FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 34, No 3 (2019), 439–457
https://doi.org/10.22190/FUMI1903439B

NONLINEAR SINGULAR STURM-LIOUVILLE PROBLEMS WITH IMPULSIVE CONDITIONS

Bilender P. Allahverdiev and Hüseyin Tuna

Abstract. In this paper, we consider a non-linear impulsive Sturm-Liouville problem on semiinfinite intervals in which the limit-circle case holds at infinity for THE Sturm-Liouville expression. We prove the existence and uniqueness theorems for this problem.
Keywords: Impulsive Sturm-Liouville problem; Singular point; Weyl limit-circle case; Completely continuous operator; Fixed point theorems.

1. Introduction

The theory of differential equations with impulses describes processes that are subjected to abrupt changes in their states at certain moments. Such processes arise in many fields of science and technology: chemical technology, biotechnology, theoretical physics, industrial robotics, etc. For an introduction to the basic theory of differential equations with impulses see Bainov and Simeonov ([3], [4], [5]), Benchohra, Henderson and Ntouyas ([6]), Lakshmikantham, Bainov and Simeonov ([18]) Samoilenko and Perestyuk ([31]) and the references therein.

Recently, much work has been done on the existence of solutions to impulsive Sturm-Liouville equations; for regular impulsive Sturm-Liouville problems see [2, 7, 9, 12-15, 25-27, 30, 33], for singular impulsive Sturm-Liouville equations see [1, 10, 18-19, 21-24, 29]. However, there is no paper concerned with the existence of solutions to singular impulsive non-linear Sturm-Liouville problems that the limit-circle case holds at infinity. In this paper, we fill the gap by using a special way to pose boundary conditions at infinity.

Let us consider the following nonlinear Sturm-Liouville equation

$$l(y) := -(p(x)y')' + q(x)y = f(x, y), \ x \in I, \tag{1.1}$$

Received September 12, 2018; accepted February 25, 2019
2010 *Mathematics Subject Classification.* Primary 34B15,34B16; Secondary 34B24,34B40

where $I := I_1 \cup I_2$, $I_1 := [a, c)$, $I_2 := (c, +\infty)$, $-\infty < a < c < +\infty$, and $y = y(x)$ is a desired solution.

Let $L^2(I)$ be a Hilbert space which is composed of all complex-valued functions y satisfying

$$\int_a^\infty |y(x)|^2 \, dx < \infty$$

in relation to the inner product

$$(y, z) := \int_a^\infty y(x) \overline{z(x)} dx.$$

Denote by $\mathcal{D}$ the linear set of all functions $y \in L^2(I)$ such that y, py' are locally absolutely continuous functions on I, one-sided limits $y(c\pm)$, $(py')(c\pm)$ exist and are finite and $l(y) \in L^2(I)$. The operator L defined by $Ly = l(y)$ is called the maximal operator on $L^2(I)$.

For two arbitrary functions $y, z \in \mathcal{D}$, we have Green's formula

$$\int_a^\infty l(y) \overline{z} dx - \int_a^\infty y \overline{l(z)} dx = [y, z]_{c-} - [y, z]_a + [y, z]_\infty - [y, z]_{c+}, \tag{1.2}$$

where $[y, z]_x = y(x)\overline{(pz')(x)} - (py')(x)\overline{z(x)}$ $(x \in I)$.

We assume that the following conditions are satisfied.

(A1) The points a and c are regular for the differential expression l. p and q are real-valued, Lebesgue measurable functions on I and $\frac{1}{p}, q \in L^1_{loc}(I)$. The point c is regular if $\frac{1}{p}, q \in L^1[c - \epsilon, c + \epsilon]$ for some $\epsilon > 0$. Moreover, the functions p and q are such that all solutions of the the equation

$$l(y) = 0 \tag{1.3}$$

belong to $L^2(I)$, i.e., Weyl limit-circle case holds for the differential expression l (see [1-3]).

(A2) The function $f(x, y)$ is real-valued and continuous in $(x, \zeta) \in I \times \mathbb{R}$, and

$$|f(x, \zeta)| \leq g(x) + \vartheta |\zeta| \tag{1.4}$$

for all (x, ζ) in $I \times \mathbb{R}$, where $g(x) \geq 0$, $g \in L^2(I)$, and ϑ is a positive constant.

If we define the operator F taking each function $y(.)$ to the function $f(., y(.))$, then the condition (4) is necessary and sufficient for F to map $L^2(I)$ into itself (see ([17], Chapter 1)).

Denote by

$$u := u(x) = \begin{cases} u^{(1)}(x), & x \in I_1 \\ u^{(2)}(x), & x \in I_2 \end{cases}, \quad v := v(x) = \begin{cases} v^{(1)}(x), & x \in I_1 \\ v^{(2)}(x), & x \in I_2 \end{cases}$$

the solutions to the equation (1.3) satisfying the initial conditions

$$u^{(1)}(a) = 0, \ (pu^{(1)\prime})(a) = 1, \ v^{(1)}(a) = -1, \ (pv^{(1)\prime})(a) = 0, \tag{1.5}$$

and impulsive conditions

$$\begin{aligned} &U\left(c+\right)=CU\left(c-\right),\ U(x):=\begin{pmatrix} u\left(x\right) \\ \left(pu'\right)\left(x\right)\end{pmatrix}, \\ &V\left(c+\right)=CV\left(c-\right),\ V(x):=\begin{pmatrix} v\left(x\right) \\ \left(pv'\right)\left(x\right)\end{pmatrix}, \end{aligned} \tag{1.6}$$

$$C\in M_2\left(\mathbb{R}\right),\ \det C=\rho>0,$$

where $M_2\left(\mathbb{R}\right)$ denotes the 2×2 matrices with entries from $\mathbb{R}$.

