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STUDY OF A NEW TYPE OF METRIC CONNECTION IN AN ALMOST HERMITIAN MANIFOLD

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Abstract. This paper contains a study of an almost Hermitian manifold equipped with a new type of metric connection. We have found the condition for an almost Hermitian manifold equipped with a new type of metric connection to be an almost Kähler manifold. We have also studied a contravariant almost analytic vector field in an almost Hermitian manifold equipped with a new type of metric connection. Also, we have shown that the Lie derivative of the metric tensor is hybrid in an almost Hermitian manifold equipped with a new type of metric connection.
Keywords: Almost Hermitian manifold, almost Kähler manifold, contravariant almost analytic vector field, hybrid.

1. Introduction

Let M be an n-dimensional almost complex manifold with an almost complex structure F. Let g be Riemannian metric, if the metric g satisfies

$$g(FX, FY) = g(X, Y), \tag{1.1}$$

for arbitrary vector fields X and Y, then the metric g is called a Hermitian metric and the almost complex manifold with metric g is called an almost Hermitian manifold. In 2013, a generalized type of non-metric connection was defined by M. Tarafdar and S. Kundu [1], who found some results on an almost Hermitian manifold. The quarter-symmetric metric connection on a Sasakian manifold was studied by A. K. Mondal and U. C. De [2]. Recently, M. M. Tripathi [3] discussed the existence of a new type of connection in Riemannian manifold. This new type of connection can be reduces to semi-symmetric metric and non-metric connections, quarter-symmetric metric and non-metric connections, Ricci-symmetric metric and non-metric connections, etc. M. M. Tripathi [3] defined a unique connection $\widetilde{\nabla}$, such

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that

$$\widetilde{\nabla}_X Y = \nabla_X Y + u(Y)\varphi_1 X - u(X)\varphi_2 Y - g(\varphi_1 X, Y)U$$
$$(1.2) \qquad - f_1[u_1(X)Y + u_1(Y)X - g(X,Y)U_1] - f_2 g(X,Y)U_2,$$

which satisfies

$$(1.3) \qquad \widetilde{T}(X,Y) = u(Y)\varphi X - u(X)\varphi Y$$

and

$$(1.4) \qquad (\widetilde{\nabla}_X g)(Y,Z) = 2f_1 u_1(X) g(Y,Z) + f_2[u_2(Y)g(X,Z) + u_2(Z)g(X,Y)],$$

where $\widetilde{T}$ denotes the torsion tensor of $\widetilde{\nabla}$. f_1, f_2 are functions in M, u, u_1, u_2 are 1-forms and φ is a tensor field of type (1,1) and defined by

$$u(X) \equiv g(U,X), \quad u_1(X) \equiv g(U_1,X), \quad u_2(X) \equiv g(U_2,X),$$
$$g(\varphi X, Y) \equiv \Phi(X,Y) = \Phi_1(X,Y) + \Phi_2(X,Y),$$

where Φ_1, Φ_2 are symmetric and skew-symmetric parts of the (0,2) type tensor Φ such that

$$\Phi_1(X,Y) \equiv g(\varphi_1 X, Y), \qquad \Phi_2(X,Y) \equiv g(\varphi_2 X, Y).$$

In 1980, R. S. Mishra and S. N. Pandey [4] gave a quarter-symmetric metric connection defined by

$$(1.5) \qquad \overline{\nabla}_X Y = \nabla_X Y + u(Y)\varphi X - g(\varphi X, Y)U,$$

where u is 1-form and φ is a tensor field of type (1,1).

Now, we define a new type of metric connection $\overline{\nabla}$ given by

$$(1.6) \qquad \overline{\nabla}_X Y = \nabla_X Y + \omega(Y)FX - g(FX,Y).$$

The torsion tensor of connection defined by (1.6) is given by

$$(1.7) \qquad \overline{T}(X,Y) = \omega(Y)FX - \omega(X)FY - 2g(FX,Y).$$

2. Almost Kähler manifold equipped with a new type of metric connection

An almost Hermitian manifold M is said to be an almost Kähler manifold if the following condition holds

$$(2.1) \qquad (\nabla_X {}'F)(Y,Z) + (\nabla_Y {}'F)(Z,X) + (\nabla_Z {}'F)(X,Y) = 0,$$

where ${}'F$ is defined by

$$(2.2) \qquad {}'F(X,Y) = g(FX,Y).$$

Taking covariant derivative of $'F$ with respect to the connection $\overline{\nabla}$, we can write

$$(\overline{\nabla}_X {}'F)(Y,Z) = \overline{\nabla}_X {}'F(Y,Z) - {}'F(\overline{\nabla}_X Y, Z) - {}'F(Y, \overline{\nabla}_X Z). \tag{2.3}$$

