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η-RICCI SOLITONS ON KENMOTSU MANIFOLD WITH GENERALIZED SYMMETRIC METRIC CONNECTION

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Abstract. The objective of the present paper is to study the η-Ricci solitons on Kenmotsu manifold with generalized symmetric metric connection of type (α, β). Ricci and η-Ricci solitons with generalized symmetric metric connection of type (α, β) have been discussed, satisfying the conditions $\bar{R}.\bar{S} = 0$, $\bar{S}.\bar{R} = 0$, $\bar{W_2}.\bar{S} = 0$ and $\bar{S}.\bar{W}_2 = 0$.. Finally, we have constructed an example of Kenmotsu manifold with generalized symmetric metric connection of type (α, β) admitting η-Ricci solitons.
Keywords: Kenmotsu manifold; Generalized symmetric metric connection; η-Ricci soliton; Ricci soliton, Einstein manifold.

1. Introduction

A linear connection $\overline{\nabla}$ is said to be generalized symmetric connection if its torsion tensor T is of the form

$$T(X, Y) = \alpha\{u(Y)X - u(X)Y\} + \beta\{u(Y)\varphi X - u(X)\varphi Y\}, \tag{1.1}$$

for any vector fields X, Y on a manifold, where α and β are smooth functions. φ is a tensor of type $(1, 1)$ and u is a $1-$form associated with a non-vanishing smooth non-null unit vector field ξ. Moreover, the connection $\overline{\nabla}$ is said to be a generalized symmetric metric connection if there is a Riemannian metric g in M such that $\overline{\nabla} g = 0$, otherwise it is non-metric.

In the equation (1.1), if $\alpha = 0$ $(\beta = 0)$, then the generalized symmetric connection is called $\beta-$ quarter-symmetric connection ($\alpha-$ semi-symmetric connection), respectively. Moreover, if we choose $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, then the generalized symmetric connection is reduced to a semi-symmetric connection and quarter-symmetric connection, respectively. Therefore, a generalized symmetric

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connections can be viewed as a generalization of semi-symmetric connection and quarter-symmetric connection. These two connections are important for both the geometry study and applications to physics. In [12], H. A. Hayden introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors (see [1], [9], [10] , [24], [26]). The idea of quarter-symmetric linear connections in a differential manifold was introduced by S.Golab [11]. In [23], Sharfuddin and Hussian defined a semi-symmetric metric connection in an almost contact manifold, by setting

$$T(X,Y) = \eta(Y)X - \eta(X)Y.$$

In [13], [25] and [19] the authors studied the semi-symmetric metric connection and semi-symmetric non-metric connection in a Kenmotsu manifold, respectively.

In the present paper, we have defined new connection for Kenmotsu manifold, generalized symmetric metric connection. This connection is the generalized form of semi-symmetric metric connection and quarter-symmetric metric connection.

On the other hand, a Ricci soliton is a natural generalization of an Einstein metric. In 1982, R. S. Hamilton [14] said that the Ricci solitons moved under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are sationary points of the Ricci flow:

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.2}$$

Definition 1.1. A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$\mathcal{L}_V g + 2S + 2\lambda = 0, \tag{1.3}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

In 1925, H. Levy [16] in Theorem 4, proved that a second order parallel symmetric non-singular tensor in real space forms is proportional to the metric tensor. Later, R. Sharma [22] initiated the study of Ricci solitons in contact Riemannian geometry. After that, Tripathi [28], Nagaraja et. al. [17] and others like C. S. Bagewadi et. al. [4] extensively studied Ricci solitons in almost contact metric manifolds. In 2009, J. T. Cho and M. Kimura [6] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. η- Ricci solitons in almost paracontact metric manifolds have been studied by A. M. Blaga et. al. [2]. A. M. Blaga and various others authors have also studied η-Ricci solitons in manifolds with different structures (see [3], [20]). It is natural and interesting to study η-Ricci solitons in almost contact metric manifolds with this new connection.

Therefore, motivated by the above studies, in this paper we will study the η-Ricci solitons in a Kenmotsu manifold with respect to a generalized symmetric metric connection. We shall consider η-Ricci solitons in the almost contact geometry, precisely, on an Kenmotsu manifold with generalized symmetric metric connection which satisfies certain curvature properties: $\bar{R}.\bar{S} = 0$, $\bar{S}.\bar{R} = 0$, $W_2.\bar{S} = 0$ and $\bar{S}.\bar{W}_2 = 0$ respectively.

