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STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS AND SOME PROPERTIES IN 2-NORMED SPACES

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Abstract. In this study, we introduced the concepts of pointwise and uniform convergence, statistical convergence and statistical Cauchy double sequences of functions in 2-normed space. Also, were studied some properties about these concepts and investigated relationships between them for double sequences of functions in 2-normed spaces.
Keywords: Uniform convergence, Statistical Convergence, Double sequences of Functions, Statistical Cauchy sequence, 2-normed Spaces.

1. Introduction and Background

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [16] and Schoenberg [35]. Gökhan et al. [21] introduced the concepts of pointwise statistical convergence and statistical Cauchy sequence of real-valued functions. Balcerzak et al. [5] studied statistical convergence and ideal convergence for sequence of functions. Duman and Orhan [7] studied μ-statistically convergent function sequences. Gökhan et al. [22] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. Dündar and Altay [8,9] studied the concepts of pointwise and uniformly $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of double sequences of functions and investigated some properties about them. Also, a lot of development have been made about double sequences of functions (see [4,14,20]).

The concept of 2-normed spaces was initially introduced by Gähler [18,19] in the 1960's. Gürdal and Pehlivan [25] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Sharma and Kumar [32] introduced statistical convergence, statistical Cauchy sequence, statistical limit points and statistical cluster points in probabilistic 2-normed space. Statistical convergence and statistical Cauchy sequence

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of functions in 2-normed space were studied by Yegül and Dündar [37]. Sarabadan and Talebi [31] presented various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $\mathcal{I}$-equistatistically convergence and study $\mathcal{I}$-equistatistically convergence of sequences of functions. Futhermore, a lot of development have been made in this area (see [1–3, 6, 15, 23, 24, 26–29, 33, 34]).

2. Definitions and Notations

Now, we recall the concepts of double sequences, density, statistical convergence, 2-normed space and some fundamental definitions and notations (See [5, 10–13, 17, 19–21, 23–25, 30–32, 36]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.

(ii) $\|x, y\| = \|y, x\|$.

(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$.

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \quad x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.

Let $(X, \|.,.\|)$ be a finite dimensional 2-normed space and $u = \{u_1, \cdots, u_d\}$ be a basis of X. We can define the norm $\|.\|_\infty$ on X by $\|x\|_\infty = \max\{\|x, u_i\| : i = 1, ..., d\}$.

Associated to the derived norm $\|.\|_\infty$, we can define the (closed) balls $B_u(x, \varepsilon)$ centered at x having radius ε by $B_u(x, \varepsilon) = \{y : \|x - y\|_\infty \leq \varepsilon\}$, where $\|x - y\|_\infty = \max\{\|x - y, u_j\|, j = 1, ..., d\}$.

Throughout the paper, we let X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f, g be two functions from X to Y.

The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to be convergent to f if $f_n(x) \to f(x)(\|.,.\|_Y)$ for each $x \in X$. We write $f_n \to f(\|.,.\|_Y)$. This can be expressed by the formula $(\forall y \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\|f_n(x) - f(x), y\| < \varepsilon$.

If $K \subseteq \mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. The natural density of K is given by $\delta(K) = \lim\limits_{n\to\infty} \frac{1}{n}|K_n|$, if it exists.

The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be (pointwise) statistical convergent to f, if for every $\varepsilon > 0$, $\lim\limits_{n\to\infty} \frac{1}{n}\big|\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\}\big| = 0$, for each $x \in X$ and each nonzero $z \in Y$. It means that for each $x \in X$ and each nonzero $z \in Y$, $\|f_n(x) - f(x), z\| < \varepsilon$, a.a. (almost all) n. In this case, we write

$$st - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \quad or \quad f_n \to_{st} f(\|.,.\|_Y).$$

The sequence of functions $\{f_n\}$ is said to be statistically Cauchy sequence, if for every $\varepsilon > 0$ and each nonzero $z \in Y$, there exists a number $k = k(\varepsilon, z)$ such that $\delta(\{n \in \mathbb{N} : \|f_n(x) - f_k(x), z\| \geq \varepsilon\}) = 0$, for each $x \in X$, i.e., $\|f_n(x) - f_k(x), z\| < \varepsilon$, a.a. n.