Using (2.2) in (2.3), we get

$$(\overline{\nabla}_X {}'F)(Y,Z) = \overline{\nabla}_X\, g(FY,Z) - g(F(\overline{\nabla}_X Y), Z) - g(FY, \overline{\nabla}_X Z). \tag{2.4}$$

Using (1.6) in (2.4), we have

$$\begin{aligned}(\overline{\nabla}_X {}'F)(Y,Z) &= g((\nabla_X F)Y, Z) + \omega(Y)g(X,Z) - \omega(Z)g(X,Y)\\ &\quad +\omega(FY)g(FX,Z) - \omega(FZ)g(FX,Y).\end{aligned} \tag{2.5}$$

We know that in an almost Hermitian manifold equipped with connection ∇

$$g((\nabla_X F)Y, Z) = (\nabla_X {}'F)(Y,Z). \tag{2.6}$$

Using (2.6) in (2.5), we have

$$\begin{aligned}(\overline{\nabla}_X {}'F)(Y,Z) &= (\nabla_X {}'F)(Y,Z) + \omega(Y)g(X,Z) - \omega(Z)g(X,Y)\\ &\quad + \omega(FY)g(FX,Z) - \omega(FZ)g(FX,Y).\end{aligned} \tag{2.7}$$

Taking X,Y,Z in cyclic order of equation (2.7), we obtained

$$\begin{aligned}(\overline{\nabla}_Y {}'F)(Z,X) &= (\nabla_Y {}'F)(Z,X) + \omega(Z)\, g(Y,X) - \omega(X)\, g(Y,Z)\\ &\quad + \omega(FZ)\, g(FY,X) - \omega(FX)\, g(FY,Z).\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}(\overline{\nabla}_Z {}'F)(X,Y) &= (\nabla_Z {}'F)(X,Y) + \omega(X)g(Z,Y) - \omega(Y)g(Z,X)\\ &\quad +\omega(FX)g(FZ,Y) - \omega(FY)g(FZ,X).\end{aligned} \tag{2.9}$$

Adding equations (2.7), (2.8) and (2.9), we get

$$\begin{aligned}&(\overline{\nabla}_X {}'F)(Y,Z) + (\overline{\nabla}_Y {}'F)(Z,X) + (\overline{\nabla}_Z {}'F)(X,Y)\\ &= (\nabla_X {}'F)(Y,Z) + (\nabla_Y {}'F)(Z,X) + (\nabla_Z {}'F)(X,Y)\\ &\quad + 2[\omega(FX)g(Y,FZ) + \omega(FY)g(Z,FX) + \omega(FZ)g(X,FY)].\end{aligned} \tag{2.10}$$

Thus we conclude:

Theorem 2.1. *If M is an almost Kähler manifold with respect to the Riemannian connection ∇, then the manifold M will be an almost Kähler manifold with respect to the new type of metric connection $\overline{\nabla}$ if and only if*

$$\omega(FX)g(Y,FZ) + \omega(FY)g(Z,FX) + \omega(FZ)g(X,FY) = 0. \tag{2.11}$$

Now we compose:

Theorem 2.2. *If M be an almost Hermitian manifold equipped with a new type of metric connection $\overline{\nabla}$ then $(\overline{\nabla}_X F)Y$ is hybrid.*

Proof : Taking a covariant derivative of FY with respect to connection $\overline{\nabla}$, we can write

$$(\overline{\nabla}_X F)Y = \overline{\nabla}_X FY - F(\overline{\nabla}_X Y). \tag{2.12}$$