2. Preliminaries

A differentiable M manifold of dimension $n = 2m+1$ is called almost contact metric manifold [5], if it admits a $(1,1)$ tensor field ϕ, a contravaryant vector field ξ, a $1-$ form η and Riemannian metric g which satisfies

$$\phi\xi = 0, \tag{2.1}$$
$$\eta(\phi X) = 0 \tag{2.2}$$
$$\eta(\xi) = 1, \tag{2.3}$$
$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.4}$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.5}$$
$$g(X,\xi) = \eta(X), \tag{2.6}$$

for all vector fields X, Y on M. If we write $g(X,\phi Y) = \Phi(X,Y)$, then the tensor field ϕ is a anti-symmetric $(0,2)$ tensor field [5]. If an almost contact metric manifold satisfies

$$(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.7}$$
$$\nabla_X\xi = X - \eta(X)\xi, \tag{2.8}$$

then M is called a Kenmotsu manifold, where ∇ is the Levi-Civita connection of g [18].

In Kenmotsu manifolds the following relations hold [18]:

$$(\nabla_X\eta)Y = g(\phi X, \phi Y) \tag{2.9}$$
$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.10}$$
$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.11}$$
$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.12}$$
$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{2.13}$$
$$S(X,\xi) = -(n-1)\eta(X), \tag{2.14}$$
$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{2.15}$$

for any vector fields X, Y and Z, where R and S are the the curvature and Ricci the tensors of M, respectively.

A Kenmotsu manifold M is said to be generalized η Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\phi X, Y), \tag{2.16}$$

for any $X, Y \in \Gamma(TM)$, where a, b and c are scalar functions such that $b \neq 0$ and $c \neq 0$. If $c = 0$ then M is called η Einstein manifold.

3. Generalized Symmetric Metric Connection in a Kenmotsu Manifold

Let $\overline{\nabla}$ be a linear connection and ∇ be a Levi-Civita connection of an almost contact metric manifold M such that

$$\overline{\nabla}_X Y = \nabla_X Y + H(X,Y), \tag{3.1}$$

for any vector field X and Y. Where H is a tensor of type $(1,2)$. For $\overline{\nabla}$ to be a generalized symmetric metric connection of ∇, we have

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T^{'}(X,Y) + T^{'}(Y,X)], \tag{3.2}$$

where T is the torsion tensor of $\overline{\nabla}$ and

$$g(T^{'}(X,Y),Z) = g(T(Z,X),Y). \tag{3.3}$$

From (1.1) and (3.3) we get

$$T^{'}(X,Y) = \alpha\{\eta(X)Y - g(X,Y)\xi\} + \beta\{-\eta(X)\phi Y - g(\phi X,Y)\xi\}. \tag{3.4}$$

Using (1.1), (3.2) and (3.4) we obtain

$$H(X,Y) = \alpha\{\eta(Y)X - g(X,Y)\xi\} + \beta\{-\eta(X)\phi Y\}. \tag{3.5}$$

Corollary 3.1. *For a Kenmotsu manifold, generalized symmetric metric connection $\overline{\nabla}$ is given by*

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha\{\eta(Y)X - g(X,Y)\xi\} - \beta\eta(X)\phi Y. \tag{3.6}$$

If we choose $(\alpha,\beta) = (1,0)$ and $(\alpha,\beta) = (0,1)$, generalized metric connection is reduced to a semi-symmetric metric connection and quarter-symmetric metric connection as follows:

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{3.7}$$

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.8}$$

From (3.6) we have the following proposition

Proposition 3.1. *Let M be a Kenmotsu manifold with generalized metric connection. We have the following relations:*

$$(\overline{\nabla}_X \phi)Y = (\alpha+1)\{g(\phi X,Y)\xi - \eta(Y)\phi X\}, \tag{3.9}$$

$$\overline{\nabla}_X \xi = (\alpha+1)\{X - \eta(X)\xi\}, \tag{3.10}$$

$$(\overline{\nabla}_X \eta)Y = (\alpha+1)\{g(X,Y) - \eta(Y)\eta(X)\}, \tag{3.11}$$

for any $X,Y,Z \in \Gamma(TM)$.

4. Curvature Tensor on Kenmotsu manifold with generalized symmetric metric connection

Let M be an $n-$ dimensional Kenmotsu manifold. The curvature tensor $\overline{R}$ of the generalized metric connection $\overline{\nabla}$ on M is defined by

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z, \tag{4.1}$$

Using the proposition 3.1, from (3.6) and (4.1) we have

$$\begin{aligned}(4.2)\,\bar{R}(X,Y)Z &= R(X,Y)Z + \{(-\alpha^2-2\alpha)g(Y,Z) + (\alpha^2+a)\eta(Y)\eta(Z)\}X \\ &+ \{(\alpha^2+2\alpha)g(X,Z) + (-\alpha^2-\alpha)\eta(X)\eta(Z)\}Y \\ &+ \{(\alpha^2+\alpha)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &+ (\beta+\alpha\beta)[g(X,\phi Z)\eta(Y) - g(Y,\phi Z)\eta(X)]\}\xi \\ &+ (\beta+\alpha\beta)\eta(Y)\eta(Z)\phi X - (\beta+\alpha\beta)\eta(X)\eta(Z)\phi Y\end{aligned}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{4.3}$$

is the curvature tensor with respect to the Levi-Civita connection ∇.

