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BOUNDEDNESS FOR TOEPLITZ TYPE OPERATOR ASSOCIATED WITH SINGULAR INTEGRAL OPERATOR WITH VARIABLE CALDERÓN-ZYGMUND KERNEL

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Abstract. In this paper, we establish sharp maximal function inequalities for the Toeplitz-type operator associated with the singular integral operator with a variable Calderón-Zygmund kernel. As an application, we obtain the boundedness of the operator on Lebesgue, Morrey and Triebel-Lizorkin spaces.
Keywords: function inequalities; Toeplitz-type operator; singular integral operator.

1. Introduction and Preliminaries

As the development of singular integral operators(see [6, 21]), their commutators have been well studied. In [3, 19, 20], the authors prove that the commutators generated by singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by fractional integral operators. In [7, 16], the boundedness for the commutators generated by singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1], Calderón and Zygmund introduce some singular integral operators with a variable kernel and discuss their boundedness. In [11, 12, 13, 22], the authors obtain the boundedness for the commutators generated by singular integral operators with a variable kernel and BMO functions. In [14], the authors prove the boundedness for the multilinear oscillatory singular integral operators generated by operators and BMO functions. In [8, 9], some Toeplitz-type operators associated with singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained. In this paper, we will study the Toeplitz-type operator generated by the singular integral operator with a variable Calderón-Zygmund kernel and Lipschitz and BMO functions.

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First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [6, 21])

$$M^{\#}(f)(x)\approx\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q|f(y)-c|dy.$$

We say that f belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^\infty(R^n)$ and define $||f||_{BMO}=||M^{\#}(f)||_{L^\infty}$. It has been known that (see [21])

$$||f-f_{2^kQ}||_{BMO}\leqslant Ck||f||_{BMO}.$$

Let

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|dy.$$

For $\eta>0$, let $M_\eta(f)(x)=M(|f|^\eta)^{1/\eta}(x)$.

For $0<\eta<n$ and $1\le r<\infty$, set

$$M_{\eta,r}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{|Q|^{1-r\eta/n}}\int_Q|f(y)|^r dy\right)^{1/r}.$$

The A_p weight is defined by (see [6])

$$A_p=\left\{w\in L^1_{loc}(R^n):\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty\right\},$$

$1<p<\infty$. and

$$A_1=\{w\in L^p_{loc}(R^n):M(w)(x)\leqslant Cw(x),a.e.\}.$$

For $\beta>0$ and $p>1$, let $\dot F_p^{\beta,\infty}(R^n)$ be a homogeneous Triebel-Lizorkin space(see [16]).

For $\beta>0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta}=\sup_{\substack{x,y\in R^n\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\beta}<\infty.$$

Definition 1. Let φ be a positive, increasing function on R^+ and there exists a constant $D>0$ such that

$$\varphi(2t)\leqslant D\varphi(t)\quad\text{for}\quad t\geqslant 0.$$

Let f be a locally integrable function on R^n. Set, for $1 \leqslant p < \infty$,

$$||f||_{L^{p,\varphi}} = \sup_{x\in R^n,\ d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey spaces (see [17, 18]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$, which is the Lebesgue spaces.

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [4, 5, 10, 15]).

In this paper, we will study some singular integral operators as follows(see [1]).

Definition 2. Let $K(x) = \Omega(x)/|x|^n \ : \ R^n \setminus \{0\} \to R$. K is said to be a Calderón-Zygmund kernel if
(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;
(b) Ω is homogeneous of degree zero;
(c) $\int_\Sigma \Omega(x)x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n \ : \ |x| = 1\}$ is the unit sphere of R^n.

Definition 3. Let $K(x,y) = \Omega(x,y)/|y|^n \ : \ R^n \times (R^n \setminus \{0\}) \to R$. K is said to be a variable Calderón-Zygmund kernel if
(d) $K(x,\cdot)$ is a Calderón-Zygmund kernel for a.e. $x \in R^n$;
(e) $max_{|\gamma|\leq 2n} \left|\left| \frac{\partial^\gamma}{\partial^\gamma y}\Omega(x,y) \right|\right|_{L^\infty(R^n\times\Sigma)} = M < \infty$.