Using (1.6) in (2.12), we have

$$\begin{aligned}(\overline{\nabla}_X F)Y &= (\nabla_X F)Y + \omega(FY)FX + \omega(Y)X \\ &\quad - g(X, Y)U - g(FX, Y)FU.\end{aligned} \tag{2.13}$$

Replacing X by FX and Y by FY in (2.13), we get

$$\begin{aligned}(\overline{\nabla}_{FX} F)(FY) &= (\nabla_{FX} F)FY + \omega(Y)X + \omega(FY)FX \\ &\quad - g(X, Y)U - g(FX, Y)FU.\end{aligned} \tag{2.14}$$

Subtracting equation (2.13) from (2.14), we have

$$(\overline{\nabla}_{FX} F)(FY) - (\overline{\nabla}_X F)Y = (\nabla_{FX} F)FY - (\nabla_X F)Y. \tag{2.15}$$

In an almost Hermitian manifold equipped with Riemannian connection ∇, we know that

$$(\nabla_{FX} F)FY = (\nabla_X F)Y. \tag{2.16}$$

Using the above equation in (2.15), we get

$$(\overline{\nabla}_{FX} F)(FY) = (\overline{\nabla}_X F)Y. \tag{2.17}$$

Hence $(\overline{\nabla}_X F)(Y)$ is hybrid.

Now we compose:

Theorem 2.3. *If M is an almost Hermitian manifold equipped with a new type of metric connection $\overline{\nabla}$, then the Nijenhuis tensors of both connections ∇ and $\overline{\nabla}$ will be equal.*

Proof : The Nijenhuis tensor in an almost Hermitian manifold is defined by

$$N(X, Y) = (\nabla_{FX} F)Y - (\nabla_{FY} F)X + (\nabla_X F)(FY) - (\nabla_Y F)(FX). \tag{2.18}$$

Nijenhuis tensor of the connection $\overline{\nabla}$ is given by

$$\overline{N}(X, Y) = (\overline{\nabla}_{FX} F)Y - (\overline{\nabla}_{FY} F)X + (\overline{\nabla}_X F)(FY) - (\overline{\nabla}_Y F)(FX). \tag{2.19}$$

Using (2.13) in (2.19), we get

$$\overline{N}(X, Y) = (\nabla_{FX} F)Y - (\nabla_{FY} F)X + (\nabla_X F)(FY) - (\nabla_Y F)(FX). \tag{2.20}$$

Using equation (2.18) in (2.20), we have

$$\overline{N}(X, Y) = N(X, Y). \tag{2.21}$$

Hence, the theorem is proved.

We know that the necessary and sufficient condition for an almost Hermitian manifold to be a Hermitian manifold is that the Nijenhuis tensor vanishes. Hence from equation (2.21) we can state

Theorem 2.4. *If M is an almost Hermitian manifold with respect to the Riemannian connection ∇ and also an almost Hermitian manifold with respect to a new type of metric connection $\overline{\nabla}$, then the manifold M is a Hermitian manifold with respect to the connection $\overline{\nabla}$ if and only if it is Hermitian manifold with respect to the connection ∇.*

3. Contravariant almost analytic vector fields

A vector field V is said to be a contravariant almost analytic vector field in an almost Hermitian manifold if

$$(L_V F)X = 0, \tag{3.1}$$

where X is an arbitrary vector field and L denotes the Lie derivative defined by

$$L_V X = [V, X] = \nabla_V X - \nabla_X V. \tag{3.2}$$

Taking the Lie derivative of FX with respect to the connection ∇, we can write easily

$$(L_V F)X = L_V(FX) - F(L_V X). \tag{3.3}$$

Using equation (3.2) in (3.3), we have

$$(L_V F)X = \nabla_V(FX) - F(\nabla_V X) - \nabla_{FX} V + F(\nabla_X V). \tag{3.4}$$

The Lie derivative of a vector field X with respect to the connection $\overline{\nabla}$, in an almost Hermitian manifold equipped with a new type of metric connection, is given by

$$\overline{L}_V X = \overline{[V, X]} = \overline{\nabla}_V X - \overline{\nabla}_X V. \tag{3.5}$$