Using (2.10), (2.11), (2.12), (2.13) and (4.2) we give the following proposition:

Proposition 4.1. *Let M be an $n-$ dimensional Kenmotsu manifold with generalized symmetric metric connection of type (α,β). Then we have the following equations:*

$$\bar{R}(X,Y)\xi = (\alpha+1)\{\eta(X)Y - \eta(Y)X + \beta[\eta(Y)\phi X - \eta(X)\phi Y]\} \tag{4.4}$$

$$\bar{R}(\xi,X)Y = (\alpha+1)\{\eta(Y)X - g(X,Y)\xi + \beta[\eta(Y)\phi X - g(X,\phi Y)\xi]\}, \tag{4.5}$$

$$\bar{R}(\xi,Y)\xi = (\alpha+1)\{Y - \eta(Y)\xi - \beta\phi Y\}, \tag{4.6}$$

$$\begin{aligned}\eta(\bar{R}(X,Y)Z &= (\alpha+1)\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z) \\ &+\beta[\eta(Y)g(X,\phi Z) - \eta(X)g(Y,\phi Z)]\}\end{aligned} \tag{4.7}$$

for any $X,Y,Z \in \Gamma(TM)$.

We know that Ricci tensor is defined by

$$\overline{S}(Y,Z) = \sum_{i=1}^{n} g(\overline{R}(e_i,Y)Z,e_i),$$

where $Y, Z \in \Gamma(TM)$, $\{e_1, e_2, ..., e_n\}$ is viewed as orthonormal frame. We can calculate the Ricci tensor with respect to generalized symmetric metric connection as follows:

$$\begin{aligned}\overline{S}(Y,Z) &= S(Y,Z) + \{(2-n)\alpha^2 + (3-2n)\alpha\}g(Y,Z) + (n-2)(\alpha^2+\alpha)\eta(Y)\eta(Z) \\ &\quad -(\beta+\alpha\beta)g(Y,\phi Z), \end{aligned} \tag{4.8}$$

where S is Ricci tensor with respect to Levi-Civita connection.

Example 4.1. We consider a 3-dimensional manifold $M = \{(x,y,z) \in R^3 : x \neq 0\}$, where (x,y,z) are the standard coordinates in R^3. Let E_1, E_2, E_3 be a linearly independent global frame on M given by

$$E_1 = x\frac{\partial}{\partial z},\ E_2 = x\frac{\partial}{\partial y},\ E_3 = -x\frac{\partial}{\partial x}. \tag{4.9}$$

Let g be the Riemannian metric defined by

$$g(E_1,E_2) = g(E_1,E_3) = g(E_2,E_3) = 0, g(E_1,E_1) = g(E_2,E_2) = g(E_3,E_3) = 1,$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$, for any $U \in TM$. Let ϕ be the $(1,1)$ tensor field defined by $\phi E_1 = E_2, \phi E_2 = -E_1$ and $\phi E_3 = 0$. Then, using the linearity of ϕ and g we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W)$ for any $U, W \in TM$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) an almost contact metric manifold is defined.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Then we have

$$[E_1,E_2] = 0, \qquad [E_1,E_3] = E_1, \qquad [E_2,E_3] = E_2, \tag{4.10}$$

Using Koszul formula for the Riemannian metric g, we can easily calculate

$$\begin{aligned} &\nabla_{E_1}E_1 = -E_3, \quad \nabla_{E_1}E_2 = 0. \quad \nabla_{E_1}E_3 = E_1, \\ &\nabla_{E_2}E_1 = 0, \quad \nabla_{E_2}E_2 = -E_3, \quad \nabla_{E_2}E_3 = 0, \\ &\nabla_{E_3}E_1 = 0, \quad \nabla_{E_3}E_2 = 0, \quad \nabla_{E_3}E_3 = 0. \end{aligned} \tag{4.11}$$

From the above relations, it can be easily seen that

$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, $\nabla_X \xi = X - \eta(X)\xi$, for all $E_3 = \xi$. Thus the manifold M is a Kenmotsu manifold with the structure (ϕ, ξ, η, g). for $\xi = E_3$. Therefore, the manifold M under consideration is a Kenmotsu manifold of dimension three.

5. Ricci and η-Ricci solitons on $(M, \phi, \xi, \eta, g,)$

Let $(M, \phi, \xi, \eta, g,)$ be an almost contact metric manifold. Consider the equation

$$\mathcal{L}_\xi g + 2\bar{S} + 2\lambda + 2\mu\eta \otimes \eta = 0, \tag{5.1}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, $\bar{S}$ is the Ricci curvature tensor field with respect to the generalized symmetric metric connection of the metric g, and λ and μ are real constants. Writing $\mathcal{L}_\xi$ in terms of the generalized symmetric metric connection $\bar{\nabla}$, we obtain:

$$2\bar{S}(X,Y) = -g(\bar{\nabla}_X\xi, Y) - g(X, \bar{\nabla}_Y\xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{5.2}$$

for any $X, Y \in \chi(M)$.

