FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 5 (2018), 721–738
https://doi.org/10.22190/FUMI1805721S

SEQUENCE SPACES OVER n-NORMED SPACES DEFINED BY MUSIELAK-ORLICZ FUNCTION OF ORDER (α, β)

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Abstract. In the present paper, we introduce sequence spaces over n-normed spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$ of order (α, β). We examine some topological properties and prove some inclusion relations between the resulting sequence spaces.

Keywords: Musielak-Orlicz function; lacunary sequence; n-normed spaces; statistical convergence; paranorm space

1. Introduction and preliminaries

Mursaleen and Noman [29] introduced the notion of λ-convergent and λ-bounded sequences as follows : Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\Lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [29] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) \right) = 0.$$

Received September 23, 2018; accepted December 19, 2018
2010 *Mathematics Subject Classification.* Primary 40A05; Secondary 40C05, 46A45

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m |\frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a)| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

The concept of 2-normed spaces was initially developed by Gähler [14] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [22]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, if we take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then, the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} given by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Let w be the space of all real or complex sequences $x = (x_k)$. Lindenstrauss and Tzafriri [20] used the idea of Orlicz function to define the following sequence space

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\},$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [20] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [33]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty}(M_k)(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm.

For some other recent works related to sequence spaces, we refer the interested reader to [4, 9, 16, 17, 18, 19, 21, 23, 24, 27, 30, 31, 32, 34, 35, 44] and reference therein.

The notion of statistical convergence was introduced by Fast [10]. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory (see [1, 2, 3, 5, 8, 12, 15, 26, 28, 36, 37]). In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. In the recent past, Çolak [6] introduced the concept of statistical convergence order α (also see [7, 38]).

By a lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience.

$$N_\theta = \Big\{ x \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0, \text{ for some } \quad l \Big\}.$$

The notion of lacunary statistically convergent sequences of order (α, β) was first defined by Şengül [40] and then studied in [41, 42, 43, 25]. Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \le \beta \le 1$ be given. We say that the sequence $x = (x_k)$ is $S_\alpha^\beta(\theta)$-statistically convergent(or lacunary statistically convergent sequences of order (α, β)) if there is a real number L such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \ge \epsilon\}|^\beta = 0,$$

where $I_r = (k_{r-1}, k_r]$ and h_r^α denotes the αth power $(h_r)^\alpha$ of h_r, that is $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, \cdots, h_r^\alpha, \cdots)$ and $|\{k \le n : k \subset E\}|^\beta$ denotes the βth power of number of elements of E not exceeding n. In the present case this convergence is indicated by $S_\alpha^\beta(\theta) - \lim x_k = L$. $S_\alpha^\beta(\theta)$ will denote the set of all $S_\alpha^\beta(\theta)$-statistically convergent sequences. If $\theta = (2^r)$, then we will write S_α^β (see [39]). If $\alpha = \beta = 1$ and $\theta = (2^r)$, then we obtain the notion of statistical convergence. The choice of $\beta = 1$ and $\theta = (2^r)$ gives the notion of statistical convergence of order α due to Çolak [6]. Further, if we take $\alpha = \beta = 1$, then we obtain the notion of lacunary statistical convergence given by Fridy and Orhan [13].

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. In the present paper, we define the following sequence spaces:

$w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 = \Big\{ x = (x_k) \in w :$

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s} \Big[\Big[M_k \Big(\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^{p_k} \Big]^\beta = 0, \ \ \rho > 0, \ s \ge 0 \Big\},$$

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)=\Big\{x=(x_k)\in w:$$

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,\ldots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta=0,\text{for some } L,\ \rho>0,\ s\geq 0\Big\}$$

and

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty=\Big\{x=(x_k)\in w:$$

$$\sup_r\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta<\infty,\ \ \rho>0,\ s\geq 0\Big\}.$$

If we take $\mathcal{M}(x)=x$, we get

$$w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0=\Big\{x=(x_k)\in w:$$

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)^{p_k}\Big]^\beta=0,\ \ \rho>0,\ s\geq 0\Big\},$$

$$w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)=\Big\{x=(x_k)\in w:$$

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}\|\Big)^{p_k}\Big]^\beta=0,\ \ \text{for some } L,\ \ \rho>0,\ s\geq 0\Big\}$$

and

$$w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty=\Big\{x=(x_k)\in w:$$

$$\sup_r\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)^{p_k}\Big]^\beta<\infty,\ \ \rho>0,\ s\geq 0\Big\}.$$

