, f and (2.31), we get $d(Tu, fu) = 0$, that is, $Tu = fu$. Therefore, u is a coincidence point of T and f. □

Corollary 2.2. *Let $(X, d, s, \preceq)$ be a partially ordered b-metric space where d is continuous in each variable and $T, f : X \longrightarrow X$ be two mappings such that*

1. *$TX \subset fX$ and fX is a complete subspace of X.*
2. *T is a monotone f-nondecreasing mapping.*
3. *There exist $\psi \in \Psi$ and an altering distance function φ such that*

$$\varphi\Big(sd(Tx, Ty)\Big) \leq \varphi\Big(M_f(x, y)\Big) - \psi\Big(M_f(x, y)\Big) \tag{2.33}$$

for all $x, y \in X$ with $fx \succ fy$, where

$$M_f(x, y) = \max\Big\{\frac{d(fx, Tx)d(fy, Ty)}{d(fx, fy)}, d(fx, fy)\Big\}.$$

4. *If $\{fx_n\}$ is a non-decreasing sequence in X such that $\lim_{n\to\infty} fx_n = fx$, then $fx = \sup fx_n$.*
5. *If z is a coincidence point of T and f, then $fz \preceq f(fz)$.*
6. *T and f are weakly compatible.*
7. *There exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$.*

Then T and f have common fixed point. Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have only one common fixed point.

Proof. By Lemma 1.1, there exists $E \subset X$ such that $fE = fX$ and $f : E \longrightarrow X$ is one-to-one. Now, define a map $h : fE \longrightarrow fE$ by $h(fx) = Tx$. Since f is one-to-one on E, h is well-defined. Then, $fE = fX$ is complete and (2.33) becomes

$$\varphi\Big(sd\big(h(fx), h(fy)\big)\Big) \leq \varphi\Big(M_f(x, y)\Big) - \psi\Big(M_f(x, y)\Big)$$

for all $x, y \in X$ with $fx \succ fy$, where

$$M_f(x, y) = \max\Big\{\frac{d(fx, h(fx))d(fy, h(fy))}{d(fx, fy)}, d(fx, fy)\Big\}.$$

Moreover, other assumptions of Theorem 2.2 are fulfilled. Therefore, by using Theorem 2.2, there exists $z \in X$ such that $fz = h(fz) = Tz$, that is, z is a coincidence point of T and f.

Now suppose that T and f are weakly compatible. Let $w = Tz = fz$. Then, $Tw = T(fz) = f(Tz) = fw$. Suppose that $Tw \neq w$. Then, $fw \neq fz$ and $fz \preceq f(fz) = fw$, from (2.33), we have

$$\begin{aligned}
\varphi\Big(d(Tw, w)\Big) &= \varphi\Big(d(Tw, Tz)\Big) \\
&\leq \varphi\Big(sd(Tw, Tz)\Big) \\
&\leq \varphi\Big(\max\Big\{\frac{d(fw, Tw)d(fz, Tz)}{d(fw, fz)}, d(fw, fz)\Big\}\Big) \\
&\quad -\psi\Big(\max\Big\{\frac{d(fw, fw)d(w, w)}{d(Tw, w)}, d(Tw, w)\Big\}\Big) \\
&= \varphi\Big(\max\Big\{0, d(Tw, w)\Big\}\Big) - \psi\Big(\max\Big\{0, d(Tw, w)\Big\}\Big) \\
&< \varphi\Big(d(Tw, w)\Big).
\end{aligned}$$

It is a contradiction. Therefore, $Tw = w$. Then, $Tw = fw = w$, that is, w is a common fixed point of T and f.

Finally, from Theorem 2.3, we conclude that the set of common fixed points of T and f is well ordered if and only if T and f have only one common fixed point. □

Remark 2.2. *By using the similar argument as in the proof of Corollary 2.1 and Corollary 2.2, we see that [7, Theorem 2.1] is a consequence of [22, Theorem 2.1] and [7, Theorem 2.2] is a consequence of [22, Theorem 2.2] and [22, Theorem 2.4].*

In what follows, we give some examples to support our results. The following example is an illustration of the existence of the fixed point in case d is continuous in each variable.