Now, we introduce the Hilbert space $H=L^2\left(I_1\right)\dot{+}L^2\left(I_2\right)$ with the inner product

$$\langle y,z\rangle:=\int_a^c y^{(1)}\overline{z^{(1)}}dx+\gamma\int_c^\infty y^{(2)}\overline{z^{(2)}}dx,\ \gamma=\frac{1}{\rho},$$

where

$$y(x)=\begin{cases} y^{(1)}(x), & x\in I_1 \\ y^{(2)}(x), & x\in I_2\end{cases},\quad z(x)=\begin{cases} z^{(1)}(x), & x\in I_1 \\ z^{(2)}(x), & x\in I_2.\end{cases}$$

We set $W_x^{(i)}:=W_x\left(u^{(i)},v^{(i)}\right)=u^{(i)}(x)(pv^{(i)\prime})(x)-(pu^{(i)\prime})(x)v^{(i)}(x)\ (x\in I_i, i=1,2)$. Then the equality $W_x^{(1)}=\rho W_x^{(2)}$ holds. For convenience, we denote $W_x:=W_x^{(1)}=\rho W_x^{(2)}$. Since the wronskian of any two solutions of Equation (1.3) is constant, we have $W_x\left(u,v\right)=1$. Then, u and v are linearly independent and they form a fundamental system of solutions of equation (1.3). By the condition A1, we get u, $v\in L^2\left(I\right)$ and moreover, u, $v\in\mathcal{D}$. So, the values $[y,u]_\infty$ and $[y,v]_\infty$ exist and are finite for every $y\in\mathcal{D}$. By using Green's formula (1.2) and the conditions (1.5)-(1.6), we can get

$$\begin{aligned} &[y,u]_\infty=y\left(a\right)+\textstyle\int_a^\infty u\left(x\right)\overline{l(y\left(x\right))}dx, \\ &[y,v]_\infty=\left(py'\right)\left(a\right)+\textstyle\int_a^\infty v\left(x\right)\overline{l(y\left(x\right))}dx. \end{aligned} \tag{1.7}$$

Now, we will add to problem (1.1) the boundary conditions

$$\begin{aligned} &y\left(a\right)\cos\alpha+\left(py'\right)\left(a\right)\sin\alpha=d_1, \\ &[y,u]_\infty\cos\beta+[y,v]_\infty\sin\beta=d_2, \end{aligned} \tag{1.8}$$

and impulsive conditions

$$Y\left(c+\right)=CY\left(c-\right),\ Y=\begin{pmatrix} y \\ py'\end{pmatrix},\ \det C=\rho>0, \tag{1.9}$$

where $\alpha,\beta\in\mathbb{R}$, and d_1,d_2 are arbitrary given real numbers, and

(**A3**) $\qquad \omega:=\cos\alpha\sin\beta-\cos\beta\sin\alpha\ \neq 0.$

Since the function y in (1.8) satisfies Equation (1.1), we have

$$\begin{aligned} &[y,u]_\infty=y\left(a\right)+\int_a^\infty u\left(x\right)f\left(x,y\left(x\right)\right)dx, \\ &[y,v]_\infty=\left(py'\right)\left(a\right)+\int_a^\infty v\left(x\right)f\left(x,y\left(x\right)\right)dx. \end{aligned}$$

2. Green's function

In this section, we construct an appropriate Green's function. So, we will reduce the boundary-value problem (1.1), (1.8), (1.9) to a fixed point problem.

Let us consider the linear boundary value problem

$$-(p(x)y')' + q(x)y = h\,(x)\,,\ x \in I,\ h \in H, \tag{2.1}$$

$$\left.\begin{array}{c} y\,(a)\cos\alpha + (py')\,(a)\sin\alpha = 0, \\ [y,u]_\infty \cos\beta + [y,v]_\infty \sin\beta = 0,\ \alpha,\beta \in \mathbb{R}, \\ Y\,(c+) = CY\,(c-)\,,\ Y := \begin{pmatrix} y \\ py' \end{pmatrix},\ \det C = \rho > 0, \end{array}\right\} \tag{2.2}$$

where y is a desired solution, u and v are solutions to the equation (1.3) under the conditions (1.5)-(1.6).

Define

$$\varphi\,(x) = \cos\alpha u\,(x) + \sin\alpha v\,(x)\,,\ \psi\,(x) = \cos\beta u\,(x) + \sin\beta v\,(x)\,, \tag{2.3}$$

where $W_x\,(\varphi,\psi) = \omega$. It is clear that these functions are solutions to the equation (1.3) and are in H. Further, we have

$$\begin{aligned} &[\varphi,u]_x = \varphi\,(a) = -\sin\alpha,\ [\varphi,v]_x = (p\varphi)'\,(a) = \cos\alpha,\ (x \in I_1), \\ &[\psi,u]_x = \psi\,(a) = -\sin\beta,\ [\psi,v]_x = (p\psi)'\,(a) = \cos\beta,\ (x \in I_1), \\ &[\psi,u]_\infty = -\rho\sin\beta,\ [\psi,v]_\infty = \rho\cos\beta, \\ &\Phi\,(c+) = C\Phi\,(c-)\,,\ \Phi(x) := \begin{pmatrix} \varphi\,(x) \\ (p\varphi')\,(x) \end{pmatrix}, \\ &\Psi\,(c+) = C\Psi\,(c-)\,,\ \Psi(x) := \begin{pmatrix} \psi\,(x) \\ (p\psi')\,(x) \end{pmatrix}. \end{aligned}$$

Let us introduce the function

$$G\,(x,t) = \begin{cases} \frac{\varphi(x)\psi(t)}{\omega}, & \text{if } a \le x \le t < \infty,\ x \ne c,\ t \ne c, \\ \frac{\varphi(t)\psi(x)}{\omega}, & \text{if } a \le t \le x < \infty,\ x \ne c,\ t \ne c. \end{cases} \tag{2.4}$$

$G\,(x,t)$ is called the Green's function of the boundary-value problem (2.1)-(2.2). Since $\varphi,\psi \in H$, we have

$$\int_a^\infty \int_a^\infty |G\,(x,t)|^2\,dxdt < \infty, \tag{2.5}$$

i.e., $G\,(x,t)$ is a Hilbert-Schmidt kernel.