The data (g, ξ, λ, μ) which satisfy the equation (4.9) is said to be an *η-Ricci soliton* on M [10]. In particular, if $\mu = 0$ then (g, ξ, λ) is called *Ricci soliton* [6] and it is called *shrinking*, *steady* or *expanding*, according as λ is negative, zero or positive respectively [6].

Here is an example of η-Ricci soliton on Kenmotsu manifold with generalized symmetric metric connection.

Example 5.1. Let $M(\phi, \xi, \eta, g)$ be the Kenmotsu manifold considered in example 4.3 .

Let $\bar{\nabla}$ be a generalized symmetric metric connection, we obtain: Using the above relations, we can calculate the non-vanishing components of the curvature tensor as follows:

$$\begin{aligned} &R(E_1,E_2)E_1 = E_2,\ R(E_1,E_2)E_2 = -E_1,\ R(E_1,E_3)E_1 = E_3 \\ &R(E_1,E_3)E_3 = -E_1,\ R(E_2,E_3)E_2 = E_3,\ R(E_2,E_3)E_3 = -E_2 \end{aligned} \tag{5.3}$$

From the equations (5.3) we can easily calculate the non-vanishing components of the Ricci tensor as follows:

$$S(E_1,E_1) = -2,\ S(E_2,E_2) = -2,\ S(E_3,E_3) = -2 \tag{5.4}$$

Now, we can make similar calculations for generalized metric connection. Using (3.6) in the above equations, we get

$$\begin{aligned} &\overline{\nabla}_{E_1}E_1 = -(1+\alpha)E_3, && \overline{\nabla}_{E_1}E_2 = 0. && \overline{\nabla}_{E_1}E_3 = (1+\alpha)E_1, \\ &\overline{\nabla}_{E_2}E_1 = 0, && \overline{\nabla}_{E_2}E_2 = -(1+\alpha)E_3, && \overline{\nabla}_{E_2}E_3 = \alpha E_2, \\ &\overline{\nabla}_{E_3}E_1 = -\beta E_2, && \overline{\nabla}_{E_3}E_2 = \beta E_1, && \overline{\nabla}_{E_3}E_3 = 0. \end{aligned} \tag{5.5}$$

From (5.5), we can calculate the non-vanishing components of curvature tensor with respect to generalized metric connection as follows:

$$\begin{aligned} &\overline{R}(E_1,E_2)E_1 = (1+\alpha)^2E_2, && \overline{R}(E_1,E_2)E_2 = -(1+\alpha)^2E_1, \\ &\overline{R}(E_1,E_3)E_1 = (1+\alpha)E_3 && \overline{R}(E_1,E_3)E_3 = (1+\alpha)(\beta E_2 - E_1), \\ &\overline{R}(E_2,E_3)E_2 = (1+\alpha)E_3, && \overline{R}(E_2,E_3)E_3 = -(1+\alpha)(-\beta E_1 + E_2) \\ &\overline{R}(E_3,E_2)E_1 = -(1+\alpha)\beta E_3, && \overline{R}(E_3,E_1)E_2 = (1+\alpha)\beta E_3, . \end{aligned} \tag{5.6}$$

From (5.6), the non-vanishing components of the Ricci tensor are as follows:

$$\overline{S}(E_1,E_1) = -(1+\alpha)(2+\alpha), \quad \overline{S}(E_2,E_2) = -(1+\alpha)(2+\alpha),$$
$$\overline{S}(E_3,E_3) = -2(1+\alpha). \tag{5.7}$$

From (5.2) and (5.5) we get

$$2(1+\alpha)[g(e_i,e_i) - \eta(e_i)\eta(e_i)] + 2\bar{S}(e_i,e_i) + 2\lambda g(e_i,e_i) + 2\mu\eta(e_i)\eta(e_i) = 0 \tag{5.8}$$

for all $i \in \{1,2,3\}$, and we have $\lambda = (1+\alpha)^2$ (*i.e.* $\lambda > 0$) and $\mu = 1-\alpha^2$, the data (g,ξ,λ,μ) is an η-Ricci soliton on (M,ϕ,ξ,η,g). If $\alpha = -1$ which is steady and if $\alpha \neq -1$ which is expanding.

6. Parallel symmetric second order tensors and η-Ricci solitons in Kenmotsu manifolds

An important geometrical object in studying Ricci solitons is well known to be a symmetric $(0,2)$-tensor field which is parallel with respect to the generalized symmetric metric connection.