Moreover, let b be a locally integrable function on R^n and T be a singular integral operator with a variable Calderón-Zygmund kernel as

$$T(f)(x) = \int_{R^n} K(x, x-y)f(y)dy,$$

where $K(x,x-y) = \frac{\Omega(x,x-y)}{|x-y|^n}$ and that $\Omega(x,y)/|y|^n$ is a variable Calderón-Zygmund kernel. The Toeplitz-type operator associated with T is defined by

$$T_b = \sum_{k=1}^m (T^{k,1}M_b I_\alpha T^{k,2} + T^{k,3} I_\alpha M_b T^{k,4}),$$

where $T^{k,1}$ is the singular integral operator with a variable Calderón-Zygmund kernel T or $\pm I$(the identity operator), $T^{k,2}$ and $T^{k,4}$ are linear operators, $T^{k,3} = \pm I$, $k = 1, ..., m$, $M_b(f) = bf$ and I_α is the fractional integral operator($0 < \alpha < n$)(see [2]).

Note that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz-type operator T_b. The Toeplitz-type operator T_b are non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [19, 20]). The main purpose of this paper is to prove sharp maximal inequalities for the Toeplitz-type operator T_b. As the application, we obtain the L^p-norm inequality, Morrey and Triebel-Lizorkin spaces boundedness for the Toeplitz-type operator T_b.

2. Theorems and Lemmas

We shall prove the following theorems.

Theorem 1. Let T be a singular integral operator as **Definition 3**, $0 < \beta < 1$, $1 < s < \infty$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^{\#}(T_b(f))(\tilde{x}) \leq C||b||_{Lip_\beta} \sum_{k=1}^{m} (M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Theorem 2. Let T be a singular integral operator as **Definition 3**, $0 < \beta < 1$, $1 < s < \infty$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$\begin{aligned} \sup_{Q \ni \tilde{x}} \inf_{c \in R^n} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - c|\, dx \quad &\leqslant \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) \\ &+ \quad M_{\alpha,s}(T^{k,4}(f))(\tilde{x})). \end{aligned}$$

Theorem 3. Let T be a singular integral operator as **Definition 3**, $1 < s < \infty$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^{\#}(T_b(f))(\tilde{x}) \leq C||b||_{BMO} \sum_{k=1}^{m} (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Theorem 4. Let T be a singular integral operator as **Definition 3**, $0 < \beta < 1$, $1 < p < n/(\alpha + \beta)$, $1/q = 1/p - (\alpha + \beta)/n$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are bounded operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^p(R^n)$ to $L^q(R^n)$.

Theorem 5. Let T be a singular integral operator as **Definition 3**, $0 < \beta < 1$, $1 < p < n/(\alpha + \beta)$, $1/q = 1/p - (\alpha + \beta)/n$, $0 < D < 2^n$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are bounded operators on $L^{p,\varphi}(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^{p,\varphi}(R^n)$ to $L^{q,\varphi}(R^n)$.

Theorem 6. Let T be a singular integral operator as **Definition 3**, $0 < \beta < 1$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any

$g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are bounded operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^p(R^n)$ to $\dot{F}_q^{\beta,\infty}(R^n)$.

Theorem 7. Let T be a singular integral operator as **Definition 3**, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are bounded operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^p(R^n)$ to $L^q(R^n)$.

Theorem 8. Let T be a singular integral operator as **Definition 3**, $0 < D < 2^n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are bounded operators on $L^{p,\varphi}(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^{p,\varphi}(R^n)$ to $L^{q,\varphi}(R^n)$.

To prove the theorems, we need the following lemmas.

Lemma 1.(see [1]) Let T be a singular integral operator as **Definition 3**. Then T is bounded on $L^p(R^n)$ for $1 < p < \infty$.

Lemma 2.(see [16]). For $0 < \beta < 1$ and $1 < p < \infty$, we have

$$\begin{aligned} ||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left|\left|\sup_{Q \ni x} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(y) - f_Q| dy\right|\right|_{L^p} \\ &\approx \left|\left|\sup_{Q \ni x} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(y) - c| dy\right|\right|_{L^p}, \end{aligned}$$

where the sup is taken all cubes Q containing $x \in R^n$.

