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CHARACTERIZATION OF ORDERED SEMIGROUPS BASED ON ($\Bbbk$,q_k)-QUASI-COINCIDENT WITH RELATION

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Abstract. Based on generalized quasi-coincident with relation, new types of fuzzy bi-ideals of an ordered semigroup S are introduced. Level subset and characteristic functions are used to linked ordinary bi-ideals and $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of an ordered semigroup S. Further, upper/lower parts of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of S are determined. Finally, some well-known classes of ordered semigroups like regular, left (resp. right) regular and completely regular ordered semigroups are characterized by the properties of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals..

Keywords: fuzzy bi-ideals; ordered semigroup; level subset; characteristic functions.

1. Introduction

Over the last few decades, the use of fuzzy set theory [29] has been accomplishing landmark achievements in contemporary mathematics. Several mathematical problems involving uncertainties in various fields like decision making, automata theory, coding theory, computer sciences, control engineering and economics cannot be dealt with through classical set theory (ordinary mathematical tools) due to crisp in nature. Crisp means dichotomous i.e., yes or no type rather than more or less type. In set theory, an element can either belong or not belong to a set. Zadeh's paper [29] on fuzzy sets has opened a new direction for researchers to tackle problems of uncertainties with a more appropriated mathematical tool. Presently, around the globe, the latest research and new investigations of fuzzy set theory is much productive due to the diverse applications in the aforementioned fields.

In algebraic framework, Rosenfeld [25] was the first to apply Zadeh's idea of fuzzy sets and introduce fuzzy subgroups. The inception of fuzzy subgroups provides a platform for other researchers to use this pioneering idea in other algebraic

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structures along with several diverse applications. Among other algebraic structures, semigroups (especially ordered semigroups) are having a lot of applications in error correcting codes, control engineering, performance of super computer and information sciences. Mordeson *et al.* idea in [23] gave birth to an up to date account of fuzzy subsemigroups and fuzzy ideals of semigroups while Kehayopulu and Tsingelis [10-12] used fuzzy sets in ordered semigroups to develop a fuzzy ideal theory. Shabir and Khan [28] gave a characterization of ordered semigroups by the properties of fuzzy ideals and fuzzy generalized bi-ideals.

In 1996, the idea of a quasi-coincidence of a fuzzy point with a fuzzy set [1, 2] was presente. It played a vital role in generating different types of fuzzy subgroups. Bhakat and Das [1] gave the concept of (α, β)-fuzzy subgroups and introduced $(\in, \in \vee q)$-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup. In fact, this is an important and useful generalization of the Rosenfeld's idea of fuzzy subgroup [25]. Since then a verity of research has been carried out using this icebreaking idea. More precisely, (α, β)-fuzzy concept was used by Ma *et al.* [21, 22] in R_0-algebras, Davvaz and Mozafar [3] in Lie algebra, Davvaz and coauthors used the idea of generalized fuzzy sets in rings [4-6], Jun *et al.* [7], Khan and Shabir [13] and Khan *et al.* [14] in ordered semigroups, Shabir *et al.* [26, 27] in semigroups. In 2009, Jun *et al.* [8] initiated a more general form of quasi-coincident with relation (q) and provide (q_k) where $k \in [0, 1)$. The notion has been further strengthened by applying it at various algebraic structures [15-18]. Recently, Jun *et al.* [9] have presented another comprehensive generalization of fuzzy subgroups in light of generalized quasi-coincident with relation. Further, Khan *et al.* [19] elaborated ordered semigroups in terms of fuzzy generalized bi-ideals using this idea [9]. Also, Khan *et al.* [20] determined fuzzy filters of ordered semigroups for the said notion.

In this paper, we apply Jun's idea [9] in ordered semigroups to build a new sort of fuzzy bi-ideals and fuzzy left (resp. right) ideals i.e., $(\in, \in \vee (\Bbbk, q_k))$-fuzzy bi-ideals and $(\in, \in \vee (\Bbbk, q_k))$-fuzzy left (resp. right) ideals. Further, bridging between ordinary bi-ideals and $(\in, \in \vee (\Bbbk, q_k))$-fuzzy bi-ideals through level subsets and characteristic functions is a key milestone of the present paper. Moreover, the lower/upper parts of $(\in, \in \vee (\Bbbk, q_k))$-fuzzy bi-ideals are determined. Finally, several classes of ordered semigroups like regular, left and right regular, and completely regular ordered semigroups are characterized by the properties of $(\in, \in \vee (\Bbbk, q_k))$-fuzzy bi-ideals.