If we take $p=(p_k)=1$ for all $k\in\mathbb{N}$, we have

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|)_0=\Big\{x=(x_k)\in w:$$

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^\beta=0,\ \ \rho>0,\ s\geq 0\Big\},$$

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|)=\Big\{x=(x_k)\in w:$$

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^\beta=0,\ \ \text{for some } L,\ \rho>0,\ s\geq 0\Big\}$$

and

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|)_\infty=\Big\{x=(x_k)\in w:$$

$$\sup_r\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^\beta<\infty,\ \ \rho>0,\ s\geq 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \le K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

2. Main results

In this section, we study some topological properties of sequence spaces over n-normed spaces defined by a Musielak-Orlicz function of order (α, β) and prove some inclusion relations between the resulting spaces.

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers the spaces $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$, $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)$ and $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x = (x_k), y = (y_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result we need to find some ρ_3 such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s} \Big[\Big[M_k\Big(\|\frac{\Lambda_k(\alpha x + \beta y)}{\rho_3}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta = 0.$$

Since $x = (x_k), y = (y_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$, there exist positive numbers $\rho_1, \rho_2 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s} \Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^\beta = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s} \Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^\beta = 0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is non-decreasing, convex function and so by using inequality (1.1), we have

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(\alpha x+\beta y)}{\rho_3},z_1,z_2,\cdots,z_{n-1}\|\Big]^{p_k}\Big]^\beta$$

$$\begin{aligned}
&\le \frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\alpha\Lambda_k(x)}{\rho_3},z_1,\cdots,z_{n-1}\|+\|\frac{\beta\Lambda_k(y)}{\rho_3},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\\
&\le K\frac{1}{h_r^\alpha}\sum_{k\in I_r}\frac{1}{2^{p_k}}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\\
&+ K\frac{1}{h_r^\alpha}\sum_{k\in I_r}\frac{1}{2^{p_k}}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\\
&\le K\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\\
&+ K\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\\
&\to 0 \text{ as } r\to\infty.
\end{aligned}$$

Thus we have $\alpha x+\beta y \in w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0$. Hence $w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0$ is a linear space. Similarly, we can prove that $w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)$ and $w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty$ are linear spaces. □

Theorem 2.2. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers. Then $w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0$ is a topological linear spaces paranormed by*

$$g(x)=\inf\Big\{\rho^{\frac{p_r}{H}}:\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}}\le 1\Big\},$$

where $H=\max(1,\sup_k p_k)<\infty$.

Proof. Clearly $g(x)\ge 0$ for $x=(x_k)\in w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0$. Since $M_k(0)=0$ we get $g(0)=0$. Again if $g(x)=0$ then

$$\inf\Big\{\rho^{\frac{p_r}{H}}:\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}}\le 1\Big\}=0.$$

This implies that for a given $\epsilon>0$ there exists some $\rho_\epsilon (0<\rho_\epsilon<\epsilon)$ such that

$$\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_\epsilon},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}}\le 1.$$

Thus

$$\begin{aligned}
&\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\epsilon},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}}\\
&\qquad\le\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_\epsilon},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}}.
\end{aligned}$$

Suppose $(x_k) \neq 0$ for each $k \in \mathbb{N}$. This implies that $\Lambda_k(x) \neq 0$ for each $k \in \mathbb{N}$. Let $\epsilon \to 0$ then

$$\|\frac{\Lambda_k(x)}{\epsilon}, z_1, z_2, \cdots, z_{n-1}\| \to \infty.$$

It follows that

$$\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\epsilon}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}} \to \infty,$$

which is a contradiction. Therefore $\Lambda_k(x) = 0$ for each k and thus $(x_k) = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}} \leq 1$$

and

$$\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by using Minkowski's inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x+y)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}}\\
&= \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)+\Lambda_k(y)}{\rho_1+\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}}\\
&\leq \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big[\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]\\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big[\|\frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}}\\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}}\\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}}\\
&\leq 1.
\end{aligned}$$