Let X be a 2-normed space. A double sequence (x_{mn}) in X is said to be convergent to $L \in X$, if for every $z \in X$, $\lim\limits_{m,n\to\infty} \|x_{mn} - L, z\| = 0$. In this case, we write $\lim\limits_{n,m\to\infty} x_{mn} = L$ and call L the limit of (x_{mn}).

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{mn} be the number of $(j, k) \in K$ such that $j \leq m$, $k \leq n$. That is, $K_{mn} = |\{(j, k) : j \leq m, k \leq n\}|$, where $|A|$ denotes the number of elements in A. If the double sequence $\{\frac{K_{mn}}{mn}\}$ has a limit then we say that K has double natural density and is denoted by $d_2(K) = \lim\limits_{m,n\to\infty} \frac{K_{mn}}{mn}$.

A double sequence $x = (x_{mn})$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$, if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

Let $\{x_{mn}\}$ be a double sequence in 2-normed space $(X, \|.,.\|)$. The double sequence (x_{mn}) is said to be statistically convergent to L, if for every $\varepsilon > 0$, the set $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L, z\| \geq \varepsilon\}$ has natural density zero for each nonzero z in X, in other words (x_{mn}) statistically converges to L in 2-normed space $(X, \|.,.\|)$ if $\lim\limits_{m,n\to\infty} \frac{1}{mn}\big|\{(m, n) : \|x_{mn} - L, z\| \geq \varepsilon\}\big| = 0$, for each nonzero z in X. It means that for each $z \in X$, $\|x_{mn} - L, z\| < \varepsilon$, $a.a.$ (m, n). In this case, we write $st - \lim\limits_{m,n\to\infty} \|x_{mn}, z\| = \|L, z\|$.

A double sequence (x_{mn}) in 2-normed space $(X, \|.,.\|)$ is said to be statistically Cauchy sequence in X, if for every $\varepsilon > 0$ and every nonzero $z \in X$ there exist two number $M = M(\varepsilon, z)$ and $N = N(\varepsilon, z)$ such that $d_2\big(\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{MN}, z\| \geq \varepsilon\}\big) = 0$, i.e., for each nonzero $z \in X$, $\|x_{mn} - x_{MN}, z\| < \varepsilon$, a.a. (m,n).

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise convergent to f on a set $S \subset \mathbb{R}$, if for each point $x \in S$ and for each $\varepsilon > 0$, there exists a positive integer $N = N(x, \varepsilon)$ such that $|f_{mn}(x) - f(x)| < \varepsilon$, for all $m, n > N$. In this case we write $\lim\limits_{m,n\to\infty} f_{mn}(x) = f(x)$ or $f_{mn} \to f$, on S.

A double sequence of functions $\{f_{mn}\}$ is said to be uniformly convergent to f on a set $S \subset \mathbb{R}$, if for each $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that for all $m,n > N$ implies $|f_{mn}(x) - f(x)| < \varepsilon$, for all $x \in S$. In this case we write $f_{mn} \rightrightarrows f$, on S.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise statistically convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{i,j\to\infty} \frac{1}{ij} |\{(m,n), m \leq i \ \ and \ \ n \leq j : |f_{mn}(x) - f(x)| \geq \varepsilon\}| = 0,$$

for each (fixed) $x \in S$, i.e., for each (fixed) $x \in S$, $|f_{mn}(x) - f(x)| < \varepsilon$, *a.a.* (m,n). In this case, we write $st - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$ *or* $f_{mn} \to_{st} f$, on S.

A double sequence of functions $\{f_{mn}\}$ is said to be uniformly statistically convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{i,j\to\infty} \frac{1}{ij} |\{(m,n), m \leq i \ \ and \ \ n \leq j : |f_{mn}(x) - f(x)| \geq \varepsilon\}| = 0,$$

for all $x \in S$, i.e., for all $x \in S$, $|f_{mn}(x) - f(x)| < \varepsilon$, *a.a.* (m,n). In this case we write $f_{mn} \rightrightarrows f$, on S.