Theorem 2.1. *The function*

$$y\,(x) = \int_a^c G\,(x,t)\,h\,(t)\,dt + \gamma\int_c^\infty G\,(x,t)\,h\,(t)\,dt,\ x \in I, \tag{2.6}$$

is the solution of the boundary-value problem (2.1)-(2.2).

Proof. By a variation of constants formula, the general solution of the equation (2.1) has the form

$$y(x)=\begin{cases} k_1\varphi^{(1)}(x)+k_2\psi^{(1)}(x) \\ +\frac{\psi^{(1)}(x)}{\omega}\int_a^x \varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{\varphi^{(1)}(x)}{\omega}\int_x^c \psi^{(1)}(t)\,h(t)\,dt,\ x\in I_1, \\ k_3\varphi^{(2)}(x)+k_4\psi^{(2)}(x) \\ +\frac{\gamma}{\omega}\psi^{(2)}(x)\int_c^x \varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\gamma}{\omega}\varphi^{(2)}(x)\int_x^\infty \psi^{(2)}(t)\,h(t)\,dt,\ x\in I_2, \end{cases} \tag{2.7}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants.

By (2.7), we get

$$(py)'(x)=\begin{cases} k_1\left(p\varphi^{(1)}\right)'(x)+k_2(p\psi^{(1)})'(x) \\ +\frac{\left(p\psi^{(1)}\right)'(x)}{\omega}\int_a^x \varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{\left(p\varphi^{(1)}\right)'(x)}{\omega}\int_x^c \psi^{(1)}(t)\,h(t)\,dt,\ x\in I_1, \\ k_3\left(p\varphi^{(2)}\right)'(x)+k_4(p\psi^{(2)})'(x) \\ +\frac{\gamma}{\omega}\left(p\psi^{(2)}\right)'(x)\int_c^x \varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\gamma}{\omega}\left(p\varphi^{(2)}\right)'(x)\int_x^\infty \psi^{(2)}(t)\,h(t)\,dt,\ x\in I_2. \end{cases}$$

Hence, we have

$$\begin{aligned} y(a) &= k_1\varphi^{(1)}(a)+k_2\psi^{(1)}(a)+\frac{\varphi^{(1)}(a)}{\omega}\int_a^c \psi^{(1)}(t)\,h(t)\,dt \\ &= -k_1\sin\alpha-k_2\sin\beta-\frac{1}{\omega}\sin\alpha\int_a^c \varphi^{(1)}(t)\,h(t)\,dt, \\ (py)'(a) &= k_1\left(p\varphi^{(1)}\right)'(a)+k_2(p\psi^{(1)})'(a) \\ &\quad +\frac{1}{\omega}\left(p\varphi^{(1)}\right)'(a)\int_a^c \psi^{(1)}(t)\,h(t)\,dt \\ &= k_1\cos\alpha+k_2\cos\beta+\frac{1}{\omega}\cos\alpha\int_a^c \varphi^{(1)}(t)\,h(t)\,dt. \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.2), we get

$$k_2(\cos\alpha\sin\beta-\sin\alpha\cos\beta)=0,\ \ k_2\omega=0,$$

i.e., $k_2 = 0$. Further, we have

$$[y,u]_x = y(x)(pu')(x) - (py')(x)u(x)$$

$$= \begin{cases} k_1[\varphi^{(1)},u]_x + \frac{1}{\omega}[[\psi^{(1)}(x),u]_x \int_a^x \varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{1}{\omega}[\varphi^{(1)}(x),u]_x \int_x^c \psi^{(1)}(t)\,h(t)\,dt,\ x \in I_1, \\ k_3[\varphi^{(2)},u]_x + k_4[\psi^{(2)},u]_x \\ +\frac{\gamma}{\omega}[\psi^{(2)},u]_x \int_c^x \varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\gamma}{\omega}[\varphi^{(2)},u]_x \int_x^\infty \psi^{(2)}(t)\,h(t)\,dt,\ x \in I_2. \end{cases}$$

Thus

$$[y,u]_\infty = -k_3\rho\sin\alpha - k_4\rho\sin\beta - \frac{\gamma}{\omega}\rho\sin\beta\int_c^\infty \varphi^{(2)}(t)\,h(t)\,dt.$$

Similarly, we get

$$[y,v]_x = y(x)(pv')(x) - (py')(x)v(x)$$

$$= \begin{cases} k_1[\varphi^{(1)},v]_x \\ +\frac{1}{\omega}[[\psi^{(1)}(x),v]_x \int_a^x \varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{1}{\omega}[\varphi^{(1)}(x),v]_x \int_x^c \psi^{(1)}(t)\,h(t)\,dt,\ x \in I_1, \\ k_3[\varphi^{(2)},v]_x + k_4[\psi^{(2)},v]_x \\ +\frac{\gamma}{\omega}[\psi^{(2)},v]_x \int_c^x \varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\gamma}{\omega}[\varphi^{(2)},v]_x \int_x^\infty \psi^{(2)}(t)\,h(t)\,dt,\ x \in I_2, \end{cases}$$

and

$$[y,v]_\infty = k_3\rho\cos\alpha + k_4\rho\cos\beta + \frac{\gamma}{\omega}\rho\cos\beta\int_c^\infty \varphi^{(2)}(t)\,h(t)\,dt.$$