2. Mathematical Formulas

Due to the overarching role of algebraic structures like ordered semigroups in advanced fields such as computer sciences, error correcting codes, automata theory, robotics, control engineering and formal languages, the researchers often develop new structures to tackle the complicated problems faced in the aforementioned fields. The research at hand is a part of new contributions.

In this section, we present some fundamental definitions and results which will be used later on.

A structure $(S, \cdot, \leq)$ is called an **ordered semigroup** if it satisfies the following conditions:

$\cdot \longrightarrow (S, \cdot)$ is a semigroup,

$\cdot \longrightarrow (S, \leq)$ is a poset,

$\cdot \longrightarrow a \leq b \longrightarrow ax \leq bx$ and $a \leq b \longrightarrow xa \leq xb$ for all $a, b, x \in S$.

For subsets A, B of an ordered semigroup S, we denote by $AB = \{ab \in S | a \in A, b \in B\}$. If $A \subseteq S$ we denote $(A] = \{t \in S \mid t \leq h$ for some $h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. If $A, B \subseteq S$, then $A \subseteq (A]$, $(A](B] \subseteq (AB]$, and $((A]] = (A]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a **subsemigroup** of S if $A^2 \subseteq A$. A non-empty subset A of S is called **left** (resp. **right**) ideal of S if

(i) $(\forall a \in S)(\forall b \in A)$ $(a \leq b \longrightarrow a \in A)$,

(ii) $SA \subseteq A$ (resp. $AS \subseteq A$).

A non-empty subset A of S is called an **ideal** if it is both left and right ideal of S.

A non-empty subset A of an ordered semigroup S is called a **bi-ideal** of S if

(i) $(\forall a \in S)(\forall b \in A)$ $(a \leq b \longrightarrow a \in A)$,

(ii) $A^2 \subseteq A$,

(iii) $ASA \subseteq A$.

An ordered semigroup S is **regular** if for every $a \in S$ there exists, $x \in S$ such that $a \leq axa$, or equivalently, we have (i) $a \in (aSa]$ $\forall a \in S$ and (ii) $A \subseteq (ASA]$ $\forall A \subseteq S$. An ordered semigroup S is called **left** (resp. **right**) **regular** if for every $a \in S$ there exists $x \in S$, such that $a \leq xa^2$(resp. $a \leq a^2x$), or equivalently, (i) $a \in (Sa^2]$(resp. $a \in (a^2S]$) $\forall a \in S$ and (ii) $A \subseteq (SA^2]$(resp. $A \subseteq (A^2S]$) $\forall A \subseteq S$. An ordered semigroup S is called **left** (resp. **right**) **simple** if for every left (resp. right) ideal A of S we have $A = S$ and S is called **simple** if it is both left and right simple. An ordered semigroup S is called **completely regular**, if it is left regular, right regular and regular.

Before 1965, the researchers were using traditional mathematical tools for modeling. Traditional tools are often dichotomous in nature. Dichotomous means yes "1" or no "0", therefore it could not handle problems involving uncertainties. In 1965, Zadeh was the first to introduce fuzzy sets (a new mathematical approach for dealing such problems of uncertainties). A function $\xi : S \longrightarrow [0, 1]$ is called a **fuzzy subset** of S. Since in classical set, the range of the function is $\{0, 1\}$ while in Zadeh's fuzzy set the range is $[0, 1]$, therefore, fuzzy sets are the generalizations of ordinary sets. The study of fuzzification of algebraic structures started in the pioneering paper of Rosenfeld [25] in 1971.

If ξ_1 and ξ_2 are fuzzy subsets of S, then $\xi_1 \preceq \xi_2$ means $\xi_1(x) \leq \xi_2(x)$ for all $x \in S$ and the symbols $\wedge$ and $\vee$ will mean the following fuzzy subsets:

$$\begin{aligned} \xi_1 \wedge \xi_2 \quad &: S \longrightarrow [0,1] | x \longmapsto (\xi_1 \wedge \xi_2)(x) = \xi_1(x) \wedge \xi_2(x) \\ \xi_1 \vee \xi_2 \quad &: S \longrightarrow [0,1] | x \longmapsto (\xi_1 \vee \xi_2)(x) = \xi_1(x) \vee \xi_2(x), \end{aligned}$$