Let $\{f_{mn}\}$ be a double sequence of functions defined on a set S. A double sequence $\{f_{mn}\}$ is said to be statistically Cauchy if for every $\varepsilon > 0$, there exist $N(= N(\varepsilon))$ and $M(= M(\varepsilon))$ such that $|f_{mn}(x) - f_{MN}(x)| < \varepsilon$ a.a. (m,n) and for each (fixed) $x \in S$, i.e.,

$$\lim_{i,j\to\infty} \frac{1}{ij} |\{(m,n), m \leq i \ \ and \ \ n \leq j : |f_{mn}(x) - f_{MN}(x)| \geq \varepsilon\}| = 0$$

for each (fixed)$x \in S$

Lemma 2.1. *[9] Let f and f_{mn}, $m,n = 1,2,...$, be continuous functions on $D = [a,b] \subset \mathbb{R}$. Then $f_{mn} \rightrightarrows f$ on D if and only if $\lim_{m,n\to\infty} c_{mn} = 0$, where $c_{mn} = \max_{x\in D} |f_{mn}(x) - f(x)|$.*

3. Main Results

In this paper, we study concepts of convergence, statistical convergence and statistical Cauchy sequence of double sequences of functions and investigate some properties and relationships between them in 2-normed spaces.

Throughout the paper, we let X and Y be two 2-normed spaces, $\{f_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ and $\{g_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ be two double sequences of functions, f and g be two functions from X to Y.

Definition 3.1. A double sequence $\{f_{mn}\}$ is said to be pointwise convergent to f if, for each point $x \in X$ and for each $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(x,\varepsilon)$ such that for all $m,n \geq k_0$ implies $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for every $z \in Y$. In this case, we write $f_{mn} \to f(\|.,.\|_Y)$.

Definition 3.2. A double sequence $\{f_{mn}\}$ is said to be uniformly convergent to f, if for each $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(\varepsilon)$ such that for all $m, n > k_0$ implies $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for all $x \in X$ and for every $z \in Y$. In this case, we write $f_{mn} \rightrightarrows f(\|.,.\|_Y)$.

Theorem 3.1. *Let D be a compact subset of X and f and f_{mn}, $(m, n = 1, 2, ...)$, be continuous functions on D. Then,*

$$f_{mn} \rightrightarrows f(\|.,.\|_Y)$$

on D if and only if

$$\lim_{m,n\to\infty} c_{mn} = 0,$$

where $c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\|$.

Proof. Suppose that $f_{mn} \rightrightarrows f(\|.,.\|_Y)$ on D. Since f and f_{mn} are continuous functions on D, so $(f_{mn}(x) - f(x))$ is continuous on D, for each $(m, n) \in \mathbb{N} \times \mathbb{N}$. Since $f_{mn} \rightrightarrows f(\|.,.\|_Y)$ on D then, for each $\varepsilon > 0$, there is a positive integer $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $m, n > k_0$ implies

$$\|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}$$

for all $x \in D$ and every $z \in Y$. Thus, when $m, n > k_0$ we have

$$c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2} < \varepsilon.$$

This implies

$$\lim_{m,n\to\infty} c_{mn} = 0.$$

Now, suppose that

$$\lim_{m,n\to\infty} c_{mn} = 0.$$

Then, for each $\varepsilon > 0$, there is a positive integer $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$0 \le c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\| < \varepsilon,$$

for $m, n > k_0$ and every $z \in Y$. This implies that $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for all $x \in D$, every $z \in Y$ and $m, n > k_0$. Hence, we have

$$f_{mn} \rightrightarrows f(\|.,.\|_Y),$$

for all $x \in D$ and every $z \in Y$. □

Definition 3.3. A double sequence $\{f_{mn}\}$ is said to be (pointwise) statistical convergent to f, if for every $\varepsilon > 0$,

$$\lim_{i,j\to\infty} \frac{1}{ij}\big|\{(m,n), m \le i, n \le j : \|f_{mn}(x) - f(x), z\| \ge \varepsilon\}\big| = 0,$$

for each (fixed) $x \in X$ and each nonzero $z \in Y$. It means that for each (fixed) $x \in X$ and each nonzero $z \in Y$,

$$\|f_{mn}(x) - f(x), z\| < \varepsilon, \quad a.a. \ (m, n).$$

In this case, we write

$$st - \lim_{m,n\to\infty} \|f_{mn}(x) - z\| = \|f(x), z\| \quad or \quad f_{mn} \longrightarrow_{st} f(\|.,.\|_Y).$$

Remark 3.1. $\{f_{mn}\}$ is any double sequence of functions and f is any function from X to Y, then set

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon, \ for \ each \ x \in X \ and \ each \ z \in Y\} = \emptyset,$$

since if $z = \overrightarrow{0}$ (0 vektor), $\|f_{mn}(x) - f(x), z\| = 0 \ngeq \varepsilon$ so the above set is empty.