Lemma 3.(see [6]). Let $0 < p < \infty$ and $w \in \cup_{1 \le r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,

$$\int_{R^n} M(f)(x)^p w(x) dx \leqslant C \int_{R^n} M^{\#}(f)(x)^p w(x) dx.$$

Lemma 4.(see [2, 6]). Suppose that $0 < \alpha < n$, $1 \le s < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Then

$$||I_\alpha(f)||_{L^q} \leqslant C||f||_{L^p}$$

and

$$||M_{\alpha,s}(f)||_{L^q} \leqslant C||f||_{L^p}.$$

Lemma 5. Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function f for which the left-hand side is finite,

$$||M(f)||_{L^{p,\varphi}} \leqslant C||M^{\#}(f)||_{L^{p,\varphi}}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(\chi_Q) \in A_1$ for any cube Q by [6]. Noticing that $M(\chi_Q) \le 1$ and $M(\chi_Q)(x) \le d^n/(|x - x_0| - d)^n$ if $x \in Q^c$, by Lemma 3, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\begin{aligned}
&\int_Q M(f)(x)^p dx = \int_{R^n} M(f)(x)^p \chi_Q(x) dx \\
\leqslant\ & \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx \leqslant C \int_{R^n} M^{\#}(f)(x)^p M(\chi_Q)(x) dx \\
\leqslant\ & C\left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q\setminus 2^k Q} M^{\#}(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} dx\right) \\
\leqslant\ & C||M^{\#}(f)||_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} 2^{-kn}\varphi(2^{k+1}d) \\
\leqslant\ & C||M^{\#}(f)||_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d) \\
\leqslant\ & C||M^{\#}(f)||_{L^{p,\varphi}}^p \varphi(d),
\end{aligned}$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M(f)(x)^p dx\right)^{1/p} \le C\left(\frac{1}{\varphi(d)}\int_Q M^{\#}(f)(x)^p dx\right)^{1/p}$$

and

$$||M(f)||_{L^{p,\varphi}} \leqslant C||M^{\#}(f)||_{L^{p,\varphi}}.$$

This finishes the proof.

Lemma 6. Let $0 < \alpha < n$, $0 < D < 2^n$, $1 \leqslant s < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Then

$$||I_\alpha(f)||_{L^{q,\varphi}} \leqslant C||f||_{L^{p,\varphi}}$$

and

$$||M_{\alpha,s}(f)||_{L^{r,\varphi}} \leqslant C||f||_{L^{p,\varphi}}.$$

The proof of the Lemma is similar to that of Lemma 5 by Lemma 4, we omit the details.

3. Proofs of Theorems

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|\, dx \leqslant C||b||_{Lip_\beta} \sum_{k=1}^{m} (M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We write, by $T_1(g) = 0$,

$$\begin{aligned}
T_b(f)(x) &= \sum_{k=1}^{m} T^{k,1} M_b I_\alpha T^{k,2}(f)(x) + \sum_{k=1}^{m} T^{k,3} I_\alpha M_b T^{k,4}(f)(x) \\
&= A_b(x) + B_b(x) = A_{b-b_Q}(x) + B_{b-b_Q}(x),
\end{aligned}$$

where

$$\begin{aligned} A_{b-b_Q}(x) &= \sum_{k=1}^m T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x) + \sum_{k=1}^m T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} I_\alpha T^{k,2}(f)(x) \\ &= A_1(x) + A_2(x) \end{aligned}$$

and

$$\begin{aligned} B_{b-b_Q}(x) &= \sum_{k=1}^m T^{k,3} I_\alpha M_{(b-b_Q)\chi_{2Q}} T^{k,4}(f)(x) + \sum_{k=1}^m T^{k,3} I_\alpha M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,4}(f)(x) \\ &= B_1(x) + B_2(x). \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{|Q|}\int_Q \left|A_{b-b_Q}(f)(x) - A_2(x_0)\right| dx &\leqslant \frac{1}{|Q|}\int_Q |A_1(x)|dx + \frac{1}{|Q|}\int_Q |A_2(x) - A_2(x_0)|dx \\ &= I_1 + I_2 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{|Q|}\int_Q \left|B_{b-b_Q}(f)(x) - B_2(x_0)\right| dx &\leqslant \frac{1}{|Q|}\int_Q |B_1(x)|dx + \frac{1}{|Q|}\int_Q |B_2(x) - B_2(x_0)|dx \\ &= I_3 + I_4. \end{aligned}$$