Since ρ, ρ_1 and ρ_2 are non-negative, so we have

$$\begin{aligned}
g(x+y) &= \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x+y)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}} \leq 1\Big\}\\
&\leq \inf\Big\{(\rho_1)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}} \leq 1\Big\}\\
&+ \inf\Big\{(\rho_2)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}} \leq 1\Big\}.
\end{aligned}$$

Therefore $g(x+y) \leq g(x) + g(y)$. Finally we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition

$$g(\mu x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(\mu x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

Thus

$$g(\mu x) = \inf\Big\{(|\mu| t)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{t}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}} \leq 1\Big\},$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{p_r} \leq \max(1, |\mu|^{\sup p_r})$, we have

$$g(\mu x) \leq \max(1, |\mu|^{\sup p_r}) \inf\Big\{t^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{t}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

So the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 2.3. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 \subseteq w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$.*

Proof. Let $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$, then there exists a positive number ρ_1 such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta} = 0.$$

Define $\rho = 2\rho_1$. Since (M_k) is non-decreasing, convex and so by using inequality (1.1), we have

$$\begin{aligned}
&\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta} \\
&= \sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)+L-L}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^{\beta} \\
&\leq K\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\frac{1}{2^{p_k}}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^{\beta} \\
&+ K\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\frac{1}{2^{p_k}}\Big[\Big[M_k\Big(\|\frac{L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta} \\
&\leq K\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta} \\
&+ K\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta} \\
&< \infty.
\end{aligned}$$

Hence $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$. □

Theorem 2.4. *Let* $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ *and* $\mathcal{M} = (M_k), \mathcal{M}' = (M'_k)$ *be Musielak-Orlicz functions satisfying* Δ_2*-condition, then we have*

(i) $w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 \subset w_\alpha^\beta(\mathcal{M} \circ \mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$;

(ii) $w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|) \subset w_\alpha^\beta(\mathcal{M} \circ \mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)$;

(iii) $w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\mathcal{M} \circ \mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$.

Proof. Let $x = (x_k) \in w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ then we have

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s} \Big[\Big[M'_k\Big(\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta = 0.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Let $(y_k) = M'_k\Big[\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]$ for all $k \in \mathbb{N}$. We can write

$$\frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s}\Big[\Big(M_k[y_k]\Big)^{p_k}\Big]^\beta = \frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r \\ y_k\leq\delta}} k^{-s}\Big[\Big(M_k[y_k]\Big)^{p_k}\Big]^\beta + \frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r \\ y_k\geq\delta}} k^{-s}\Big[\Big(M_k[y_k]\Big)^{p_k}\Big]^\beta.$$

So we have

$$\frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r \\ y_k\leq\delta}} k^{-s}\Big[\big(M_k[y_k]\big)^{p_k}\Big]^\beta \leq [M_k(1)]^H \frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r \\ y_k\leq\delta}} k^{-s}\Big[\big(M_k[y_k]\big)^{p_k}\Big]^\beta$$

$$\leq [M_k(2)]^H \frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r \\ y_k\leq\delta}} k^{-s}\Big[\big(M_k[y_k]\big)^{p_k}\Big]^\beta \tag{2.1}$$

for $y_k > \delta, y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$. Since $(M_k)'s$ are non-decreasing and convex, it follows that

$$M_k(y_k) < M_k(1 + \frac{y_k}{\delta}) < \frac{1}{2}M_k(2) + \frac{1}{2}M_k(\frac{2y_k}{\delta}).$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2-condition, we can write

$$M_k(y_k) < \frac{1}{2}T\frac{y_k}{\delta}M_k(2) + \frac{1}{2}T\frac{y_k}{\delta}M_k(2) = T\frac{y_k}{\delta}M_k(2).$$

Hence,

$$\frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r \\ y_k\geq\delta}} k^{-s}\big(M_k[y_k]^{p_k}\big)^\beta \leq \max\Big(1, (T\frac{M_k(2)}{\delta})^H\Big) \frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r \\ y_k\leq\delta}} k^{-s}\big([y_k]^{p_k}\big)^\beta \tag{2.2}$$

From equation (2.1) and (2.2), we have $x = (x_k) \in w_\alpha^\beta(\mathcal{M}\circ\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. This completes the proof of (i). Similarly we can prove that

$$w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|) \subset w_\alpha^\beta(\mathcal{M}\circ\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|).$$

and

$$w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\mathcal{M}\circ\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty.$$

□

Theorem 2.5. *Let* $0 < h = \inf p_k = p_k < \sup p_k = H < \infty$. *Then for a Musielak-Orlicz function* $\mathcal{M} = (M_k)$ *which satisfies* Δ_2*-condition, we have*
(i) $w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 \subset w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$;
(ii) $w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|) \subset w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)$;
(iii) $w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$.