Now, let fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to generalized symmetric metric connection$\bar{\nabla}$ that is $\bar{\nabla}h = 0$. By applying Ricci identity [7]

$$\bar{\nabla}^2 h(X,Y;Z,W) - \bar{\nabla}^2 h(X,Y;Z,W) = 0, \tag{6.1}$$

we obtain the relation

$$h(\bar{R}(X,Y)Z,W) + h(Z,\bar{R}(X,Y)W) = 0. \tag{6.2}$$

Replacing $Z = W = \xi$ in (6.2) and by using (4.4) and by the symmetry of h it follows $h(\bar{R}(X,Y)\xi,\xi) = 0$ for any $X,Y \in \chi(M)$ and

$$(\alpha+1)\eta(X)h(Y,\xi) - (\alpha+1)\eta(Y)h(X,\xi) \tag{6.3}$$

$$+(\alpha+1)\eta(X)h(\xi,Y) - (\alpha+1)\eta(Y)h(\xi,X) \tag{6.4}$$

$$+\beta\eta(Y)h(\phi X,\xi) - \beta\eta(X)h(\phi Y,\xi) + \beta\eta(Y)h(\xi,\phi X) - \beta\eta(X)h(\xi,\phi Y) = 0 \tag{6.5}$$

Putting $X = \xi$ in (6.3) and by the virtue of (2.4), we obtain

$$2(\alpha+1)[h(Y,\xi) - \eta(Y)h(\xi,\xi)] - 2\beta h(\phi Y,\xi) = 0. \tag{6.6}$$

or

$$2(\alpha+1)[h(Y,\xi) - g(Y,\xi)h(\xi,\xi)] - 2\beta(\phi Y,\xi) = 0. \tag{6.7}$$

Suppose $(\alpha+1) \neq 0$, $\beta = 0$ it results

$$h(Y,\xi) - \eta(Y)h(\xi,\xi) = 0, \tag{6.8}$$

for any $Y \in \chi(M)$, equivalent to

$$h(Y,\xi) - g(Y,\xi)h(\xi,\xi) = 0, \tag{6.9}$$

for any $Y \in \chi(M)$. Differentiating the equation (6.9) covariantly with respect to the vector field $X \in \chi(M)$, we obtain

$$h(\bar{\nabla}_X Y,\xi) + h(Y,\bar{\nabla}_X \xi) = h(\xi,\xi)[g(\bar{\nabla}_X Y,\xi) + g(Y,\bar{\nabla}_X \xi)]. \tag{6.10}$$

Using (4.4) in (6.10), we obtain

$$h(X,Y) = h(\xi,\xi)g(X,Y), \tag{6.11}$$

for any $X, Y \in \chi(M)$. The above equation gives the conclusion:

Theorem 6.1. *Let $(M,\phi,\xi,\eta,g,)$ be a Kenmotsu manifold with generalized symmetric metric connection also with non-vanishing ξ-sectional curvature and endowed with a tensor field of type $(0,2)$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to $\bar{\nabla}$, then it is a constant multiple of the metric tensor g.*

On a Kenmotsu manifold with generalized symmetric metric connection using equation (3.10) and $\mathcal{L}_\xi g = 2(g - \eta \otimes \eta)$, the equation (5.2) becomes:

$$\bar{S}(X,Y) = -(\lambda + \alpha + 1)g(X,Y) + (\alpha + 1 - \mu)\eta(X)\eta(Y). \tag{6.12}$$

In particular, $X = \xi$, we obtain

$$\bar{S}(X,\xi) = -(\lambda + \mu)\eta(X). \tag{6.13}$$

In this case, the Ricci operator $\bar{Q}$ defined by $g(\bar{Q}X,Y) = \bar{S}(X,Y)$ has the expression

$$\bar{Q}X = -(\lambda + \alpha + 1)X + (\alpha + 1 - \mu)\eta(X)\eta(X)\xi. \tag{6.14}$$

Remark that on a Kenmostu manifold with generalized symmetric metric connection, the existence of an η-Ricci soliton implies that the characteristic vector field ξ is an eigenvector of Ricci operator corresponding to the eigenvalue $-(\lambda+\mu)$.

Now we shall apply the previous results on η-Ricci solitons.

Theorem 6.2. *Let (M,ϕ,ξ,η,g) be a Kenmotsu manifold with generalized symmetric metric connection. Assume that the symmetric $(0,2)$-tensor filed $h = \mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta$ is parallel with respect to the generalized symmetric metric connection associated to g. Then $(g,\xi,-\frac{1}{2}h(\xi,\xi),\mu)$ yields an η-Ricci soliton.*

Proof. Now, we can calculate

$$h(\xi,\xi) = \mathcal{L}_\xi g(\xi,\xi) + 2\bar{S}(\xi,\xi) + 2\mu\eta(\xi)\eta(\xi) = -2\lambda, \tag{6.15}$$

so $\lambda = -\frac{1}{2}h(\xi,\xi)$. From (6.11) we conclude that $h(X,Y) = -2\lambda g(X,Y)$, for any $X, Y \in \chi(M)$. Therefore $\mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta = -2\lambda g$. □