For I_1, by Hölder's inequality and Lemma 1, we obtain

$$\begin{aligned} & \frac{1}{|Q|}\int_Q |T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|dx \\ \leqslant\ & \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s} \\ \leqslant\ & C|Q|^{-1/s}\left(\int_{R^n} |M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s} \\ \leqslant\ & C|Q|^{-1/s}\left(\int_{2Q} (|b(x) - b_Q||I_\alpha T^{k,2}(f)(x)|)^s dx\right)^{1/s} \\ \leqslant\ & C|Q|^{-1/s}||b||_{Lip_\beta}|2Q|^{\beta/n}|2Q|^{1/s-\beta/n}\left(\frac{1}{|2Q|^{1-s\beta/n}}\int_{2Q} |I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s} \\ \leq\ & C||b||_{Lip_\beta} M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}), \end{aligned}$$

thus

$$\begin{aligned} I_1 &\leq \sum_{k=1}^m \frac{1}{|Q|}\int_{R^n} |T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|dx \\ &\leq C||b||_{Lip_\beta} \sum_{k=1}^m M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}). \end{aligned}$$

For I_2, by [1][14], we know that

$$T(f)(x)=\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}a_{uv}(x)\int_{R^n}\frac{Y_{uv}(x-y)}{|x-y|^n}f(y)dy,$$

where $g_u\leqslant Cu^{n-2}$, $||a_{uv}||_{L^\infty}\leqslant Cu^{-2n}$, $|Y_{uv}(x-y)|\leqslant Cu^{n/2-1}$ and

$$\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right|\leqslant Cu^{n/2}|x-x_0|/|x_0-y|^{n+1}$$

for $|x-y|>2|x_0-x|>0$, we get, for $x\in Q$,

$$\begin{aligned}
&|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x)-T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x_0)|\\
\leqslant\ &\int_{(2Q)^c}|b(y)-b_{2Q}||K(x,x-y)-K(x_0,x_0-y)||I_\alpha T^{k,2}(f)(y)|dy\\
=\ &\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|H_1|I_\alpha T^{k,2}(f)(y)|dy\\
\leqslant\ &C\sum_{j=1}^{\infty}||b||_{Lip_\beta}|2^{j+1}Q|^{\beta/n}\int_{2^jd\le|y-x_0|<2^{j+1}d}|I_\alpha T^{k,2}(f)(y)|H_2dy\\
\le\ &C||b||_{Lip_\beta}\sum_{u=1}^{\infty}u^{-2n}\cdot u^{n/2}\sum_{j=1}^{\infty}|2^{j+1}Q|^{\beta/n}\int_{2^jd\le|y-x_0|<2^{j+1}d}H_3dy\\
\le\ &C||b||_{Lip_\beta}\sum_{j=1}^{\infty}2^{-j}\left(\frac{1}{|2^{j+1}Q|^{1-\beta/n}}\int_{2^{j+1}Q}|I_\alpha T^{k,2}(f)(y)|dy\right)\\
\le\ &C||b||_{Lip_\beta}M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}2^{-j}\\
\le\ &C||b||_{Lip_\beta}M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

where

$$H_1=\left|\frac{\Omega(x,x-y)}{|x-y|^n}-\frac{\Omega(x_0,x_0-y)}{|x_0-y|^n}\right|,H_3=\frac{|x-x_0|}{|x_0-y|^{n+1}}|I_\alpha T^{k,2}(f)(y)|,$$

$$H_2=\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}|a_{uv}(x)|\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right|,$$

thus

$$\begin{aligned}
I_2\ &\leqslant\ \frac{1}{|Q|}\int_Q\sum_{k=1}^{m}|H_4|dx\\
&\leqslant\ C||b||_{Lip_\beta}\sum_{k=1}^{m}M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

where

$$H_4 = T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x) - T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x_0).$$

Similarly, by Lemma 4, for $1/r = 1/s - \alpha/n$,

$$\begin{aligned}
I_3 &\leq \sum_{k=1}^m \left(\frac{1}{|Q|}\int_{R^n} |I_\alpha M_{(b-b_Q)\chi_{2Q}} T^{k,4}(f)(x)|^r dx\right)^{1/r} \\
&\leqslant C\sum_{k=1}^m |Q|^{-1/r}\left(\int_{2Q} (|b(x)-b_Q||T^{k,4}(f)(x)|)^s dx\right)^{1/s} \\
&\leqslant C||b||_{Lip_\beta}\sum_{k=1}^m |Q|^{-1/r}|2Q|^{\beta/n}|2Q|^{1/s-(\beta+\alpha)/n}H_5 \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^m M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x}),
\end{aligned}$$