Proof. The proof is on similar lines. We omit the details. □

Theorem 2.6. *Let* $\mathcal{M} = (M_k)$ *be a Musielak-Orlicz function and* $0 < h = \inf p_k$. *Then* $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ *if and only if*

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s}\big(\big(M_k(t)\big)^{p_k}\big)^\beta = \infty \tag{2.3}$$

for some $t > 0$.

Proof. Let $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. Suppose that (2.3) does not hold. Therefore there are subinterval $I_{r(j)}$ of the set of interval I_r and a number $t_0 > 0$, where

$$t_0 = \|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\| \text{ for all } k,$$

such that

$$\frac{1}{h_{r(j)}^\alpha} = \sum_{k\in I_{r(j)}} k^{-s}\big(\big(M_k(t_0)\big)^{p_k}\big)^\beta \le K < \infty, m = 1, 2, 3, \cdots \tag{2.4}$$

Let us define $x = (x_k)$ as follows :

$$\Lambda_k(x) = \begin{cases} \rho t_0, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases}.$$

Thus, by (2.4), $x \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$. But $x \notin w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. Hence (2.3) must hold.

Conversely, suppose that (2.3) holds and let $x \in w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty$. Then for each r,

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \le K<\infty. \tag{2.5}$$

Suppose that $x \notin w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0$. Then for some number $\epsilon>0$, there is a number k_0 such that for a subinterval $I_{r(j)}$, of the set of interval I_r,

$$\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\| \;>\epsilon \text{ for } k\ge k_0.$$

From properties of sequence of Orlicz functions, we obtain

$$\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \ge M_k(\epsilon)^{p_k},$$

which contradicts (2.3), by using (2.5). Hence we get

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty \subset w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0.$$

This completes the proof. □

Theorem 2.7. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent :*
(i) $w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty \subset w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty$;
(ii) $w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0 \subset w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty$;
(iii) $\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\big((M_k(t))^{p_k}\big)^\beta<\infty$ *for all* $t>0$.

Proof. (i) ⇒ (ii). Let (i) holds. To verify (ii), it is enough to prove

$$w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0 \subset w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty.$$

Let $x=(x_k)\in w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0$. Then for $\epsilon>0$ there exists $r\ge 0$, such that

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big]^{p_k}\Big]^\beta<\epsilon.$$

Hence there exists $K>0$ such that

$$\sup_r\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big]^{p_k}\Big]^\beta<K.$$

So we get $x=(x_k)\in w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty$.
(ii) ⇒ (iii). Let (ii) holds. Suppose (iii) does not hold. Then for some $t>0$

$$\sup_r\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\big[(M_k(t))^{p_k}\big]^\beta=\infty$$

and therefore we can find a subinterval $I_{r(j)}$, of the set of interval I_r such that

$$\frac{1}{h_{r(j)}^{\alpha}} \sum_{k \in I_{r(j)}} k^{-s} \Big[\Big(M_k(\frac{1}{j}) \Big)^{p_k} \Big]^{\beta} > j, \; j = 1, 2, 3, \cdots \tag{2.6}$$

Let us define $x = (x_k)$ as follows :

$$\Lambda_k(x) = \begin{cases} \frac{\rho}{j}, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases}.$$

Then $x = (x_k) \in w_{\alpha}^{\beta}(\Lambda, p, s, \|\cdot, \cdots, \cdot\|)_0$. But by (2.6), $x \notin w_{\alpha}^{\beta}(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_{\infty}$, which contradicts (ii). Hence (iii) must holds.
(iii) $\Rightarrow$ (i). Let (iii) holds and suppose $x = (x_k) \in w_{\alpha}^{\beta}(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_{\infty}$. Suppose that $x = (x_k) \notin w_{\alpha}^{\beta}(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_{\infty}$, then

$$\sup_r \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} k^{-s} \Big[\Big[M_k \Big(\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^{p_k} \Big]^{\beta} = \infty. \tag{2.7}$$