For $\mu = 0$ follows $\mathcal{L}_\xi g + 2S - S(\xi, \xi)g = 0$ and this gives

Corollary 6.1. *On a Kenmotsu manifold (M, ϕ, ξ, η, g) with generalized symmetric metric connection with property that the symmetric $(0, 2)$-tensor field $h = \mathcal{L}_\xi g + 2S$ is parallel with respect to generalized symmetric metric connection associated to g, the relation (5.1), for $\mu = 0$, defines a Ricci soliton.*

Conversely, we shall study the consequences of the existence of η-Ricci solitons on a Kenmotsu manifold with generalized symmetric metric connection. From (6.12), we give the conclusion:

Theorem 6.3. *If equation (4.9) defines an η-Ricci soliton on a Kenmotsu manifold (M, ϕ, ξ, η, g) with generalized symmetric metric connection, then (M, g) is quasi-Einstein.*

Recall that the manifold is called *quasi-Einstein* [8] if the Ricci curvature tensor field S is a linear combination (with real scalars λ and μ respectively, with $\mu \neq 0$) of g and the tensor product of a non-zero 1-from η satisfying $\eta = g(X, \xi)$, for ξ a unit vector field and respectively, *Einstein* [8] if S is collinear with g.

Theorem 6.4. *If (ϕ, ξ, η, g) is a Kenmotsu structure with generalized symmetric metric connection on M and (4.9) defines an η-Ricci soliton on M, then*

1. $Q \circ \phi = \phi \circ Q$
2. *Q and S are parallel along ξ.*

Proof. The first statement follows from a direct computation and for the second one, note that

$$(\bar{\nabla}_\xi Q)X = \bar{\nabla}_\xi QX - Q(\bar{\nabla}_\xi X) \tag{6.16}$$

and

$$(\bar{\nabla}_\xi S)(X, Y) = \xi(S(X, Y)) - S(\bar{\nabla}_\xi X, Y) - S(X, \bar{\nabla}_\xi Y). \tag{6.17}$$

Replacing Q and S from (6.14) and (6.13) we get the conclusion. □

A particular case arises when the manifold is ϕ-Ricci symmetric, which means that $\phi^2 \circ \nabla Q = 0$, as stated in the next theorem.

Theorem 6.5. *Let (M, ϕ, ξ, η, g) be a Kenmotsu manifold with generalized symmetric metric connection. If M is ϕ-Ricci symmetric and (4.9) defines an η-Ricci soliton on M, then $\mu = 1$ and (M, g) is Einstein manifold [8].*

Proof. Replacing Q from (6.14) in (6.16) and applying ϕ^2 we obtain

$$(\alpha + 1 - \mu)\eta(Y)[X - \eta(X)\xi] = 0, \tag{6.18}$$

for any $X, Y \in \chi(M)$. Follows $\mu = \alpha + 1$ and $S = -(\lambda + \alpha + 1)g$. □

Remark 6.1. In particular, the existence of an η-Ricci soliton on a Kenmotsu manifold with generalized symmetric metric connection which is *Ricci symmetric* (i.e. $\bar{\nabla}S = 0$) implies that M is *Einstein* manifold. The class of Ricci symmetric manifold represents an extension of class of Einstein manifold to which the locally symmetric manifold also belong ($i.e.\ satisfying\ \bar{\nabla}R = 0$). The condition $\bar{\nabla}S = 0$ implies $\bar{R}.\bar{S} = 0$ and the manifolds satisfying this condition are called *Ricci semi-symmetric* [7].

In what follows we shall consider η-Ricci solitons requiring for the curvature to satisfy $\bar{R}(\xi, X).\bar{S} = 0$, $\bar{S}.\bar{R}(\xi, X) = 0$, $\bar{W}_2(\xi, X).\bar{S} = 0$ and $\bar{S}.\bar{W}_2(\xi, X) = 0$ respectively, where the W_2-curvature tensor field is the curvature tensor introduced by G. P. Pokhariyal and R. S. Mishra in [21]:

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{dimM - 1}[g(X, Z)QY - g(Y, Z)QX]. \tag{6.19}$$

7. η-Ricci solitions on a Kenmotsu manifold with generalized symmetric metric connection satisfying $\bar{R}(\xi, X).\bar{S} = 0$

Now we consider a Kenmotsu manifold with with a generalized symmetric metric connection $\bar{\nabla}$ satisfying the condition

$$\bar{S}(\bar{R}(\xi, X)Y, Z) + \bar{S}(Y, \bar{R}(\xi, X)Z) = 0, \tag{7.1}$$

for any $X, Y \in \chi(M)$.