Theorem 3.2. *If for each $x \in X$ and each nonzero $z \in Y$,*

$$st - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad and \quad st - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|g(x), z\|$$

then, for each $x \in X$ and each nonzero $z \in Y$

$$\|f_{mn}(x), z\| = \|g_{mn}(x), z\|$$

(i.e., $f = g$).

Proof. Assume $f \neq g$. Then, $f - g \neq \overrightarrow{0}$, so there exists a $z \in Y$ such that f, g and z are linearly independent (such a z exists since $d \geq 2$). Therefore, for each $x \in X$ and each nonzero $z \in Y$,

$$\|f(x) - g(x), z\| = 2\varepsilon, \quad with \quad \varepsilon > 0.$$

Now, for each $x \in X$ and each nonzero $z \in Y$, we get

$$\begin{aligned} 2\varepsilon = \|f(x) - g(x), z\| &= \|(f(x) - f_{mn}(x)) + (f_{mn}(x) - g(x)), z\| \\ &\leq \|f_{mn}(x) - g(x), z\| + \|f_{mn}(x) - f(x), z\| \end{aligned}$$

and so

$$\{(m, n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x)-g(x), z\| < \varepsilon\} \subseteq \{(m, n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x)-f(x), z\| \geq \varepsilon\}.$$

But, for each $x \in X$ and each nonzero $z \in Y$,

$$d_2 (\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - g(x), z\| < \varepsilon\}) = 0,$$

then contradicting the fact that $f_{mn} \longrightarrow_{st} g(\|.,.\|_Y)$. □

Theorem 3.3. *If $\{g_{mn}\}$ is a convergent sequence of double sequences of functions such that $f_{mn} = g_{mn}$, a.a. (m,n) then, $\{f_{mn}\}$ is statistically convergent.*

Proof. Suppose that for each $x \in X$ and each nonzero $z \in Y$,

$d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \neq g_{mn}(x)\}) = 0$ *and* $\lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\|$,

then for every $\varepsilon > 0$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}$$

$$\subseteq \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon\}$$

$$\cup\{(m,n) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \neq g_{mn}(x)\}.$$

Therefore,

$$\begin{aligned} d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}) \\ \leq d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon) \\ +d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \neq g_{mn}\}). \end{aligned} \tag{3.1}$$

Since $\lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$, the set $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon\}$ contains finite number of integers and so

$$d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon\}) = 0.$$

Using inequality (3.1) we get for every $\varepsilon > 0$

$$d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}) = 0,$$

for each $x \in X$ and each nonzero $z \in Y$ and so consequently

$$st - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

□

Theorem 3.4. *If* $st - \lim \|f_{mn}(x), z\| = \|f(x), z\|$ *for each* $x \in X$ *and each nonzero* $z \in Y$*, then* $\{f_{mn}\}$ *has a subsequence of function* $\{f_{m_i n_i}\}$ *such that*

$$\lim_{i\to\infty} \|f_{m_i n_i}(x), z\| = \|f(x), z\|$$

for each $x \in X$ *and each nonzero* $z \in Y$.

Proof. Proof of this Theorem is as an immediate consequence of Theorem 3.3. □

Theorem 3.5. *Let* $\alpha \in \mathbb{R}$*. If for each* $x \in X$ *and each nonzero* $z \in Y$*,*

$$st - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad \text{and} \quad st - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|g(x), z\|,$$

then

(i) $st - \lim_{m,n\to\infty} \|f_{mn}(x) + g_{mn}(x), z\| = \|f(x) + g(x), z\|$ *and*

(ii) $st - \lim_{m,n\to\infty} \|\alpha f_{mn}(x), z\| = \|\alpha f(x), z\|$.