Taking a Lie derivative of FX with respect to a new type of metric connection $\overline{\nabla}$, we can write

$$(\overline{L}_V F)X = \overline{L}_V(FX) - F(\overline{L}_V X). \tag{3.6}$$

Using equation (3.5) in (3.6), we get

$$(\overline{L}_V F)X = \overline{\nabla}_V(FX) - \overline{\nabla}_{FX} V - F(\overline{\nabla}_V X) + F(\overline{\nabla}_X V). \tag{3.7}$$

Using equation (1.6) in (3.7), we have

$$(\overline{L}_V F)X = \nabla_V(FX) - F(\nabla_V X) - \nabla_{FX} V + F(\nabla_X V) \\ +\omega(FX)FV + \omega(X)V - 2g(X, V)U - 2g(FX, V)FU. \tag{3.8}$$

Using (3.4) in (3.8), we can write

$$(\overline{L}_V F)X = (L_V F)X + \omega(FX)FV + \omega(X)V - 2g(X, V)U - 2g(FX, V). \tag{3.9}$$

Thus we conclude:

Theorem 3.1. *If M is an almost Hermitian manifold equipped with a new type of metric connection $\overline{\nabla}$, then a vector field V is a contravariant almost analytic vector field if and only if*

$$\omega(FX)FV + \omega(X)V = 2[g(X, V)U + g(FX, V)FU]. \tag{3.10}$$

Now we compose:

Theorem 3.2. *If M is an almost Hermitian manifold equipped with a new type of metric connection $\overline{\nabla}$, then*

$$(\overline{\nabla}_X {}'F)(FY, Z) + (\overline{\nabla}_X {}'F)(FZ, Y) = 0. \tag{3.11}$$

Proof: Replacing Y by FY in equation (2.7), we get

$$(\overline{\nabla}_X {}'F)(FY, Z) = (\nabla_X {}'F)(FY, Z) + \omega(FY)g(X, Z) - \omega(Z)g(X, FY) - \omega(Y)g(FX, Z) - \omega(FZ)g(X, Y). \tag{3.12}$$

Interchanging Y and Z in equation (3.12), we have

$$(\overline{\nabla}_X {}'F)(FZ, Y) = (\nabla_X {}'F)(FZ, Y) + \omega(FZ)g(X, Y) - \omega(Y)g(X, FZ) \\ -\omega(Z)g(FX, Y) - \omega(FY)g(X, Z). \tag{3.13}$$

Adding equations (3.12) and (3.13),we get

$$(\overline{\nabla}_X {}'F)(FY, Z) + (\overline{\nabla}_X {}'F)(FZ, Y) = (\nabla_X {}'F)(FY, Z) + (\nabla_X {}'F)(FZ, Y).$$

Using the symmetric property of ${}'F$ in an almost Hermitian manifold, we have

$$(\overline{\nabla}_X {}'F)(FY, Z) + (\overline{\nabla}_X {}'F)(FZ, Y) = 0. \tag{3.14}$$

Now we compose:

Theorem 3.3. *If M is an almost Hermitian manifold equipped with a new type of metric connection $\overline{\nabla}$, then $\overline{L}_V g$ is hybrid if and only if*

$$\omega(FY)g(V, Z) + \omega(FZ)g(V, Y) + \omega(Y)g(V, FZ) + \omega(Z)g(V, FY) = 0. \tag{3.15}$$

Proof : Taking the Lie derivative of g with respect to the new type of metric connection, we can write

$$(\overline{L}_V g)(Y,Z) = \overline{L}_V g(Y,Z) - g(\overline{L}_V Y, Z) - g(Y, \overline{L}_V Z).$$

Using (3.5) in (3.17), we get

$$(\overline{L}_V g)(Y,Z) = \overline{\nabla}_V g(Y,Z) - g(\overline{\nabla}_V Y, Z) + g(\overline{\nabla}_Y V, Z) - g(Y, \overline{\nabla}_V Z) + g(Y, \overline{\nabla}_Z V). \tag{3.16}$$

Using (1.6) in (3.18), we have

$$(\overline{L}_V g)(Y,Z) = g(\nabla_Y V, Z) + g(Y, \nabla_Z V) - \omega(Y) g(V, FZ) - \omega(Z) g(V, FY). \tag{3.17}$$