Replacing the expression of $\bar{S}$ from (6.12) and from the symmetries of $\bar{R}$ we get

$$(\alpha + 1)(\alpha + 1 - \mu)[\eta(Y)g(X, Z) + \eta(Z)g(X, Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0, \tag{7.2}$$

for any $X, Y \in \chi(M)$.
For $Z = \xi$ we have

$$(\alpha + 1)(\alpha + 1 - \mu)g(\phi X, \phi Y) = 0, \tag{7.3}$$

for any $X, Y \in \chi(M)$.

Hence we can state the following theorem:

Theorem 7.1. *If a Kenmotsu manifold with a generalized symmetric metric connection $\bar{\nabla}$, (g, ξ, λ, μ) is an η-Ricci soliton on M and it satisfies $\bar{R}(\xi, X).\bar{S} = 0$, then the manifold is an η-Einstein manifold.*

For $\mu = 0$, we deduce:

Corollary 7.1. *On a Kenmotsu manifold with a generalized symmetric metric connection satisfying $\bar{R}(\xi, X).\bar{S} = 0$, there is no η-Ricci soliton with the potential vector field ξ.*

8. η-Ricci solitons on Kenmotsu manifold with generalized symmetric metric connection satisfying $\bar{S}.\bar{R}(\xi, X) = 0$

In this section, we have considered Kenmotsu manifold with a generalized symmetric metric connection $\bar{S}$ satisfying the condition

$$\bar{S}(X, \bar{R}(Y,Z)W)\xi - \bar{S}(\xi, \bar{R}(Y,Z)W)X + \bar{S}(X,Y)\bar{R}(\xi,Z)W - \tag{8.1}$$

$$-\bar{S}(\xi,Y)\bar{R}(X,Z)W + \bar{S}(X,Z)\bar{R}(Y,\xi)W - \bar{S}(\xi,Z)\bar{R}(Y,X)W + \tag{8.2}$$

$$+\bar{S}(X,W)\bar{R}(Y,Z)\xi - \bar{S}(\xi,W)\bar{R}(Y,Z)X = 0 \tag{8.3}$$

for any $X, Y, Z, W \in \chi(M)$.

Taking the inner product with ξ, the equation (8.1) becomes

$$\bar{S}(X, \bar{R}(Y,Z)W) - \bar{S}(\xi, \bar{R}(Y,Z)W)\eta(X) + \bar{S}(X,Y)\eta(\bar{R}(\xi,Z)W) - \tag{8.4}$$

$$-\bar{S}(\xi,Y)\eta(\bar{R}(X,Z)W) + \bar{S}(X,Z)\eta(\bar{R}(Y,\xi)W) - \bar{S}(\xi,Z)\eta(\bar{R}(Y,X)W) + \tag{8.5}$$

$$+\bar{S}(X,W)\eta(\bar{R}(Y,Z)\xi) - \bar{S}(\xi,W)\eta(\bar{R}(Y,Z)X) = 0 \tag{8.6}$$

for any $X, Y, Z, W \in \chi(M)$.

For $W = \xi$, using the equation (4.4), (4.5), (4.7) and (6.12) in (8.4), we get

$$(\alpha+1)(2\lambda+\mu+\alpha+1)[g(X,Y)\eta(Z) - g(X,Z)\eta(Y) + \beta g(\phi X,Y)\eta(Z) - g(\phi X,Z)\eta(Y)] \tag{8.7}$$

for any $X, Y, Z, W \in \chi(M)$.

Hence we can state the following theorem:

Theorem 8.1. *If (M, ϕ, ξ, η, g) is a Kenmotsu manifold with a generalized symmetric metric connection, (g, ξ, λ, μ) is an η-Ricci soliton on M and it satisfies $\bar{S}.\bar{R}(\xi, X) = 0$. Then*

$$(\alpha+1)(2\lambda+\mu+\alpha+1) = 0. \tag{8.8}$$

For $\mu = 0$ follows $\lambda = -\frac{\alpha+1}{2}$,$(\alpha \neq -1)$, therefore, we have the following corollary:

Corollary 8.1. *On a Kenmotsu manifold with a generalized symmetric metric connection, satisfying $\bar{S}.\bar{R}(\xi, X) = 0$, the Ricci soliton defined by (5.1), $\mu = 0$ is either shrinking or expanding.*

9. η-Ricci soliton on (ε)-Kenmotsu manifold with a semi-symmetric metric connection satisfying $\bar{W}_2(\xi, X).\bar{S} = 0$

The condition that must be satisfied by $\bar{S}$ is

$$\bar{S}(\bar{W}_2(\xi,X)Y, Z) + \bar{S}(Y, \bar{W}_2(\xi,X)Z) = 0, \tag{9.1}$$

for any $X, Y, Z \in \chi(M)$.