From the conditions (2.2), we obtain

$$k_3(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = 0.$$

Hence, $k_3 = 0$. Similarly, we have

$$Y(c+) = \begin{pmatrix} y(c+) \\ (py')(c+) \end{pmatrix} = \begin{pmatrix} k_4\psi^{(2)}(c+) \\ k_4(p\psi^{(2)})'(c+) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{\gamma}{\omega}\varphi^{(2)}(c+)\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \\ \frac{\gamma}{\omega}\left(p\varphi^{(2)}\right)'(c+)\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \end{pmatrix}$$

$$= k_4 \begin{pmatrix} \psi^{(2)}(c+) \\ (p\psi^{(2)})'(c+) \end{pmatrix}$$

$$+ \frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \begin{pmatrix} \varphi^{(2)}(c+) \\ \left(p\varphi^{(2)}\right)'(c+) \end{pmatrix}$$

$$= k_4\Psi(c+) + \left\{\frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt\right\}\Phi(c+)$$

and

$$Y(c-) = \begin{pmatrix} y(c-) \\ (py')(c-) \end{pmatrix}$$

$$= \begin{pmatrix} k_1\varphi^{(1)}(c-) \\ k_1\left(p\varphi^{(1)}\right)'(c-) \end{pmatrix} + \begin{pmatrix} \frac{\psi^{(1)}(c-)}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt \\ \frac{\left(p\psi^{(1)}\right)'(c-)}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt \end{pmatrix}$$

$$= k_1 \begin{pmatrix} \varphi^{(1)}(c-) \\ \left(p\varphi^{(1)}\right)'(c-) \end{pmatrix} + \frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt \begin{pmatrix} \psi^{(1)}(c-) \\ \left(p\psi^{(1)}\right)'(c-) \end{pmatrix}$$

$$= k_1\Phi(c-) + \left\{\frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt\right\}\Psi(c-).$$

By the conditions (2.2), we obtain

$$k_4\Psi(c+) + \left\{\frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt\right\}\Phi(c+)$$

$$= C\left\{k_1\Phi(c-) + \left\{\frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt\right\}\Psi(c-)\right\}.$$

Using the conditions (2.) and (2.), we get

$$\Phi\left(c-\right)\left\{\frac{\gamma}{\omega}\int_{c}^{\infty}\psi^{(2)}\left(t\right)h\left(t\right)dt-k_{1}\right\}$$

$$=\Psi\left(c-\right)\left\{\frac{1}{\omega}\int_{a}^{c}\varphi^{(1)}\left(t\right)h\left(t\right)dt-k_{4}\right\},$$

$$\begin{pmatrix}\varphi^{(1)}\left(c-\right)\\(p\varphi^{(1)\prime})\left(c-\right)\end{pmatrix}\left\{\frac{\gamma}{\omega}\int_{c}^{\infty}\psi^{(2)}\left(t\right)h\left(t\right)dt-k_{1}\right\}$$

$$=\begin{pmatrix}\psi^{(1)}\left(c-\right)\\(p\psi^{(1)\prime})\left(c-\right)\end{pmatrix}\left\{\frac{1}{\omega}\int_{a}^{c}\varphi^{(1)}\left(t\right)h\left(t\right)dt-k_{4}\right\}.$$

So, we have the following linear equation system

$$k_{4}\psi^{(1)}\left(c-\right)-k_{1}\varphi^{(1)}\left(c-\right)$$

$$=\left\{\frac{1}{\omega}\int_{a}^{c}\varphi^{(1)}\left(t\right)h\left(t\right)dt\right\}\psi^{(1)}\left(c-\right)$$

$$-\left\{\frac{\gamma}{\omega}\int_{c}^{\infty}\psi^{(2)}\left(t\right)h\left(t\right)dt\right\}\varphi^{(1)}\left(c-\right),$$

$$k_{4}(p\psi^{(1)\prime})\left(c-\right)-k_{1}(p\varphi^{(1)\prime})\left(c-\right)$$

$$=\left\{\frac{1}{\omega}\int_{a}^{c}\varphi^{(1)}\left(t\right)h\left(t\right)dt\right\}(p\psi^{(1)\prime})\left(c-\right)$$

$$-\left\{\frac{\gamma}{\omega}\int_{c}^{\infty}\psi^{(2)}\left(t\right)h\left(t\right)dt\right\}(p\varphi^{(1)\prime})\left(c-\right),$$

i.e.,

$$\begin{pmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)\prime})(c-) & (p\varphi^{(1)\prime})(c-) \end{pmatrix} \begin{pmatrix} k_4 \\ -k_1 \end{pmatrix}$$

$$= \begin{pmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)\prime})(c-) & (p\varphi^{(1)\prime})(c-) \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\, h(t)\, dt \\ \\ -\frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\, h(t)\, dt \end{pmatrix}.$$

Hence, we have the following determinant of this linear equation system

$$\begin{vmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)\prime})(c-) & (p\varphi^{(1)\prime})(c-) \end{vmatrix} = -\omega.$$

Since this determinant is different from zero, the solution of this system is unique. If we solve this system, we have the following equalities

$$k_1 = \frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\, h(t)\, dt, k_4 = \frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\, h(t)\, dt.$$

From what has already been done, we have

$$y(x) = \begin{cases} \varphi^{(1)}(x)\frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\, h(t)\, dt \\ +\frac{\psi^{(1)}(x)}{\omega}\int_a^x \varphi^{(1)}(t)\, h(t)\, dt \\ +\frac{\varphi^{(1)}(x)}{\omega}\int_x^c \psi^{(1)}(t)\, h(t)\, dt,\ x \in I_1, \\ \psi^{(2)}(x)\frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\, h(t)\, dt \\ +\frac{\gamma}{\omega}\psi^{(2)}(x)\int_c^x \varphi^{(2)}(t)\, h(t)\, dt \\ +\frac{\gamma}{\omega}\varphi^{(2)}(x)\int_x^\infty \psi^{(2)}(t)\, h(t)\, dt,\ x \in I_2, \end{cases}$$

i.e., (2.4) and (2.6) hold. □

Thus we have a

Theorem 2.2. *The unique solution to the equation (2.1) under the conditions (1.8)-(1.9) is given by the formula*

$$y(x) = w(x) + \langle G(x,.), \overline{h(.)}\rangle,$$

where

$$w(x) = \frac{d_1}{\omega}\varphi(x) - \frac{d_2}{\omega}\psi(x).$$

Proof. By the conditions (2.)-(2.), the function $w(x)$ is a unique solution of the equation (1.3) satisfying the conditions (1.8)-(1.9). By Theorem 1 the function $\langle G(x,.), \overline{h(.)}\rangle$ a unique solution to the equation (2.1) satisfying the conditions (2.2). This finishes the proof. □

From Theorem 2, the boundary-value problem (1.1), (1.8), (1.9) in H is equivalent to the non-linear integral equation

$$y(x) = w(x) + \langle G(x,.), f(., y(.))\rangle, \; x \in I, \tag{2.9}$$

where the functions $w(x)$ and $G(x,t)$ are defined above. Hence, we shall study the equation (2.9).