Proof. **(i)** Suppose that

$$st-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\| \quad and \quad st-\lim_{m,n\to\infty}\|g_{mn}(x),z\|=\|g(x),z\|$$

for each $x\in X$ and each nonzero $z\in Y$. Then, $\delta(K_1)=0$ and $\delta(K_2)=0$ where

$$K_1=K_1(\varepsilon,z):\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|\geq\frac{\varepsilon}{2}\right\}$$

and

$$K_2=K_2(\varepsilon,z):\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|g_{mn}(x)-g(x),z\|\geq\frac{\varepsilon}{2}\right\}$$

for every $\varepsilon>0$, each $x\in X$ and each nonzero $z\in Y$. Let

$$K=K(\varepsilon,z)=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|(f_{mn}(x)+g_{mn}(x))-(f(x)+g(x)),z\|\geq\varepsilon\}.$$

To prove that $\delta(K)=0$, it suffices to show that $K\subset K_1\cup K_2$. Let $(m_0,n_0)\in K$ then, for each $x\in X$ and each nonzero $z\in Y$,

$$\|(f_{m_0n_0}(x)+g_{m_0n_0}(x))-(f(x)+g(x)),z\|\geq\varepsilon. \tag{3.2}$$

Suppose to the contrary, that $(m_0,n_0)\notin K_1\cup K_2$. Then, $(m_0,n_0)\notin K_1$ and $(m_0,n_0)\notin K_2$. If $(m_0,n_0)\notin K_1$ and $(m_0,n_0)\notin K_2$ then, for each $x\in X$ and each nonzero $z\in Y$,

$$\|f_{m_0n_0}(x)-f(x),z\|<\frac{\varepsilon}{2} \quad and \quad \|g_{m_0n_0}(x)-g(x),z\|<\frac{\varepsilon}{2}.$$

Then, we get

$$\begin{aligned}\|(f_{m_0n_0}(x)&+g_{m_0n_0}(x))-(f(x)+g(x)),z\|\\&\leq\|f_{m_0n_0}(x)-f(x),z\|+\|g_{m_0n_0}(x)-g(x),z\|\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\&=\varepsilon,\end{aligned}$$

for each $x\in X$ and each nonzero $z\in Y$, which contradicts (3.2). Hence, $(m_0,n_0)\in K_1\cup K_2$ and so $K\subset K_1\cup K_2$.

(ii) Let $\alpha\in\mathbb{R}$ $(\alpha\neq 0)$ and for each $x\in X$ and each nonzero $z\in Y$,

$$st-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|.$$

Then, we get

$$d_2\left(\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|\geq\frac{\varepsilon}{|\alpha|}\right\}\right)=0.$$

Therefore, for each $x\in X$ and each nonzero $z\in Y$, we have

$$\begin{aligned}\{(m,n)\in\mathbb{N}\times\mathbb{N}&:\|\alpha f_{mn}(x)-\alpha f(x),z\|\geq\varepsilon\}\\&=\{(m,n)\in\mathbb{N}\times\mathbb{N}:|\alpha|\|f_{mn}(x)-f(x),z\|\geq\varepsilon\}\\&=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|\geq\frac{\varepsilon}{|\alpha|}\right\}.\end{aligned}$$

Hence, density of the right hand side of above equality equals 0. Therefore, for each $x \in X$ and each nonzero $z \in Y$, we have

$$st - \lim_{m,n\to\infty} \|\alpha f_{mn}(x), z\| = \|\alpha f(x), z\|.$$

□

Theorem 3.6. *A double sequence of functions $\{f_{mn}\}$ is pointwise statistically convergent to a function f if and only if there exists a subset $K_x = \{(m,n)\} \subseteq \mathbb{N}\times\mathbb{N}$, $m,n = 1,2,...$ for each (fixed) $x \in X$ $d_2(K_x) = 1$ and $\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ for each (fixed) $x \in X$ and each nonzero $z \in Y$.*

Proof. Let $st_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$. For $r = 1,2,...$ put

$$K_{r,x} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x), z\| \geq \frac{1}{r}\}$$

and

$$M_{r,x} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x), z\| < \frac{1}{r}\}$$

for each (fixed) $x \in X$ and each nonzero $z \in Y$. Then, $d_2(K_{r,x}) = 0$ and

$$M_{1,x} \supset M_{2,x} \supset ... \supset M_{i,x} \supset M_{i+1,x} \supset ... \tag{3.3}$$

and

$$d_2(M_{r,x}) = 1, \quad r = 1,2,... \tag{3.4}$$

for each (fixed) $x \in X$ and each nonzero $z \in Y$.