where

$$H_5 = \left(\frac{1}{|2Q|^{1-s(\beta+\alpha)/n}}\int_{2Q} |T^{k,4}(f)(x)|^s dx\right)^{1/s}.$$

$$\begin{aligned}
I_4 &\leq \sum_{k=1}^m \frac{1}{|Q|}\int_Q\int_{(2Q)^c} |b(y)-b_{2Q}|\left|\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}}\right||T^{k,4}(f)(y)|dydx \\
&\leq C\sum_{k=1}^m\sum_{j=1}^\infty ||b||_{Lip_\beta}|2^{j+1}Q|^{\beta/n}\int_{2^jd\leq|y-x_0|<2^{j+1}d} \frac{d}{|x_0-y|^{n-\alpha+1}}|T^{k,4}(f)(y)|dy \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^m\sum_{j=1}^\infty (2^jd)^\beta d(2^jd)^{-n+\alpha-1}(2^jd)^{n(1-1/s)}(2^jd)^{n/s-\beta-\alpha} \\
&\quad\times\left(\frac{1}{|2^{j+1}Q|^{1-s(\beta+\alpha)/n}}\int_{2^{j+1}Q} |T^{k,4}(f)(y)|^s dy\right)^{1/s} \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^m M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})\sum_{j=1}^\infty 2^{-j} \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^m M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 1.

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q |T_b(f)(x)-C_0|\,dx \leqslant C||b||_{Lip_\beta}\sum_{k=1}^m (M_s(I_\alpha T^{k,2}(f))(\tilde{x})+M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k=1,...,m)$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Similar to the proof of Theorem 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |T_b(f)(x)-A_2(x_0)-B_2(x_0)|\,dx\\
\le\ & \frac{1}{|Q|^{1+\beta/n}}\int_Q |A_1(x)|dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q |A_2(x)-A_2(x_0)|dx\\
+\ & \frac{1}{|Q|^{1+\beta/n}}\int_Q |B_1(x)|dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q |B_2(x)-B_2(x_0)|dx\\
=\ & I_5+I_6+I_7+I_8.
\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get, for $1/r=1/s-\alpha/n$,

$$\begin{aligned}
I_5\ \le\ & |Q|^{-\beta/n}\sum_{k=1}^m\left(\frac{1}{|Q|}\int_{R^n}|T^{k,1}M_{(b-b_Q)\chi_{2Q}}I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
\leqslant\ & C|Q|^{-\beta/n}\sum_{k=1}^m|Q|^{-1/s}\left(\int_{2Q}(|b(x)-b_Q||I_\alpha T^{k,2}(f)(x)|)^s dx\right)^{1/s}\\
\leqslant\ & C|Q|^{-\beta/n}\sum_{k=1}^m|Q|^{-1/s}||b||_{Lip_\beta}|2Q|^{\beta/n}|Q|^{1/s}\left(\frac{1}{|Q|}\int_{2Q}|I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
\le\ & C||b||_{Lip_\beta}\sum_{k=1}^m M_s(I_\alpha T^{k,2}(f))(\tilde{x}),\\
I_6\ \le\ & |Q|^{-\beta/n}\sum_{k=1}^m\frac{1}{|Q|}\int_Q\sum_{j=1}^\infty\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\\
& \times|K(x,x-y)-K(x_0,x_0-y)||I_\alpha T^{k,2}(f)(y)|dydx\\
\le\ & |Q|^{-\beta/n}\sum_{k=1}^m\frac{C}{|Q|}\int_Q\sum_{j=1}^\infty||b||_{Lip_\beta}|2^{j+1}Q|^{\beta/n}\int_{2^jd\le|y-x_0|<2^{j+1}d}\sum_{u=1}^\infty\sum_{v=1}^{g_u}|a_{uv}(x)|\\
& \times\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right||I_\alpha T^{k,2}(f)(y)|dydx\\
\le\ & C||b||_{Lip_\beta}|Q|^{-\beta/n}\sum_{k=1}^m\frac{1}{|Q|}\\
& \times\int_Q\sum_{j=1}^\infty|2^{j+1}Q|^{\beta/n}\int_{2^jd\le|y-x_0|<2^{j+1}d}\frac{|x-x_0|}{|x_0-y|^{n+1}}|I_\alpha T^{k,2}(f)(y)|dydx
\end{aligned}$$