By (1.4) and (2.5), we can define the operator $T : H \to H$ by the formula

$$(Ty)(x) = w(x) + \langle G(x,.), f(., y(.))\rangle, \; x \in I, \tag{2.10}$$

where $y, w \in H$. Then the equation (2.9) can be written as $y = Ty$.

Now, we search the fixed points of the operator T because it is equivalent to solving the equation (2.9).

3. The fixed points of the operator T

In this section, we investigate the fixed points of the operator T by using the following Banach fixed point theorem:

Definition 3.1. [[16]]Let A be a mapping of a metric space R into itself. Then x is called a fixed point of A if $Ax = x$. Suppose there exists a number $\alpha < 1$ such that

$$\rho(Ax, Ay) \leq \alpha\rho(x,y)$$

for every pair of points $x, y \in R$. Then A is said to be a contraction mapping.

Theorem 3.1. [16] *Every contraction mapping A defined on a complete metric space R has a unique fixed point.*

Theorem 3.2. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. Further, let the function $f(x,y)$ satisfy the following Lipschitz condition: there* *exists a constant $K > 0$ such that*

$$\int_a^c \left| f^{(1)}\left(x, y^{(1)}(x)\right) - f^{(1)}\left(x, z^{(1)}(x)\right)\right|^2 dx$$

$$+\gamma \int_c^\infty \left| f^{(2)}\left(x, y^{(2)}(x)\right) - f^{(2)}\left(x, z^{(2)}(x)\right)\right|^2 dx$$

$$\leq K^2 \left(\int_a^c \left| y^{(1)}(x) - z^{(1)}(x)\right|^2 dx + \gamma \int_c^\infty \left| y^{(2)}(x) - z^{(2)}(x)\right|^2 dx \right)$$

$$= K^2 \left\| y - z \right\|^2$$

for all $y, z \in H$. If

$$K \left(\int_a^c \int_a^c \left| G(x,t)\right|^2 dxdt + \gamma \int_c^\infty \int_c^\infty \left| G(x,t)\right|^2 dxdt \right) < 1, \tag{3.1}$$

then the boundary-value problem (1.1), (1.8), (1.9) has a unique solution in H.

Proof. It suffices to prove that the operator T is a contraction operator. For $y, z \in H$, we have

$$\begin{aligned}\left| Ty(x) - Tz(x)\right|^2 &= \left| \langle G(x,.), [f(., y(.)) - f(., z(.))]\rangle \right|^2 \\ &\leq \left\| G(x,.)\right\|^2 \left\| f(., y(.)) - f(., z(.))\right\|^2 \\ &\leq K^2 \left\| G(x,.)\right\|^2 \left\| y - z\right\|^2, \ x \in I.\end{aligned}$$

Thus, we get

$$\left\| Ty - Tz \right\| \leq \alpha \left\| y - z \right\|,$$

where

$$\alpha = K \left(\int_a^c \int_a^c \left| G(x,t)\right|^2 dxdt + \gamma \int_c^\infty \int_c^\infty \left| G(x,t)\right|^2 dxdt \right)^{\frac{1}{2}} < 1,$$

i.e., T is a contraction mapping. □

Now, our next claim is that the function $f(x, y)$ satisfies a Lipschitz condition on a subset of H but not of the whole space.

Theorem 3.3. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, let the function $f(x, y)$ satisfy the following Lipschitz condition: there* *exist constants M, $K>0$ such that*

$$\int_a^c \left|f^{(1)}\left(x,y^{(1)}(x)\right)-f^{(1)}\left(x,z^{(1)}(x)\right)\right|^2 dx$$

$$+\gamma\int_c^\infty \left|f^{(2)}\left(x,y^{(2)}(x)\right)-f^{(2)}\left(x,z^{(2)}(x)\right)\right|^2 dx$$

$$\leq K^2\left(\int_a^c \left|y^{(1)}(x)-z^{(1)}(x)\right|^2 dx+\gamma\int_c^\infty \left|y^{(2)}(x)-z^{(2)}(x)\right|^2 dx\right)$$

$$=K^2\left\|y-z\right\|^2$$

for all y and z in $S_M=\{t\in H:\|t\|\leq M\}$, where K may depend on M. If

$$\left\{\int_a^c \left|w^{(1)}(x)\right|^2 dx+\gamma\int_c^\infty \left|w^{(2)}(x)\right|^2 dx\right\}^{1/2}$$

$$+\left(\int_a^c\int_a^c \left|G(x,t)\right|^2 dxdt+\gamma\int_c^\infty\int_c^\infty \left|G(x,t)\right|^2 dxdt\right)^{\frac{1}{2}}$$

$$\times\sup_{y\in S_M}\left\{\begin{array}{c}\int_a^c \left|f^{(1)}\left(t,y^{(1)}(t)\right)-f^{(1)}\left(t,z^{(1)}(t)\right)\right|^2 dt\\ +\gamma\int_c^\infty \left|f^{(2)}\left(t,y^{(2)}(t)\right)-f^{(2)}\left(t,z^{(2)}(t)\right)\right|^2 dt\end{array}\right\}^{1/2}$$

$$\leq M$$

and

$$K\left(\int_a^c\int_a^c \left|G(x,t)\right|^2 dxdt+\gamma\int_c^\infty\int_c^\infty \left|G(x,t)\right|^2 dxdt\right)^{\frac{1}{2}}<1, \tag{3.2}$$

then the boundary-value problem (1.1), (1.8), (1.9) has a unique solution with

$$\int_a^c \left|y^{(1)}(x)\right|^2 dx+\gamma\int_c^\infty \left|y^{(2)}(x)\right|^2 dx\leq M^2.$$