Let $t = \|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|$ for each k, then by (2.7)

$$\sup_r \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} k^{-s} \big[\big(M_k(t) \big)^{p_k} \big]^{\beta} = \infty$$

which contradicts (iii). Hence (i) must holds. □

Theorem 2.8. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent :*
(i) $w_{\alpha}^{\beta}(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 \subset w_{\alpha}^{\beta}(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$;
(ii) $w_{\alpha}^{\beta}(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 \subset w_{\alpha}^{\beta}(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_{\infty}$;
(iii) $\inf_r \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} k^{-s} \big[\big(M_k(t) \big)^{p_k} \big]^{\beta} > 0$ *for all* $t > 0$.

Proof. (i) $\Rightarrow$ (ii). It is obvious.
(ii) $\Rightarrow$ (iii). Let the inclusion in (ii) hold. Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} k^{-s} \big[\big(M_k(t) \big)^{p_k} \big]^{\beta} = 0 \text{ for some } t > 0,$$

and we can find a subinterval $I_{r(j)}$, of the set of interval I_r such that

$$\frac{1}{h_{r(j)}^{\alpha}} \sum_{k \in I_{r(j)}} k^{-s} \big[\big(M_k(j) \big)^{p_k} \big]^{\beta} < \frac{1}{j}, \; j = 1, 2, 3, \cdots \tag{2.8}$$

Let us define $x = (x_k)$ as follows :

$$\Lambda_k(x) = \begin{cases} \rho j, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases}.$$

Thus by (2.8), $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ but $x = (x_k) \notin w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$, which contradicts (ii). Hence (iii) must hold.
(iii) $\Rightarrow$ (i). Let (iii) holds. Suppose that $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. Then

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[\Big[M_k \Big(\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^{p_k} \Big]^\beta \to 0 \text{ as } r \to \infty. \tag{2.9}$$

Again suppose that $x = (x_k) \notin w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ for some number $\epsilon > 0$ and a subinterval $I_{r(j)}$, of the set of interval I_r, we have

$$\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \geq \epsilon \text{ for all } k.$$

Then from properties of the Orlicz function, we can write

$$\Big[\Big[M_k \Big(\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^{p_k} \Big]^\beta \geq \big[\big(M_k(\epsilon) \big)^{p_k} \big]^\beta.$$

Consequently, by (2.9), we have

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \big[\big(M_k(\epsilon) \big)^{p_k} \big]^\beta = 0,$$

which contradicts (iii). Hence (i) must hold. $\square$

Theorem 2.9. *(i) If* $0 < \inf p_k \leq p_k \leq 1$ *for all* $k \in \mathbb{N}$, *then*

$$w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|) \subseteq w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, s, \|\cdot, \cdots, \cdot\|).$$

(ii) If $1 \leq p_k \leq \sup p_k = H < \infty$, *for all* $k \in \mathbb{N}$, *then*

$$w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, s, \|\cdot, \cdots, \cdot\|) \subseteq w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|).$$

Proof. (i) Let $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)$, then

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[\Big[M_k \Big(\| \frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^{p_k} \Big]^\beta = 0.$$

Since $0 < \inf p_k \leq p_k \leq 1$. This implies that

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[M_k \Big(\| \frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^\beta$$

$$\leq \lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[\Big[M_k \Big(\| \frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^{p_k} \Big]^\beta,$$

therefore,

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^\beta=0.$$

Hence

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)\subseteq w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|).$$

(ii) Let $p_k\geq 1$ for each k and $\sup p_k<\infty$. Let $x=(x_k)\in w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|)$, then for each $\rho>0$, we have

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta=0<1.$$

Since $1\leq p_k\leq\sup p_k<\infty$, we have

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\\ &\leq\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^\beta\\ &=0\\ &<1.\end{aligned}$$

Therefore $x=(x_k)\in w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)$, for each $\rho>0$. Hence

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|)\subseteq w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|).$$

This completes the proof of the theorem. □

Theorem 2.10. *If* $0<\inf p_k\leq p_k\leq\sup p_k=H<\infty$, *for all* $k\in\mathbb{N}$, *then*

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)=w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|).$$

Proof. The proof is on similar lines, we omit the details. □

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