Now, we have to show that for $(m,n) \in M_{r,x}$, $\{f_{mn}\}$ is convergent to f. Suppose that $\{f_{mn}\}$ is not convergent to f. Therefore, there is $\varepsilon > 0$ such that

$$\|f_{mn}(x), z\| = \|f(x), z\| \geq \varepsilon$$

for infinitely many terms and some $x \in X$ and each nonzero $z \in Y$. Let

$$M_{\varepsilon,x} = \{(m,n) : \|f_{mn}(x) - f(x), z\| < \varepsilon\}$$

and $\varepsilon > \frac{1}{r}$ $(r = 1,2,...)$. Then, $d_2(M_{\varepsilon,x}) = 0$ and by (3.3) $M_{r,x} \subset (M_{\varepsilon,x})$. Hence, $d_2(M_{r,x}) = 0$ which contradicts (3.4). Therefore, $\{f_{mn}\}$ is convergent to f.

Conversely, suppose that there exists a subset $K_x = \{(m,n)\} \subseteq \mathbb{N}\times\mathbb{N}$ for each (fixed) $x \in X$ and each nonzero $z \in Y$ such that $d_2(K_x) = 1$ and $\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$, i.e., there exist an $N(x,\varepsilon)$ such that for each (fixed) $x \in X$, each nonzero $z \in Y$ and each $\varepsilon > 0$, $m,n \geq N$ implies $\|f_{mn}(x), z\| = \|f(x), z\| < \varepsilon$. Now,

$$K_{\varepsilon,x} = \{(m,n) : \|f_{mn}(x), z\| \geq \varepsilon\} \subseteq \mathbb{N}\times\mathbb{N} - \{(m_{N+1}, n_{N+1}), (m_{N+2}, n_{N+2}), ...\}$$

for each (fixed) $x \in X$ and each nonzero $z \in Y$. Therefore, $d_2(K_{\varepsilon,x}) \leq 1 - 1 = 0$ for each (fixed) $x \in X$ and each nonzero $z \in Y$. Hence, $\{f_{mn}\}$ is pointwise statistically convergent to f. □

Definition 3.4. A double sequence of functions $\{f_{mn}\}$ is said to uniformly statistically convergent to f, if for every $\varepsilon > 0$ and for each nonzero $z \in Y$,

$$\lim_{i,j\to\infty} \frac{1}{ij} |\{(m,n), m \leq i, n \leq j : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}| = 0,$$

for all $x \in X$. That is, for all $x \in X$ and for each nonzero $z \in Y$

$$\|f_{mn}(x) - f(x), z\| < \varepsilon, \quad a.a \ (m,n). \tag{3.5}$$

In this case, we write $f_{mn} \rightrightarrows_{st} f(\|.,.\|_Y)$.

Theorem 3.7. *Let D be a compact subset of X and f and $\{f_{mn}\}$, $m,n = 1,2,...$ be continuous functions on D. Then,*

$$f_{mn} \rightrightarrows_{st} f(\|.,.\|_Y)$$

on D if and only if

$$st_2 - \lim_{m,n\to\infty} \|c_{mn}(x), z\| = 0,$$

where $c_{mn} = \max_{x\in S} \|f_{mn}(x) - f(x), z\|$.

Proof. Suppose that $\{f_{mn}\}$ uniformly statistically convergent to f on D. Since f and $\{f_{mn}\}$ are continuous functions on D, so $(f_{mn}(x) - f(x))$ is continuous on D, for each $m,n \in \mathbb{N}$. By statistically convergence for $\varepsilon > 0$

$$d_2(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}) = 0,$$

for each $x \in D$ and for each nonzero $z \in Y$. Hence, for $\varepsilon > 0$ it is clear that

$$c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\| \geq \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2}$$

for each $x \in D$ and for each nonzero $z \in Y$. Thus we have

$$st - \lim_{m,n\to\infty} c_{mn} = 0.$$

Now, suppose that $st - \lim_{m,n\to\infty} c_{mn} = 0$. We let following set

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \max_{x\in D} \|f_{mn}(x) - f(x), z\| \geq \varepsilon\},$$

for $\varepsilon > 0$ and for each nonzero $z \in Y$. Then, by hypothesis we have $d_2(A(\varepsilon)) = 0$. Since for $\varepsilon > 0$

$$\max_{x\in D} \|f_{mn}(x) - f(x), z\| \geq \|f_{mn}(x) - f(x), z\| \geq \varepsilon$$

we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \subset A(\varepsilon)$$

and so

$$d_2(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}) = 0,$$

for each $x \in D$ and for each nonzero $z \in Y$. This proves the theorem. □

Now, we can give the relations between well-known convergence models and our studied models as the following result.