for all $x \in S$. A fuzzy subset ξ of S is called a **fuzzy subsemigroup** if $\xi(xy) \geq \xi(x) \wedge \xi(y)$ for all $x, y \in S$. A fuzzy subset ξ of S is called a **fuzzy left (resp. right)-ideal** of S if (i) $x \leq y \longrightarrow \xi(x) \geq \xi(y)$, (ii) $\xi(xy) \geq \xi(y)$ (resp. $\xi(xy) \geq \xi(x)$) for all $x, y \in S$. A fuzzy subset ξ of S is called a **fuzzy ideal** if it is both a fuzzy left and a fuzzy right ideal of S. A fuzzy subsemigroup ξ is called a **fuzzy bi-ideal** of S if (i) $x \leq y \longrightarrow \xi(x) \geq \xi(y)$, (ii) $\xi(xyz) \geq \xi(x) \wedge \xi(z)$ for all $x, y, z \in S$. Let S be an ordered semigroup and ξ is a fuzzy subset of S. Then, for all $t \in (0,1]$, the set $U(\xi; t) = \{x \in S | \xi(x) \geq t\}$ is called a **level set** of ξ.

Theorem 2.1. *[7] A fuzzy subset ξ of an ordered semigroup S is a fuzzy bi-ideal of S if and only if $U(\xi; t) (\neq \emptyset)$ where $t \in (0,1]$ is a bi-ideal of S.*

Proof. If S is an ordered semigroup and A be any subset of S, then the characteristic function $\mathbb{C}_A$ of A is a function i.e., $\mathbb{C}_A : S \longrightarrow [0,1]$ and defined as

$$\begin{array}{cc} \{1 & if x \in A \\ 0 & otherwise\} \end{array}$$

□

Theorem 2.2. [7] A non-empty subset A of an ordered semigroup S is a bi-ideal of S if and only if the characteristic function $\mathbb{C}_A$ of A is a fuzzy bi-ideal of S.

Proof. if $a \in S$ and $\mathbb{A}$ is a non empty subset of $\mathbb{S}$. then,

$$A_a = \{(y, z) \in \mathbb{S} \times \mathbb{S} | a \leq yz\}$$

if ξ_1 and ξ_2 are two fuzzy subset of $\mathbb{S}$, then the product $\xi_1 \circ \xi_2$ of ξ_1 and ξ_2 is a function i.e, $\xi_1 \circ \xi_2 : \mathbb{S} \longmapsto [0,1]$ and defined as

$$\begin{array}{ll} (\xi_1 \circ \xi_2)(a) = \bigvee\limits_{(y,z) \in A_a} (\xi_1(y) \cap \xi_2(z)) & if A_a \neq \emptyset \\ \qquad\qquad 0 & otherwise \end{array}$$

Let ξ be a fuzzy subset of $\mathbb{S}$, then the set of the form

$$\begin{array}{ll} \xi(y) = a \in (0,1] & if y = x \\ \qquad 0 & if y \neq x \end{array}$$

is called a **fuzzy point** [24] with support x and value a and is denoted by x_a. A fuzzy point x_a is said to **belong to** (resp. **quasi-coincident with**) a fuzzy set ξ, written as $x_a \in \xi$ (resp. $x_a q \xi$) if $\xi(x) \geq a$ (resp. $\xi(x) + a > 1$). If $x_a \in \xi$ or $x_a q \xi$, then we write $x_a \in \vee q \xi$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold. □

3. Fuzzy bi-ideals based on $(\Bbbk, q_k)$-quasi-coincident with relation

Aiming to describe the fuzzy ideals of an ordered semigroups in a more realistic way, the idea of quasi-coincident with relation [1, 2] has been proposed. The said notion played a vital role in generating several type of fuzzy subsystems which have already been used in a variety of productive research [3, 5-9, 13-21, 26, 27]. Jun *et al.* [9] further generalized his idea [8] and initiated an essential generalization of $(\in, \in \vee q)$-fuzzy subgroups. Keeping in view Jun's idea [9], we have introduced a new generalization of quasi-coincident with relation called $(\Bbbk, q_k)$-quasi-coincident with relation. In this section, new classification of an ordered semigorup S based on $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals where $\Bbbk$, $k \in [0,1]$ such that $0 \leq k < \Bbbk \leq 1$ is determined. A fuzzy point x_a is said to belong to (resp. $(\Bbbk, q)$-quasi-coincident with) a fuzzy set ξ, written as $x_a \in \xi$ (resp. $x_a(\Bbbk, q)\xi$) if $\xi\,(x) \geq a$ (resp. $\xi\,(x) + a > \Bbbk$). If $x_a \in \xi$ or $x_a(\Bbbk, q)\xi$, then we write $x_a \in \vee(\Bbbk, q)\xi$. The symbol $\overline{\in \vee(\Bbbk, q)}$ means $\in \vee(\Bbbk, q)$ does not hold. In ordered semigroups, generalizing the concept of $x_a(\Bbbk, q)\xi$, we define $x_a(\Bbbk, q_k)\xi$, as $\xi\,(x) + a + k > \Bbbk$, where $k, \Bbbk \in [0, 1)$ and $0 \leq k < \Bbbk \leq 1$. Note that $x_a q_k \xi$ implies $x_a(\Bbbk, q_k)\xi$, but the converse of the statement is not always true. Particularly, if $\Bbbk = 1$, then every $(\Bbbk, q_k)$-quasi-coincident with relation will lead to quasi-coincident with relation, symbolically $x_a(1, q_k)\xi = x_a q_k \xi$. Also, $x_a \in \bigvee(\Bbbk, q_k)\xi$ (resp. $x_a \in \wedge(\Bbbk, q_k)\xi$) means that $x_a \in \xi$ or $x_a(\Bbbk, q_k)\xi$) (resp. $x_a \in \xi$ and $x_a(\Bbbk, q_k)\xi$). In what follows, let S denote an ordered semigroup unless otherwise stated.