$$
\begin{aligned}
&\leq C||b||_{Lip_\beta} d^{-\beta} \sum_{k=1}^{m} \sum_{j=1}^{\infty} (2^j d)^\beta \frac{d}{(2^j d)^{n+1}} (2^j d)^n \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |I_\alpha T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
&\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_s(I_\alpha T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} 2^{j(\beta-1)} \\
&\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
I_7 &\leq |Q|^{-\beta/n} \sum_{k=1}^{m} \left(\frac{1}{|Q|} \int_{R^n} |I_\alpha M_{(b-b_Q)\chi_{2Q}} T^{k,4}(f)(x)|^r dx \right)^{1/r} \\
&\leqslant C|Q|^{-\beta/n-1/r} \sum_{k=1}^{m} \left(\int_{2Q} (|b(x)-b_Q||T^{k,4}(f)(x)|)^s dx \right)^{1/s} \\
&\leqslant C||b||_{Lip_\beta} \sum_{k=1}^{m} |Q|^{-\beta/n-1/r} |2Q|^{\beta/n} |Q|^{1/s-\alpha/n} \\
&\quad \times \left(\frac{1}{|2Q|^{1-s\alpha/n}} \int_{2Q} |T^{k,4}(f)(x)|^s dx \right)^{1/s} \\
&\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,4}(f))(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
I_8 &\leq |Q|^{-\beta/n-1} \sum_{k=1}^{m} \int_Q \int_{(2Q)^c} |b(y)-b_{2Q}| \\
&\quad \times \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}} \right| |T^{k,4}(f)(y)| dy dx \\
&\leq C|Q|^{-\beta/n} \sum_{k=1}^{m} \sum_{j=1}^{\infty} ||b||_{Lip_\beta} |2^{j+1}Q|^{\beta/n} \\
&\quad \times \int_{2^j d \leq |y-x_0| < 2^{j+1} d} \frac{d}{|x_0-y|^{n-\alpha+1}} |T^{k,4}(f)(y)| dy \\
&\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} \sum_{j=1}^{\infty} d^{-\beta} (2^j d)^\beta d (2^j d)^{-n+\alpha-1} (2^j d)^{n(1-1/s)} (2^j d)^{n/s-\alpha} \\
&\quad \times \left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}} \int_{2^{j+1}Q} |T^{k,4}(f)(y)|^s dy \right)^{1/s} \\
&\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,4}(f))(\tilde{x}) \sum_{j=1}^{\infty} 2^{j(\beta-1)} \\
&\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,4}(f))(\tilde{x}).
\end{aligned}
$$

These complete the proof of Theorem 2.

Proof of Theorem 3. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|\,dx \leqslant C||b||_{BMO}\sum_{k=1}^m (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T_b(f)(x) - A_2(x_0) - B_2(x_0)|\,dx \le \frac{1}{|Q|}\int_Q |A_1(x)|dx \\
&+\frac{1}{|Q|}\int_Q |A_2(x) - A_2(x_0)|dx + \frac{1}{|Q|}\int_Q |B_1(x)|dx + \frac{1}{|Q|}\int_Q |B_2(x) - B_2(x_0)|dx \\
&= I_9 + I_{10} + I_{11} + I_{12}.
\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get, for $1 < r_1 < s$, $1 < p < \min(s, n/\alpha)$ with $1/r_2 = 1/p - \alpha/n$,