Proof. It is clear that S_M is a closed set of H. We first prove that the operator T

maps S_M into itself. For $y \in S_M$ we have

$$\|Ty\| = \|w(x) + \langle G(x,.), f(., y(.))\rangle\| \le \|w\| + \|\langle G(x,.), f(., y(.))\rangle\|$$

$$\le \|w\| + \left(\int_a^c \int_a^c |G(x,t)|^2 \, dxdt + \gamma \int_c^\infty \int_c^\infty |G(x,t)|^2 \, dxdt \right)^{\frac{1}{2}}$$

$$\times \sup_{y \in S_M} \left\{ \begin{array}{c} \int_a^c \left| f^{(1)}\left(t, y^{(1)}(t)\right) - f^{(1)}\left(t, z^{(1)}(t)\right) \right|^2 dt \\ +\gamma \int_c^\infty \left| f^{(2)}\left(t, y^{(2)}(t)\right) - f^{(2)}\left(t, z^{(2)}(t)\right) \right|^2 dt \end{array} \right\}^{1/2} \le M.$$

Consequently, $T : S_M \to S_M$.

We can now proceed analogously to the proof of Theorem 5. So, we can get

$$\|Ty - Tz\| \le \alpha \|y - z\|, \ y, z \in S_M.$$

If we apply the Banach fixed point theorem, then we obtain a unique solution of the boundary-value problem (1.1), (1.8), (1.9) in S_M. □

4. An existence theorem without uniqueness

In this section, we get an existence theorem without uniqueness of solution. Therefore, we will use the following Schauder fixed point theorem:

Definition 4.1. [[11]]An operator acting in a Banach space is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Theorem 4.1. [11] *Let* $\mathbf{B}$ *be a Banach space and* $\mathbf{S}$ *a nonemty bounded, convex, and closed subset of* $\mathbf{B}$. *Assume* $A : \mathbf{B} \to \mathbf{B}$ *is a completely continuous operator. If the operator* A *leaves the set* $\mathbf{S}$ *invariant, i.e., if* $A(\mathbf{S}) \subset \mathbf{S}$, *then* A *has at least one fixed point in* $\mathbf{S}$.

A set $S \subset H$ is relatively compact iff S is bounded and for every $\varepsilon > 0$ (i) there exists $\delta > 0$ such that $\|y(x+h) - y(x)\| < \varepsilon$ for all $y \in S$ and all $h \ge 0$ with $h < \delta$, (ii) there exists a number $N > 0$ such that $\int_N^\infty |y(x)|^2 dx < \varepsilon$ for all $y \in S$ ([11]).

Now, we give

Theorem 4.2. *The operator* T *defined by (2.10) is completely continuous operator under the conditions (A1), (A2) and (A3).*

Proof. Let $y_0 \in H$. Then, we have

$$
\begin{aligned}
&\left|(Ty)(x) - (Ty_0)(x)\right|^2 \\
&= \left|\langle G(x,.), [f(.,y(.)) - f(.,y_0(.))]\rangle\right|^2 \\
&\leq \left\|G(x,.)\right\|^2 \left\{ \begin{array}{c} \int_a^c \left|f^{(1)}\left(t,y^{(1)}(t)\right) - f^{(1)}\left(t,y_0^{(1)}(t)\right)\right|^2 dt \\ +\gamma \int_c^\infty \left|f^{(2)}\left(t,y^{(2)}(t)\right) - f^{(2)}\left(t,y_0^{(2)}(t)\right)\right|^2 dt \end{array} \right\}^2 .
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left\|Ty - Ty_0\right\|^2 \\
&\leq K \left\{ \begin{array}{c} \int_a^c \left|f^{(1)}\left(t,y^{(1)}(t)\right) - f^{(1)}\left(t,y_0^{(1)}(t)\right)\right|^2 dt \\ +\gamma \int_c^\infty \left|f^{(2)}\left(t,y^{(2)}(t)\right) - f^{(2)}\left(t,y_0^{(2)}(t)\right)\right|^2 dt \end{array} \right\}^2 ,
\end{aligned}
$$

where

$$
K = \left(\int_a^c \int_a^c \left|G(x,t)\right|^2 dxdt + \gamma \int_c^\infty \int_c^\infty \left|G(x,t)\right|^2 dxdt \right).
$$

We know that an operator F defined by $Fy(x) = f(x, y(x))$ is continuous in H under the condition (A2) (see [17]). Hence, for a given $\epsilon > 0$, we can find a $\delta > 0$ such that $\|y - y_0\| < \delta$ implies

$$
\left\{ \begin{array}{c} \int_a^c \left|f^{(1)}\left(t,y^{(1)}(t)\right) - f^{(1)}(t,y_0^{(1)}(t))\right|^2 dt \\ +\gamma \int_c^\infty \left|f^{(2)}\left(t,y^{(2)}(t)\right) - f^{(2)}(t,y_0^{(2)}(t))\right|^2 dt \end{array} \right\} < \frac{\epsilon^2}{K^2}.
$$

From (4.), we get

$$
\|Ty - Ty_0\| < \epsilon,
$$

i.e., T is continuous.