Definition 3.1. A fuzzy subset ξ of S is called an $(\in, \in \vee(K, q_k))$**-fuzzy bi-ideal** of S if it satisfies the conditions: (1) $(\forall x, y \in S)(\forall a \in (0,1])(x \leq y, y_a \in \xi \longrightarrow x_a \in \vee(K, q_k)\xi)$,
(2) $(\forall x, y \in S)(\forall a, b \in (0,1])(x_a \in \xi, y_a \in \xi \longrightarrow (xy)_{a \wedge b} \in \vee(K, q_k)\xi)$,
(3) $(\forall x, y, z \in S)(\forall a, b \in (0,1])(x_a \in \xi, z_b \in \xi \longrightarrow (xyz)_{a \wedge b} \in \vee(K, q_k)\xi)$.

Theorem 3.1. *Let A be a bi-ideal of S and ξ a fuzzy subset in S defined by:*

$$\xi(x) = \begin{array}{ll} \geq \dfrac{\Bbbk - k}{2} & if x \in A, \\ 0 & otherwise \end{array}$$

Then (1) ξ is a $((\Bbbk, q), \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.
(2) ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.

Proof. (1) Assume that $x, y \in S$, $x \leq y$ and $a \in (0,1]$ such that $y_a(\Bbbk, q)\xi$. Then $y \in A$, $\xi\,(y) + a > \Bbbk$. Since A is a bi-ideal of S and $x \leq y \in A$, therefore $x \in A$. Thus $\xi(x) \geq \frac{\Bbbk - k}{2}$. If $a \leq \frac{\Bbbk - k}{2}$, then $\xi(x) \geq a$ and so $x_a \in \xi$. If $a > \frac{\Bbbk - k}{2}$, then $\xi(x) + a + k > \frac{\Bbbk - k}{2} + \frac{\Bbbk - k}{2} + k = \Bbbk$ and so $x_a(\Bbbk, q_k)\xi$. Therefore, $x_a \in \vee(\Bbbk, q_k)\xi$.

Let $x, y \in S$ and $a, b \in (0,1]$ be such that $x_a(\Bbbk, q)\xi$ and $y_b(\Bbbk, q)\xi$. Then $x, y \in A$, so $\xi(x) + a > \Bbbk$ and $\xi(y) + b > \Bbbk$. Since A is a bi-ideal of S, hence $xy \in A$. Thus $\xi(xy) \geq \frac{\Bbbk - k}{2}$. If $a \wedge b > \frac{\Bbbk - k}{2}$, then $\xi(xy) + a \wedge b + k > \frac{\Bbbk - k}{2} + \frac{\Bbbk - k}{2} + k = \Bbbk$