Example 2.1. Let $X = \{1, 2, 3, 4\}$ with the usual order $\leq$. Define a b-metric d on X as follows.

$$d(1, 1) = d(2, 2) = d(3, 3) = d(4, 4) = 0,$$

$$d(1,2) = d(2,1) = d(1,3) = d(3,1) = d(2,3) = d(3,2) = 1,$$
$$d(1,4) = d(4,1) = d(2,4) = d(4,2) = 4,$$
$$d(3,4) = d(4,3) = 10.$$

Then (X,d,s) is a complete b-metric space with $s = 2$. Let $T : X \longrightarrow X$ be defined by

$$T1 = T2 = T3 = 1, T4 = 2.$$

Define two functions $\varphi(t) = t$ and $\psi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$. We consider the following two cases.

Case 1. For $x, y \in \{1,2,3\}$ and $x > y$. Then

$$\varphi\Big(2d(Tx,Ty)\Big) = 0 \leq \varphi\Big(M(x,y)\Big) - \psi\Big(M(x,y)\Big).$$

Case 2. For $x = 4$ and $y \in \{1,2,3\}$. Then $d(Tx,Ty) = d(2,1) = 1$, $M(4,3) = 10$ and $M(4,y) = 4$ if $y \in \{1,2\}$. Thus

$$\varphi\Big(2d(Tx,Ty)\Big) = 2 \leq \frac{M(x,y)}{2} = \varphi\Big(M(x,y)\Big) - \psi\Big(M(x,y)\Big).$$

By the above two cases, we conclude that the condition (2.1) of Theorem 2.1 and the condition (2.21) of Theorem 2.2 hold. Moreover, other assumptions of Theorem 2.1 and Theorem 2.2 are fulfilled. Therefore, Theorem 2.1 and Theorem 2.2 are applicable to T, φ, ψ and $(X,d,s,\leq)$.

However, it is easy to see that d is not a metric on X. This proves that [22, Theorem 2.1] and [22, Theorem 2.2] are not applicable to T, φ, ψ and $(X,d,s,\leq)$.

The following example is an illustration of the existence of the fixed point in case d is a non-continuous b-metric.

Example 2.2. Let $X = \Big\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\Big\}$ with the usual order $\leq$. Define a b-metric d on X as follows.

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{0,1\} \\ |x-y| & \text{if } x, y \in \Big\{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \geq 1\Big\} \\ 4 & \text{if otherwise.} \end{cases}$$

Then, (X,d,s) is a complete b-metric space with $s = \frac{8}{3}$ but d is a non-continuous b-metric, see [21, Example 13]. Let $T : X \longrightarrow X$ be defined by $T0 = 0$, $T\frac{1}{n} = \frac{1}{8n}$ for all $n \geq 1$. Define two functions $\varphi(t) = t$ and $\psi(t) = \frac{2t}{3}$ for all $t \in [0,\infty)$. Then, for $x, y \in X$, $x > y$, we consider the following two cases.

Case 1. For $x = \frac{1}{n}$ with $n \geq 1$ and $y = 0$, we have $d(Tx,Ty) = d\Big(\frac{1}{8n}, 0\Big) = \frac{1}{8n}$, $M(x,y) = \frac{1}{n}$ or $M(x,y) \in \{1,4\}$. Thus

$$\varphi\Big(\frac{8}{3}d(Tx,Ty)\Big) \leq \frac{M(x,y)}{3} = \varphi\Big(M(x,y)\Big) - \psi\Big(M(x,y)\Big).$$

Case 2. For $x = \frac{1}{n}$ and $y = \frac{1}{m}$ with $m > n \geq 1$, we have

$$d(Tx,Ty) = d\Big(\frac{1}{8n}, \frac{1}{8m}\Big) = \frac{1}{8}\Big(\frac{1}{n} - \frac{1}{m}\Big),$$

$$M(x,y) \geq \frac{1}{n} - \frac{1}{m} \quad \text{or} \quad M(x,y) = 4.$$

Hence

$$\varphi\Big(\frac{8}{3}d(Tx,Ty)\Big) \leq \frac{M(x,y)}{3} = \varphi\Big(M(x,y)\Big) - \psi\Big(M(x,y)\Big).$$

By the above two cases, we conclude that the condition (2.1) of Theorem 2.1 holds. Moreover, other assumptions of Theorem 2.1 are fulfilled. Therefore, Theorem 2.1 is applicable to T, φ, ψ and $(X, d, s, \preceq)$.

However, it is easy to see that d is not a metric on X. This proves that [22, Theorem 2.1] and [22, Theorem 2.2] are not applicable to T, φ, ψ and $(X, d, s, \preceq)$.

Finally, we apply Theorem 2.3 to study the existence and uniqueness of solutions to the nonlinear integral equation.