For $X = \xi$, using (4.4), (4.5), (4.7), (6.12) and (6.19) in (9.1), we get

$$\frac{(\alpha+1-\mu)(-2\mu-2\lambda+(4\alpha+4)n)}{n}\eta(Y)\eta(Z) \tag{9.2}$$

for any $X, Y, Z \in \chi(M)$. Hence, we can state the following:

Theorem 9.1. *If (M, ϕ, ξ, η, g) is an $(2n+1)$-dimensional Kenmotsu manifold with a generalized symmetric metric connection, (g, ξ, λ, μ) is an η-Ricci soliton on M and $\bar{W}_2(\xi, X).\bar{S} = 0$, then*

$$(\alpha+1-\mu)(-2\mu-2\lambda+(4\alpha+4)n) = 0. \tag{9.3}$$

For $\mu = 0$ follows that $\lambda = \frac{(4\alpha+4)n}{2}$,$(\alpha \neq -1)$, therefore, we have the following corollary:

Corollary 9.1. *On a Kenmotsu manifold with a generalized symmetric metric connection, satisfying $\bar{W}_2(\xi, X).\bar{S} = 0$, the Ricci soliton defined by (5.1), $\mu = 0$ is either shrinking or expanding.*

10. η-Ricci soliton on Kenmotsu manifold with a generalized symmetric metric connection satisfying $\bar{S}.\bar{W}_2(\xi, X) = 0$

In this section, we have considered an (ε)-Kenmotsu manifold with a semi-symmetric metric connection $\bar{\nabla}$ satisfying the condition

$$\bar{S}(X, \bar{W}_2(Y,Z)V)\xi - \bar{S}(\xi, \bar{W}_2(Y,Z)V)X + \bar{S}(X,Y)\bar{W}_2(\xi,Z)V - \tag{10.1}$$

$$-\bar{S}(\xi,Y)\bar{W}_2(X,Z)V + \bar{S}(X,Z)\bar{W}_2(Y,\xi)V - \bar{S}(\xi,Z)\bar{W}_2(Y,X)V + \tag{10.2}$$

$$+\bar{S}(X,V)\bar{W}_2(Y,Z)\xi - \bar{S}(\xi,V)\bar{W}_2(Y,Z)X = 0, \tag{10.3}$$

for any $X, Y, Z, V \in \chi(M)$.

Taking the inner product with ξ, the equation (10.1) becomes

$$\bar{S}(X, \bar{W}_2(Y,Z)V) - \bar{S}(\xi, \bar{W}_2(Y,Z)V)\eta(X) + \bar{S}(X,Y)\eta(\bar{W}_2(\xi,Z)V) - \tag{10.4}$$

$$-\bar{S}(\xi,Y)\eta(\bar{W}_2(X,Z)V) + \bar{S}(X,Z)\eta(\bar{W}_2(Y,\xi)V) - \bar{S}(\xi,Z)\eta(\bar{W}_2(Y,X)V) + \tag{10.5}$$

$$+\bar{S}(X,V)\eta(\bar{W}_2(Y,Z)\xi) - \bar{S}(\xi,V)\eta(\bar{W}_2(Y,Z)X) = 0, \tag{10.6}$$

for any $X, Y, Z, V \in \chi(M)$.

For $X = V = \xi$, using (4.4), (4.5), (4.7), (6.12) and (6.19) in (10.4), we get

$$\{-(\alpha+1)(2\lambda+\alpha+1+\mu) + \frac{(\lambda+\alpha+1)^2+(\lambda+\mu)^2}{2n}\}\{\eta(X)\eta(Y) - g(X,Y)\} \tag{10.7}$$

$$+\beta(\alpha+1)(2\lambda+\alpha+1+\mu)g(\phi X, Y) = 0, \tag{10.8}$$

for any $X, Y, Z \in \chi(M)$. Hence, we can state:

Theorem 10.1. *If (M, ϕ, ξ, η, g) is a $(2n+1)$-dimensional Kenmotsu manifold with generalized symmetric metric connection, (g, ξ, λ, μ) is an η-Ricci soliton on M and $\bar{S}.\bar{W}_2(\xi, X) = 0$, then*

$$-(\alpha+1)(2\lambda+\alpha+1+\mu) + \frac{(\lambda+\alpha+1)^2 + (\lambda+\mu)^2}{2n} = 0, \tag{10.9}$$

and

$$\beta(\alpha+1)(2\lambda+\alpha+1+\mu) = 0. \tag{10.10}$$

For $\mu = 0$ we get the following corollary:

Corollary 10.1. *On a Kenmotsu manifold with a generalized symmetric metric connection satisfying $\bar{S}.\bar{W}_2(\xi, X) = 0$, the Ricci soliton defined by (5.1), for $\mu = 0$, we have the following expressions:*
(i) $-(\alpha+1)(2\lambda+\alpha+1) + \frac{(\lambda+\alpha+1)^2+(\lambda)^2}{2n} = 0$ *and* $\beta(\alpha+1)(2\lambda+\alpha+1) = 0$.
(ii) *If $\alpha = -1$ or $\alpha = -2\lambda - 1$ which is steady.*

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