$$\begin{aligned}
I_9 &\le \sum_{k=1}^m \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1}M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|^{r_1}dx\right)^{1/r_1} \\
&\leqslant C\sum_{k=1}^m |Q|^{-1/r_1}\left(\int_{2Q} (|b(x) - b_Q||I_\alpha T^{k,2}(f)(x)|)^{r_1}dx\right)^{1/r_1} \\
&\leqslant C\sum_{k=1}^m \left(\frac{1}{|Q|}\int_{2Q} |I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\left(\frac{1}{|Q|}\int_{2Q} |b(x) - b_Q|^{sr_1/(s-r_1)}dx\right)^{(s-r_1)/sr_1} \\
&\le C||b||_{BMO}\sum_{k=1}^m M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
I_{10} &\le \sum_{k=1}^m \frac{1}{|Q|}\int_Q \sum_{j=1}^\infty \int_{2^jd\le|y-x_0|<2^{j+1}d} |b(y) - b_{2Q}||K(x, x-y) - K(x_0, x_0-y)||I_\alpha T^{k,2}(f)(y)|dydx \\
&\le \sum_{k=1}^m \frac{C}{|Q|}\int_Q \sum_{j=1}^\infty \int_{2^jd\le|y-x_0|<2^{j+1}d} |b(y) - b_{2Q}|\sum_{u=1}^\infty\sum_{v=1}^{g_u} |a_{uv}(x)| \\
&\quad \times \left|\frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right| |I_\alpha T^{k,2}(f)(y)|dydx
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{m}\frac{C}{|Q|}\int_Q\sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{|x-x_0|}{|x_0-y|^{n+1}}|I_\alpha T^{k,2}(f)(y)|dydx \\
&\leq C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\frac{d}{(2^jd)^{n+1}}\left(\int_{2^{j+1}Q}|b(y)-b_Q|^{s'}dy\right)^{1/s'} \\
&\quad\times\left(\int_{2^{j+1}Q}|I_\alpha T^{k,2}(f)(y)|^s dy\right)^{1/s}dx \\
&\leq C||b||_{BMO}\sum_{k=1}^{m}\sum_{j=1}^{\infty}j2^{-j}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|I_\alpha T^{k,2}(f)(y)|^s dy\right)^{1/s} \\
&\leq C||b||_{BMO}\sum_{k=1}^{m}M_s(I_\alpha T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}j2^{-j} \\
&\leq C||b||_{BMO}\sum_{k=1}^{m}M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
I_{11} &\leq \sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}|I_\alpha M_{(b-b_Q)\chi_{2Q}}T^{k,4}(f)(x)|^{r_2}dx\right)^{1/r_2} \\
&\leqslant C|Q|^{-1/r_2}\sum_{k=1}^{m}\left(\int_{2Q}(|b(x)-b_Q||T^{k,4}(f)(x)|)^p dx\right)^{1/p} \\
&\leqslant C\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{2Q}|b(x)-b_Q|^{ps/(s-p)}dx\right)^{(s-p)/ps}\left(\frac{1}{|Q|^{1-s\alpha/n}}\int_{2Q}|T^{k,4}(f)(x)|^s dx\right)^{1/s} \\
&\leq C||b||_{BMO}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
I_{12} &\leq |Q|^{-1}\sum_{k=1}^{m}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}|\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right||T^{k,4}(f)(y)|dydx \\
&\leq C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{d}{|x_0-y|^{n-\alpha+1}}|T^{k,4}(f)(y)|dy \\
&\leq C\sum_{k=1}^{m}\sum_{j=1}^{\infty}d(2^jd)^{-n+\alpha-1}(2^jd)^{n(1-1/s)}(2^jd)^{n/s-\alpha}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_Q|^{s'}dy\right)^{1/s'} \\
&\quad\times\left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}}\int_{2^{j+1}Q}|T^{k,4}(f)(y)|^s dy\right)^{1/s} \\
&\leq C||b||_{BMO}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x})\sum_{j=1}^{\infty}j2^{-j}\leq C||b||_{BMO}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x}).
\end{aligned}$$

This completes the proof of Theorem 3.

Proof of Theorem 4. Choose $1 < s < p$ in Theorem 1 and set $1/r = 1/p - \alpha/n$. We have, by Lemmas 3 and 4,

$$\begin{aligned}
&||T_b(f)||_{L^q} \leqslant \|M(T_b(f))\|_{L^q} \leqslant C\|M^{\#}(T_b(f))\|_{L^q} \\
\leqslant\ & C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|M_{\beta,s}(I_\alpha T^{k,2}(f))\|_{L^q} + \|M_{\beta+\alpha,s}(T^{k,4}(f))\|_{L^q}) \\
\leqslant\ & C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|I_\alpha T^{k,2}(f)\|_{L^r} + \|T^{k,4}(f)\|_{L^p}) \\
\leqslant\ & C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|T^{k,2}(f)\|_{L^p} + \|f\|_{L^p}) \leq C||b||_{Lip_\beta} \|f\|_{L^p}.
\end{aligned}$$

This completes the proof of Theorem 4.