Example 2.3. Let $C[a,b]$ be the set of all continuous functions on $[a,b]$, the b-metric d with $s = 2^{p-1}$ defined by

$$d(x,y) = \sup_{t\in[a,b]}\{|x(t) - y(t)|^p\}$$

for all $x, y \in C[a,b]$ and some $p \geq 1$ and the partial order $\preceq$ given by $x \preceq y$ if $a \leqslant x(t) \leq y(t) \leqslant b$ for all $t \in [a,b]$. Consider the nonlinear integral equation

$$x(t) = g(t) + \int_a^b K(t,s,x(s))ds \tag{2.34}$$

where $t \in [a,b]$, $g : [a,b] \longrightarrow \mathbb{R}$ and $K : [a,b]\times[a,b]\times\mathbb{R} \longrightarrow \mathbb{R}$. Suppose that the following statements hold.

1. g is continuous and $K(t,s,x(s))$ is integrable with respect to s on $[a,b]$.
2. $Tx \in C[a,b]$ for all $x \in C[a,b]$, where $Tx(t) = g(t) + \displaystyle\int_a^b K(t,s,x(s))ds$ for all $t \in [a,b]$.
3. For all $s, t \in [a,b]$ and $x, y \in X$ with $x \succeq y$,
$$0 \leq K(t,s,x(s)) - K(t,s,y(s)) \leq \xi(t,s)|x(s) - y(s)|$$
where $\xi : [a,b]\times[a,b] \longrightarrow [0,\infty)$ is a continuous function satisfying
$$\sup_{t\in[a,b]}\Big(\int_a^b \xi^p(t,s)ds\Big) < \frac{1}{2^{p-1}(b-a)^{p-1}}.$$
4. There exists $x_0 \in C[a,b]$ such that $x_0(t) \leq g(t) + \displaystyle\int_a^b K(t,s,x_0(s))ds$ for all $t \in [a,b]$.

Then, the nonlinear integral equation (2.34) has a unique solution $x \in C[a,b]$.

Proof. Consider $T : C[a,b] \longrightarrow C[a,b]$ defined by $Tx(t) = g(t) + \displaystyle\int_a^b K(t,s,x(s))ds$ for all $x \in C[a,b]$ and $t \in [a,b]$. It follows from the conditions (1) and (2) that T is well-defined. Notice that the existence of a solution to (2.34) is equivalent to the existence of a fixed point of T. Now, we prove that all assumptions of Theorem 2.3 are satisfied.

(1). Let $x, y \in C[a, b]$ with $x \succeq y$. From the condition (3), $K(t, s, x(s)) \geq K(t, s, y(s))$ for all $s, t \in [a, b]$. It implies that $Tx(t) \geq Ty(t)$ for all $t \in [a, b]$ and hence $Tx \succeq Ty$. Then, T is a non-decreasing mapping.

(2). Let $1 \leq q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For all $x, y \in C[a, b]$ with $x \succ y$ and $t \in [a, b]$, from condition (3), we have

$$
\begin{aligned}
2^{p-1}|Tx(t) - Ty(t)|^p &\leq 2^{p-1}\Big(\int_a^b |K(t,s,x(s)) - K(t,s,y(s))|ds\Big)^p \\
&\leq 2^{p-1}\Big[\Big(\int_a^b ds\Big)^{\frac{1}{q}}\Big(\int_a^b |K(t,s,x(s)) - K(t,s,y(s))|^p ds\Big)^{\frac{1}{p}}\Big]^p \\
&\leq 2^{p-1}(b-a)^{\frac{p}{q}}\Big(\int_a^b \xi^p(t,s)|x(s) - y(s)|^p ds\Big) \\
&\leq 2^{p-1}(b-a)^{p-1}\Big(\int_a^b \xi^p(t,s)ds\Big)M(x,y) \\
&\leq \lambda M(x,y) \\
&= M(x,y) - (1-\lambda)M(x,y)
\end{aligned}
$$

where $\lambda = 2^{p-1}(b-a)^{p-1}\Big(\sup\limits_{t\in[a,b]} \int_a^b \xi^p(t,s)ds\Big) \in [0, 1)$. It implies that

$$2^{p-1}d(Tx, Ty) \leq M(x, y) - (1 - \lambda)M(x, y).$$

Therefore, the condition (2.21) in Theorem 2.2 holds with $\psi(t) = (1-\lambda)t$ and $\varphi(t) = t$ for all $t \geq 0$.

(3). By using the similar argument as in the proof of [23, Thoerem 3.1], we see that the assumption (3) in Theorem 2.2 also holds.

(4). From the assumption (4), there exists $x_0 \in C[a, b]$ such that $x_0 \preceq Tx_0$.