Corollary 3.1. ***(i)*** $f_{mn} \rightrightarrows f(\|.,.\|_Y) \Rightarrow f_{mn} \longrightarrow f(\|.,.\|_Y) \Rightarrow f_{mn} \longrightarrow_{st} f(\|.,.\|_Y)$.
(ii)$f_{mn} \rightrightarrows f(\|.,.\|_Y) \Rightarrow f_{mn} \rightrightarrows_{st} f(\|.,.\|_Y) \Rightarrow f_{mn} \longrightarrow_{st} f(\|.,.\|_Y)$.

Now, we give the concept of statistical Cauchy sequence and investigate relationships between statistical Cauchy sequence and statistical convergence of double sequences of functions in 2-normed space.

Definition 3.5. The double sequences of functions $\{f_{mn}\}$ is said to be statistically Cauchy sequence, if for every $\varepsilon > 0$ and each nonzero $z \in Y$, there exist two numbers $k = k(\varepsilon, z)$, $t = t(\varepsilon, z)$ such that

$$d_2(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{kt}(x), z\| \geq \varepsilon\}) = 0, \text{ for each (fixed) } x \in X,$$

i.e., for each nonzero $z \in Y$,

$$\|f_{nm}(x) - f_{kt}(x), z\| < \varepsilon, \quad a.a. \ (m,n).$$

Theorem 3.8. *Let $\{f_{mn}\}$ be a statistically Cauchy sequence of double sequence of functions in a finite dimensional 2-normed space $(X, \|.,.\|)$. Then, there exists a convergent sequence of double sequences of functions $\{g_{mn}\}$ in $(X, \|.,.\|)$ such that $f_{mn} = g_{mn}$, for a.a. (m,n).*

Proof. First note that $\{f_{mn}\}$ is a statistically Cauchy sequence of functions in $(X, \|.\|_\infty)$. Choose a natural number $k(1)$ and $j(1)$ such that the closed ball $B_u^1 = B_u(f_{k(1)j(1)}(x), 1)$ contains $f_{mn}(x)$ for a.a. (m,n) and for each $x \in X$. Then, choose a natural number $k(2)$ and $j(2)$ such that the closed ball $B_2 = B_u(f_{k(2)j(2)}(x), \frac{1}{2})$ contains $f_{mn}(x)$ for a.a. (m,n) and for each $x \in X$. Note that $B_u^2 = B_u^1 \cap B_2$ also contains $f_{mn}(x)$ for a.a. (m,n) and for each $x \in X$. Thus, by continuing of this process, we can obtain a sequence $\{B_u^r\}_{r \geq 1}$ of nested closed balls such that diam $(B_u^r) \leq \frac{1}{2^r}$. Therefore,

$$\bigcap_{r=1}^{\infty} B_u^r = \{h(x)\},$$

where h is a function from X to Y. Since each B_u^r contains $f_{mn}(x)$ for a.a. (m,n) and for each $x \in X$, we can choose a sequence of strictly increasing natural numbers $\{S_r\}_{r \geq 1}$ such that for each $x \in X$,

$$\frac{1}{mn}|\{(m,n) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \notin B_u^r\}| < \frac{1}{r}, \ \ if \ \ m,n > S_r.$$

Put $T_r = \{(m,n) \in \mathbb{N} \times \mathbb{N} : m,n > S_r, \ \ f_{mn}(x) \notin B_u^r\}$ for each $x \in X$, for all $r \geq 1$ and $R = \bigcup_{r=1}^{\infty} R_r$. Now, for each $x \in X$, define the sequence of functions $\{g_{mn}\}$ as following

$$g_{mn}(x) = \begin{cases} h(x) & , \quad \text{if} \ \ (m,n) \in R \times R \\ f_{mn}(x) & , \quad \text{otherwise.} \end{cases}$$