Proof of Theorem 5. Choose $1 < s < p$ in Theorem 1 and set $1/r = 1/p - \alpha/n$. We have, by Lemmas 5 and 6,

$$\begin{aligned}
&||T_b(f)||_{L^{q,\varphi}} \leqslant \|M(T_b(f))\|_{L^{q,\varphi}} \leqslant C\|M^{\#}(T_b(f))\|_{L^{q,\varphi}} \\
\leqslant\ & C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|M_{\beta,s}(I_\alpha T^{k,2}(f))\|_{L^{q,\varphi}} + \|M_{\beta+\alpha,s}(T^{k,4}(f))\|_{L^{q,\varphi}}) \\
\leqslant\ & C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|I_\alpha T^{k,2}(f)\|_{L^{r,\varphi}} + \|T^{k,4}(f)\|_{L^{p,\varphi}}) \\
\leqslant\ & C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|T^{k,2}(f)\|_{L^{p,\varphi}} + \|f\|_{L^{p,\varphi}}) \leq C||b||_{Lip_\beta} \|f\|_{L^{p,\varphi}}.
\end{aligned}$$

This completes the proof of Theorem 5.

Proof of Theorem 6. Choose $1 < s < p$ in Theorem 2. We have, by Lemmas 2, 3 and 4,

$$\begin{aligned}
&||T_b(f)||_{\dot{F}_q^{\beta,\infty}} \leqslant C \left\| \sup_{Q \ni x} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(y) - C_0| \, dy \right\|_{L^q} \\
\leqslant\ & C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|M_s(I_\alpha T^{k,2}(f))\|_{L^q} + \|M_{\alpha,s}(T^{k,4}(f))\|_{L^q}) \\
\leqslant\ & C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|I_\alpha T^{k,2}(f)\|_{L^q} + \|T^{k,4}(f)\|_{L^p}) \\
\leqslant\ & C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|T^{k,2}(f)\|_{L^p} + \|f\|_{L^p}) \leqslant C||b||_{Lip_\beta} ||f||_{L^p}.
\end{aligned}$$

This completes the proof of the theorem.

Proof of Theorem 7. Choose $1 < s < p$ in Theorem 3, we have, by Lemmas 3 and 4,

$$\begin{aligned} &||T_b(f)||_{L^q} \leqslant \|M(T_b(f))\|_{L^q} \leqslant C\|M^{\#}(T_b(f))\|_{L^q} \\ \leqslant\;& C||b||_{BMO}\sum_{k=1}^{m}(\|M_s(I_\alpha T^{k,2}(f))\|_{L^q} + \|M_{\alpha,s}(T^{k,4}(f))\|_{L^q}) \\ \leqslant\;& C||b||_{BMO}\sum_{k=1}^{m}(\|I_\alpha T^{k,2}(f)\|_{L^q} + \|T^{k,4}(f)\|_{L^p}) \\ \leqslant\;& C||b||_{BMO}\sum_{k=1}^{m}(\|T^{k,2}(f)\|_{L^p} + \|f\|_{L^p}) \leqslant C||b||_{BMO}\|f\|_{L^p}. \end{aligned}$$

This completes the proof of Theorem 7.

Proof of Theorem 8. Choose $1 < s < p$ in Theorem 3, we have, by Lemmas 5 and 6,

$$\begin{aligned} &||T_b(f)||_{L^{q,\varphi}} \leqslant \|M(T_b(f))\|_{L^{q,\varphi}} \leqslant C\|M^{\#}(T_b(f))\|_{L^{q,\varphi}} \\ \leqslant\;& C||b||_{BMO}\sum_{k=1}^{m}(\|M_s(I_\alpha T^{k,2}(f))\|_{L^{q,\varphi}} + \|M_{\alpha,s}(T^{k,4}(f))\|_{L^{q,\varphi}}) \\ \leqslant\;& C||b||_{BMO}\sum_{k=1}^{m}(\|I_\alpha T^{k,2}(f)\|_{L^{q,\varphi}} + \|T^{k,4}(f)\|_{L^{p,\varphi}}) \\ \leqslant\;& C||b||_{BMO}\sum_{k=1}^{m}(\|T^{k,2}(f)\|_{L^{p,\varphi}} + \|f\|_{L^{p,\varphi}}) \leqslant C||b||_{BMO}\|f\|_{L^{p,\varphi}}. \end{aligned}$$

This completes the proof of Theorem 8.

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