Set $Y = \{y \in H : \|y\| \leq m\}$. By (3.3), we have

$$
\|Ty\| \leq \|w\| + \left\{ \begin{array}{c} K \int_a^c \left|f^{(1)}\left(t,y^{(1)}(t)\right)\right|^2 dt \\ +\gamma K \int_c^\infty \left|f^{(2)}\left(t,y^{(2)}(t)\right)\right|^2 dt \end{array} \right\}^{1/2}, \text{ for all } y \in Y.
$$

Furthermore, using (1.4), we get

$$\int_a^c \left|f^{(1)}\left(t, y^{(1)}(t)\right)\right|^2 dt + \gamma \int_c^\infty \left|f^{(2)}\left(t, y^{(2)}(t)\right)\right|^2 dt$$

$$\leq \int_a^c \left[g^{(1)}(t) + \vartheta \left|y^{(1)}(t)\right|\right]^2 dt + \gamma \int_c^\infty \left[g^{(2)}(t) + \vartheta \left|y^{(2)}(t)\right|\right]^2 dt$$

$$\leq 2\int_a^c [\left(g^{(1)}\right)^2(t) + \vartheta^2 \left|y^{(1)}(t)\right|^2]dt$$

$$+ 2\gamma \int_c^\infty [\left(g^{(2)}\right)^2(t) + \vartheta^2 \left|y^{(2)}(t)\right|^2]dt$$

$$= 2(\|g\|^2 + \vartheta^2 \|y\|^2) \leq 2(\|g\|^2 + \vartheta^2 m^2).$$

Thus, for all $y \in Y$, we obtain

$$\|Ty\| \leq \|w\| + \left[2K\left(\|g\|^2 + \vartheta^2 m\right)\right]^{1/2},$$

i.e., $T(Y)$ is a bounded set in H.

Moreover, for all $y \in Y$, we have

$$\begin{aligned}
&\int_a^c \left|(Ty^{(1)})(x+h) - (Ty^{(1)})(x)\right|^2 dx \\
&+ \gamma \int_c^\infty \left|(Ty^{(2)})(x+h) - (Ty^{(2)})(x)\right|^2 dx \\
&= \|\langle [G(x+h,.) - G(x,.)], f(., y(.))\rangle\|^2 \\
&\leq \begin{pmatrix} \int_a^c \int_a^c |G(x+h,t) - G(x,t)|^2 dxdt \\ +\gamma \int_c^\infty \int_c^\infty |G(x+h,t) - G(x,t)|^2 dxdt \end{pmatrix} \\
&\times \begin{Bmatrix} \int_a^c \left|f^{(1)}\left(t, y^{(1)}(t)\right)\right|^2 dt \\ +\gamma \int_c^\infty \left|f^{(2)}\left(t, y^{(2)}(t)\right)\right|^2 dt \end{Bmatrix}^2 \\
&\leq 2\left(\|g\|^2 + \vartheta^2 m\right) \begin{pmatrix} \int_a^c \int_a^c |G(x+h,t) - G(x,t)|^2 dxdt \\ +\gamma \int_c^\infty \int_c^\infty |G(x+h,t) - G(x,t)|^2 dxdt \end{pmatrix}
\end{aligned}$$

From (2.5), there exists a $\delta > 0$ such that

$$\int_a^c \left| Ty^{(1)}(x+h) - Ty^{(1)}(x) \right|^2 dx$$

$$+ \gamma \int_c^\infty \left| Ty^{(2)}(x+h) - Ty^{(2)}(x) \right|^2 dx < \epsilon^2,$$

for given $\epsilon > 0$, all $y \in Y$ and all $h < \delta$.

Further, for all $y \in Y$, we have $(N > c)$

$$\int_N^\infty \left| \left(Ty^{(2)}\right)(x) \right|^2 dx$$

$$\leq \int_N^\infty \left| w^{(2)}(x) \right|^2 dx + 2\left(\|g\|^2 + \vartheta^2 m \right) \int_N^\infty \|G(x,.)\|^2 dx.$$

So, from (2.5), we see that for a given $\epsilon > 0$ there exists a positive number N, depending only on ϵ such that

$$\int_N^\infty \left| \left(Ty^{(2)}\right)(x) \right|^2 dx < \epsilon^2,$$

for all $y \in Y$.

Thus $T(Y)$ is a relatively compact in H, i.e., the operator T is completely continuous. $\square$

Theorem 4.3. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, let there exist constants $M > 0$ such that*

$$\left\{ \int_a^c \left| w^{(1)}(x) \right|^2 dx + \gamma \int_c^\infty \left| w^{(2)}(x) \right|^2 dx \right\}^{1/2}$$

$$+ \left(\int_a^c \int_a^c |G(x,t)|^2 dxdt + \gamma \int_c^\infty \int_c^\infty |G(x,t)|^2 dxdt \right)$$

$$\times \sup_{y \in S_M} \left\{ \begin{array}{c} \int_a^c \left| f^{(1)}\left(t, y^{(1)}(t)\right) - f^{(1)}\left(t, z^{(1)}(t)\right) \right|^2 dt \\ +\gamma \int_c^\infty \left| f^{(2)}\left(t, y^{(2)}(t)\right) - f^{(2)}\left(t, z^{(2)}(t)\right) \right|^2 dt \end{array} \right\}^{1/2}$$

$$\leq M,$$

where $S_M = \{ y \in H : \|y\| \leq M \}$. Then the boundary-value problem (1.1), (1.8), (1.9) has at least one solution with

$$\int_a^c \left| y^{(1)}(x) \right|^2 dx + \gamma \int_c^\infty \left| y^{(2)}(x) \right|^2 dx \leq M^2.$$

Proof. Let us define an operator $T : H \to H$ by (2.10). From theorems 6, 9 and (4.3), we conclude that T maps the set S_M into itself. It is clear that the set S_M is bounded, convex and closed. Using Theorem 8, the theorem follows. □