Replacing Y by FY and Z by FZ in the above equation, we get

$$(\overline{L}_V g)(FY,FZ) = g(\nabla_{FY} V, FZ) + g(FY, \nabla_{FZ} V) + \omega(FY) g(V, Z) + \omega(FZ) g(V, Y). \tag{3.18}$$

From equation (3.1), we can write easily

$$\nabla_{FX} V = (\nabla_V F)X + F(\nabla_X V) \tag{3.19}$$

Using (3.21) in (3.20), we have

$$(\overline{L}_V g)(FY,FZ) = g((\nabla_V F)Y, FZ) + g(\nabla_Y V, Z) + g(FY, (\nabla_V F)Z) + g(Y, \nabla_Z V) + \omega(FY) g(V, Z) + \omega(FZ) g(V, Y). \tag{3.20}$$

The above equation implies

$$(\overline{L}_V g)(FY,FZ) = g((\nabla_V F)(FY), Z) + g(Y, (\nabla_V F)(FZ)) + g(\nabla_Y V, Z) + g(Y, \nabla_Z V) + \omega(FY) g(V, Z) + \omega(FZ) g(V, Y). \tag{3.21}$$

We know that in an almost Hermitian manifold

$$g((\nabla_X F)Y, Z) = (\nabla_X {}'F)(Y, Z). \tag{3.22}$$

Using the above equation in (3.23), we have

$$(\overline{L}_V g)(FY,FZ) = (\nabla_V {}'F)(FY, Z) + (\nabla_V {}'F)(FZ, Y) + g(\nabla_Y V, Z) + g(Y, \nabla_Z V) + \omega(FY) g(V, Z) + \omega(FZ) g(V, Y). \tag{3.23}$$

Using the symmetric property of $'F$ in (3.25), we can write

$$(\overline{L}_V g)(FY,FZ) = g(\nabla_Y V, Z) + g(Y, \nabla_Z V) + \omega(FY) g(V, Z) + \omega(FZ) g(V, Y).$$

Subtracting (3.19) from (3.26), we have

$$(\overline{L}_V g)(FY, FZ) - (\overline{L}_V g)(Y, Z) = \omega(FY)g(V, Z) + \omega(FZ)g(V, Y) + \omega(Y)g(V, FZ) + \omega(Z)g(V, FY). \tag{3.24}$$

From the above equation we can say that if

$$\omega(FY)g(V, Z) + \omega(FZ)g(V, Y) + \omega(Y)g(V, FZ) + \omega(Z)g(V, FY) = 0.$$

then $(\overline{L}_V g)$(Y,Z) is hybrid in Y and Z.

Hence the theorem is proved.

REFERENCES

1. M. Tarafdar and S. Kundu, *Almost Hermitian manifold with a generalised type of non-metric connection*,, Lobachevskii J. Math. **34** (2013), 181-186.
2. A.K. Mondal and U. C. De, *Some curvature properties of a quarter-symmetric metric connection on a Sasakian manifold*, Bull. Math. Anal. Appl. **1**(3) (2009), 99-108.
3. M. M. Tripathi, *A new connection in a Riemannian manifold*, arXiv:0802.0569v1[math.DG] (2008).
4. R.S. Mishra and S. N. Pandey, *On quarter-symmetric metric F-connections*, Tensor N. S., **34** (1980), 1-7.
5. P.N. Pandey and B. B. Chaturvedi, *Almost Hermitian manifold with semi-symmetric recurrent connection*, J. Internat Acad Phy. Sci. **10** (2006), 69-74.
6. P.N. Pandey and B. B. Chaturvedi, *Semi-symmetric metric connection on a Kähler manifold*, Bull. Alld. Math. Soc. **22** (2007), 51-57.
7. U.C. De and Joydeep Sengupta, *On a type of semi-symmetric metric connection on an almost contact metric manifold*, Facta Universitatis (NIS), Ser. Math. Inform. **16** (2001), 87-96.
8. R.S. Mishra, *Structures on a Differentiable Manifold and their Applications*, Chandrama Prakashan, Allahabad, 1984.

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