Note that, $\lim_{m,n\to\infty} g_{mn}(x) = h(x)$, for each $x \in X$. In fact, for each $\varepsilon > 0$ and for each $x \in X$, choose a natural number m such that $\varepsilon > \frac{1}{r} > 0$. Then, for each $m, n > S_r$ and for each $x \in X$, $g_{mn}(x) = h(x)$ or $g_{mn}(x) = f_{mn}(x) \in B_u^r$ and so in each case

$$\|g_{mn}(x) - h(x)\|_\infty \leq diam(B_u^r) \leq \frac{1}{2^{r-1}}.$$

Since, for each $x \in X$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : g_{mn}(x) \neq f_n(x)\} \subseteq \{(m,n) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \notin B_u^r\},$$

we have

$$\begin{aligned}\frac{1}{mn}|\{(m,n) \in \mathbb{N}\times\mathbb{N} : g_{mn}(x) \neq f_{mn}(x)\}| \\ &\leq \frac{1}{mn}|\{(n,m) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \notin B_u^r\}| \\ &< \frac{1}{r},\end{aligned}$$

and so

$$d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : g_{mn}(x) \neq f_{mn}(x)\}) = 0.$$

Thus, $g_{mn}(x) = f_{mn}(x)$ for a.a. m,n and for each $x \in X$ in $(X, \|.\|_\infty)$. Suppose that $\{u_1, ..., u_d\}$ is a basis for $(X, \|.,.\|)$. Since, for each $x \in X$,

$$\lim_{m,n\to\infty} \|g_{mn}(x) - h(x)\|_\infty = 0 \quad and \quad \|g_{mn}(x) - h(x), u_i\| \leq \|g_{mn}(x) - h(x)\|_\infty$$

for all $1 \leq i \leq d$, then we have

$$\lim_{m,n\to\infty} \|g_{mn}(x) - h(x), z\|_\infty = 0,$$

for each $x \in X$ and each nonzero $z \in X$. It completes the proof. □

Theorem 3.9. *The sequence $\{f_{mn}\}$ is statistically convergent if and only if $\{f_{mn}\}$ is a statistically Cauchy sequence of double sequence of functions.*

Proof. Assume that f be function from X to Y and $st - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$ and $\varepsilon > 0$. Then, for each $x \in X$ and each nonzero $z \in Y$, we have

$$\|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}, \quad a.a. \quad (m,n).$$

If $k = k(\varepsilon, z)$ and $t = t(\varepsilon, z)$ are chosen so that for each $x \in X$ and each nonzero $z \in Y$,

$$\|f_{kt}(x) - f(x), z\| < \frac{\varepsilon}{2},$$

and so we have

$$\begin{aligned}\|f_{mn}(x)-f_{kt}(x),z\| &\leq \|f_{mn}(x)-f(x),z\|+\|f(x)-f_{kt}(x),z\| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon, \quad a.a. \ (m,n).\end{aligned}$$

Hence, $\{f_{mn}\}$ is statistically Cauchy sequence of double sequence of functions.

Now, assume that $\{f_{mn}\}$ is statistically Cauchy sequence of double sequence of function. By Theorem 3.8, there exists a convergent sequence $\{g_{mn}\}$ from X to Y such that $f_{mn}=g_{mn}$ for a.a. (m,n). By Theorem 3.3, we have

$$st-\lim\|f_{mn}(x),z\|=\|f(x),z\|,$$

for each $x\in X$ and each nonzero $z\in Y$. □

Theorem 3.10. *Let $\{f_{mn}\}$ be a double sequence of functions. The following statements are equivalent*

(i) $\{f_{mn}\}$ is (pointwise) statistically convergent to $f(x)$,

(ii) $\{f_{mn}\}$ is statistically Cauchy,

(iii) There exisits a subsequence $\{g_{mn}\}$ of $\{f_{mn}\}$ such that $\lim_{m,n\to\infty}\|g_{mn}(x),z\|=\|f(x),z\|$.

Proof. Proof of this Theorem is as an immediate consequence of Theorem 3.6 and Theorem 3.9. □

Definition 3.6. Let D be a compact subset of X and $\{f_{mn}\}$ be a double sequence of functions on D. $\{f_{mn}\}$ is said to be statistically uniform Cauchy if for every $\varepsilon>0$ and each nonzero $z\in Y$, there exists $k=k(\varepsilon,z)$, $t=t(\varepsilon,z)$ such that

$$d_2(\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f_{kt}(x),z\|\geq\varepsilon\})=0$$

for all $x\in X$.

Theorem 3.11. *Let D be a compact subset of X and $\{f_{mn}\}$, be a sequence of bounded functions on D. Then, $\{f_{mn}\}$ is uniformly statistically convergent if and only if it is uniformly statistically Cauchy on D.*

Proof. Proof of this theorem is similar the Theorem 3.9. So, we omit it. □

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