REFERENCES

1. R. P. Agarwal and D. O'Regan: Boundary value problems of nonsingular type on the semi-infinite interval, Tohoku Math. Journal, 51, 3 (1999), 391–397.
2. R. P. Agarwal and D. O'Regan: Multiple nonnegative solutions for second order impulsive differential equations, Appl. Math. Comput., 114 (2000), 51-59.
3. D. D. Bainov and P. S. Simeonov: Impulsive Differential Equations: Periodic Solutions and Applications, Longman: Harlow, 1993.
4. D. D. Bainov and P. S. Simeonov: Impulsive Differential Equations: Asymptotic Properties of the Solutions, World Scientific: Singapore, 1995.
5. D. D. Bainov and P. S. Simeonov: Oscillation Theory of Impulsive Differeential Equations, International Publications: Orlando, 1998.
6. M. Benchohra, J. Henderson and S. Ntouyas: Impulsive Differential Equations and Inclusions, Hindawi Publishing Corporation: New York, 2006.
7. W. Ding and M. Han: Periodic boundary value problem for the second order impulsive functional differential equations, Appl. Math. Comput., 155, 3 (2004), 709-726.
8. N. Dunford and J. T. Schwartz: Linear Operators. Part II, Interscience, New York, 1964.
9. M. Feng and D. Xie: Multiple positive solutions of multi-point boundary value problem for second order impulsive differential equations, J. Comput. Appl. Math., 223, 1 (2009), 438–448.
10. D. Guo: Second order impulsive integro-differential equations on unbounded domains in Banach spaces, Nonlinear Analysis: Theory Meth. Appl., 35, 4 (1999), 413–423.
11. G. Sh. Guseinov and I. Yaslan: Boundary value problems for second order nonlinear differential equations on infinite intervals. J. Math. Anal. Appl., 290 (2004), 620–638.
12. S. G. Hristova and D. D. Bainov: Monotone-iterative techniques of V. Lakshmikantham for a boundary value problem for systems of impulsive differential-difference equations, J. Math. Anal. Appl., 197, 1 (1996), 1-13.
13. T. Jankowski: Positive solutions to second order four-point boundary value problems for impulsive differential equations, Appl. Math. Comput., 202, 2 (2008), 550–561.
14. T. Jankowski: Positive solutions of three-point boundary value problems for second order impulsive differential equations with advanced arguments, Appl. Math. Comput., 197, 1 (2008), 179–189.
15. T. Jankowski: Existence of solutions for second order impulsive differential equations with deviating arguments, Nonlinear Analysis: Theory Meth. Appl., 67, 6 (2007), 1764–1774.
16. A. N. Kolmogorov and S. V. Fomin: Introductory Real Analysis, Translated by R.A. Silverman, Dover Publications, New York, 1970.

17. M. A. Krasnosel'skii: Topological Methods in the Theory of Nonlinear Integral Equations, Gostekhteoretizdat, Moscow, 1956, English transl. Pergamon Press, New York, 1964.
18. V. Lakshmikantham D. D. Bainov and P. S. Simeonov: Theory of Impulsive Differential Equations. World Scientific: Singapore, 1989.
19. I. K. C. Leung and P. N. Shivakumar: On the eigenvalue problem $-y'' + f(x)\, y = \lambda y$ on a semi infinite interval. Math. Comput. Mod., 46 (2007), 316–330.
20. B. M. Levitan and I. S. Sargsjan: Sturm-Liouville and Dirac Operators. Mathematics and its Applications (Soviet Series, translated from the Russian), Kluwer Academic Publishers Group, Dordrecht, 1991.
21. J. Li and J. Shen: Existence of positive solution for second-order impulsive boundary value problems on infinity intervals. Bound. Value Probl., vol. 2006, Article ID 14594, (2006), 11 pages.
22. J. Li and J. J. Nieto: Existence of positive solutions for multipoint boundary value problem on the half-line with impulses, Bound. Value Probl., vol. 2009, Article ID 834158, (2009), 12 pages.
23. H. Lian and W. Ge: Existence of positive solutions for Sturm-Liouville boundary value problems on the half-line, J. Math. Anal. Appl., 321 (2006), 781–792.
24. H. Lian, H. Pang and W. Ge: Triple positive solutions for boundary value problems on infinite intervals, Nonlinear Anal., 67 (2007) 2199–2207.
25. X. Lin and D. Jiang: Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, J. Math. Anal. Appl., 321, 2 (2006), 501–514.
26. L. Liu and F. Y. Li: Multiple positive solution of nonlinear two-point boundary value problems, J. Math. Anal. Appl., 203 (1996), 610-625.
27. X. Liu and D. Guo: Periodic Boundary value problems for a class of second-Order impulsive integro-differential equations in Banach spaces, J. Math. Anal. Appl., 216, 1 (1997), 284-302.
28. Y. Liu: Boundary value problems for second order differential equations on unbounded domains in a Banach space, Appl. Math. Comput., 135, 2-3 (2003), 569–583.
29. R. Ma: Existence of positive solutions for second-order boundary value problems on infinity intervals. Appl. Math. Letters, 16, 1 (2003), 33–39.
30. I. Rachunkova and J. Tomecek: Impulsive BVPs with nonlinear boundary conditions for the second order differential equations without growth restrictions, J Math Anal Appl., 292, 2 (2004), 525-539.
31. A. M. Samoilenko and N. A. Perestyuk: Impulsive Differential Equations, World Scientific: Singapore, 1995.
32. E. C. Titchmarsh: Eigenfunction Expansions Associated with Second-Order Differential Equations. Part I. Second Edition, Clarendon Press, Oxford, 1962
33. Z. Wei: Periodic boundary value problems for second order impulsive integrodifferential equations of mixed type in Banach spaces, J Math Anal Appl., 195, 1 (1995), 214-229.

Bilender P. Allahverdiev
Faculty of Science and Letter
Department of Mathematics
Süleyman Demirel University
32260 Isparta, Turkey
bilenderpasaoglu@sdu.edu.tr

Hüseyin Tuna
Faculty of Science and Letter
Department of Mathematics
Mehmet Akif Ersoy University
15030 Burdur, Turkey
hustuna@gmail.com