(5). For each $x, y \in C[a, b]$, put $z = \max\{x, y\}$, we have $z \in C[a, b]$ and z is comparable to x and y

From the above, all assumptions of Theorem 2.3 hold. Therefore, by Theorem 2.3, T has a unique fixed point $x \in C[a, b]$ and hence the integral equation (2.34) has a unique solution $x \in C[a, b]$. □

The following example guarantees the existence of the function K that satisfies all assumptions in Example 2.3.

Example 2.4. Let $C[0, \frac{\pi}{2}]$ be the set of all continuous functions on $[0, \frac{\pi}{2}]$, the b-metric d with $s = 2$ defined by

$$d(x, y) = \sup_{t\in[0,\frac{\pi}{2}]} \{|x(t) - y(t)|^2\}$$

for all $x, y \in C[0, \frac{\pi}{2}]$ and the partial order $\preceq$ given by

$$x \preceq y \text{ if } 0 \leq x(t) \leq y(t) \leq \tfrac{\pi}{2} \text{ for all } t \in [0, \tfrac{\pi}{2}].$$

Consider the nonlinear integral equation

$$x(t) = -\sqrt{\frac{96}{\pi^7}}t^2 + t + \sqrt{\frac{96}{\pi^7}}\int_0^{\frac{\pi}{2}} t^2 s \sin x(s) ds$$

for all $x \in C[0, \frac{\pi}{2}]$. Put $Tx(t) = -\sqrt{\frac{96}{\pi^7}}t^2 + t + \sqrt{\frac{96}{\pi^7}}\int_0^{\frac{\pi}{2}} t^2 s \sin x(s) ds$, $g(t) = -\sqrt{\frac{96}{\pi^7}}t^2 + t$ and $K(t, s, x(s)) = \sqrt{\frac{96}{\pi^7}}t^2 s \sin x(s)$ for all $x \in C[0, \frac{\pi}{2}]$ and all $s, t \in [0, \frac{\pi}{2}]$. Then

(1). g is continuous on $[0, \frac{\pi}{2}]$. Since $x \in C[0, \frac{\pi}{2}]$, $K(t, s, x(s))$ is integrable with respect to s on $[0, \frac{\pi}{2}]$.

(2). For every $t \in [0, \frac{\pi}{2}]$ and the sequence $\{t_n\} \subset [0, \frac{\pi}{2}]$ with $\lim\limits_{n\to\infty} t_n = t$. Then, for all $x \in C[0, \frac{\pi}{2}]$, we have

$$\begin{aligned} |Tx(t_n) - Tx(t)| &\leq |g(t_n) - g(t)| + \sqrt{\frac{96}{\pi^7}}\int_0^{\frac{\pi}{2}} s|t_n^2 - t^2|| \sin x(s)| ds \\ &\leq |g(t_n) - g(t)| + \sqrt{\frac{3}{2\pi^3}}|t_n^2 - t^2|. \end{aligned}$$

It implies that $\lim\limits_{n\to\infty} Tx(t_n) = Tx(t)$ and hence $Tx \in C[0, \frac{\pi}{2}]$ for all $x \in C[0, \frac{\pi}{2}]$.

(3). For all $s, t \in [0, \frac{\pi}{2}]$ and $x, y \in C[0, \frac{\pi}{2}]$ with $x \geq y$, we have $0 \leq y(s) \leq x(s) \leq \frac{\pi}{2}$. Therefore,

$$\begin{aligned} 0 \leq K(t, s, x(s)) - K(t, s, y(s)) &= \sqrt{\frac{96}{\pi^7}}t^2 s| \sin x(s) - \sin y(s)| \\ &\leq \xi(t, s)|x(s) - y(s)| \end{aligned}$$

where $\xi : [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \longrightarrow [0, \infty)$ defined by $\xi(t, s) = \sqrt{\frac{96}{\pi^7}}t^2 s$. It easy to check that ξ is a continuous function and $\sup\limits_{t\in[0,\frac{\pi}{2}]} \Big(\int_0^{\frac{\pi}{2}} \xi^2(t, s) ds\Big) = \frac{1}{4} < \frac{1}{\pi}$.

(4). By choosing $x_0(t) = t$ for all $t \in [0, \frac{\pi}{2}]$, we have $Tx_0(t) = t$ for all $t \in [0, \frac{\pi}{2}]$. Therefore, $x_0(t) \leq Tx_0(t)$ for all $t \in [0, \frac{\pi}{2}]$.

From the above, all assumptions to K and g in Example 2.3 are satisfied.

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