and so $(xy)_{a\wedge b}(\Bbbk, q_k)\xi$. If $a \wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xy) \geq a \wedge b$ and so $(xy)_{a\wedge b} \in \xi$. It implies that, $(xy)_{a\wedge b} \in \vee(\Bbbk, q_k)\xi$. Let $x, y, z \in S$ and $a, b \in (0, 1]$ be such that $x_a(\Bbbk, q)\xi$ and $z_b(\Bbbk, q)\xi$. Then $x, z \in A$, $\xi(x) + a > \Bbbk$ and $\xi(z) + b > \Bbbk$. Since A is a bi-ideal of S, so $xyz \in A$. Hence $\xi(xyz) \geq \frac{\Bbbk-k}{2}$. If $a \wedge b > \frac{\Bbbk-k}{2}$, then $\xi(xyz)+a\wedge b+k > \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$ and so $(xyz)_{a\wedge b}(\Bbbk, q_k)\xi$. If $a\wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xyz) \geq a\wedge b$ and so $(xyz)_{a\wedge b} \in \xi$.Therefore, $(xyz)_{a\wedge b} \in \vee(\Bbbk, q_k)\xi$. Implies that ξ is a $((\Bbbk, q), \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. (2) Let $x, y \in S$, $x \leq y$ and $t \in (0, 1]$ be such that $y_a \in \xi$. Then $\xi(y) \geq a$ and $y \in A$. Since A is a bi-ideal of S and $x \leq y \in A$, we have $x \in A$. Thus $\xi(x) \geq \frac{\Bbbk-k}{2}$. If $a \leq \frac{\Bbbk-k}{2}$, then $\xi(x) \geq a$ and so $x_a \in \xi$. If $a > \frac{\Bbbk-k}{2}$, then $\xi(x) + a + k > \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$ and so $x_a(\Bbbk, q_k)\xi$. Therefore, $x_a \in \vee(\Bbbk, q_k)\xi$. Let $x, y \in S$ and $a, b \in (0, 1]$ be such that $x_a \in \xi$ and $y_b \in \xi$. Then $x, y \in A$. Since A is a bi-ideal of S, it leads to $xy \in A$. Thus $\xi(xy) \geq \frac{\Bbbk-k}{2}$. If $a \wedge b > \frac{\Bbbk-k}{2}$, then $\xi(xy) + a \wedge b + k > \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$ and so $(xy)_{a\wedge b}(\Bbbk, q_k)\xi$. If $a \wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xy) \geq a \wedge b$ and so $(xy)_{a\wedge b} \in \xi$. Thus, $(xy)_{a\wedge b} \in \vee(\Bbbk, q_k)\xi$. Let $x, y, z \in S$ and $a, b \in (0, 1]$ be such that $x_a \in \xi$ and $z_b \in \xi$. Then $x, z \in A$. Since A is a bi-ideal of S, we have, $xyz \in A$. Hence $\xi(xyz) \geq \frac{\Bbbk-k}{2}$. If $a \wedge b > \frac{\Bbbk-k}{2}$, then $\xi(xyz)+a\wedge b+k > \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$ and so $(xyz)_{a\wedge b}(\Bbbk, q_k)\xi$. If $a\wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xyz) \geq a \wedge b$ and so $(xyz)_{a\wedge b} \in \xi$. Hence, $(xyz)_{a\wedge b} \in \vee(\Bbbk, q_k)\xi$. Consequently, ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. □

If we take $\Bbbk = 1$ in Theorem (3.1), then we get the following corollary:

Corollary 3.1. *Let A be a bi-ideal of S and ξ a fuzzy subset in S defined by $\xi(x) \geq \frac{1-k}{2}$ if $x \in A, \xi(x) = 0$ if $x \notin A$, then, ξ is both $(q, \in \vee q_k)$ and $(\in, \in \vee q_k)$ type of fuzzy bi-ideal of S.*

Theorem 3.2. *Let ξ be a fuzzy subset of S. Then the following conditions are equivalent: (1) ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. (2) (i) $(\forall x, y \in S)(x \leq y \longrightarrow \xi(x) \geq \xi(y)\wedge\frac{\Bbbk-k}{2})$, (ii) $(\forall x, y \in S)(\xi(xy) \geq \xi(x)\wedge\xi(y)\wedge\frac{\Bbbk-k}{2})$, (iii) $(\forall x, y, z \in S)(\xi(xyz) \geq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2})$.*

Proof. (1) $\Longrightarrow$ (2): Suppose ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. On the contrary assume that, there exists $x, y \in S$, $x \leq y$ such that $\xi(x) < \xi(y) \wedge \frac{\Bbbk-k}{2}$. Choose $a \in (0, 1]$ such that $\xi(x) < a \leq \xi(y) \wedge \frac{\Bbbk-k}{2}$. Then, $y_a \in \xi$, but $\xi(x) < a$ and $\xi(x)+a+k < \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$, so $x_a \overline{\in \vee(\Bbbk, q_k)}\xi$, which is a contradiction. Hence, $\xi(x) \geq \xi(y) \wedge \frac{\Bbbk-k}{2}$ for all $x, y \in S$ with $x \leq y$. If there exists $x, y \in S$ such that $\xi(xy) < \xi(x)\wedge\xi(y)\wedge\frac{\Bbbk-k}{2}$. Choose $a \in (0, 1]$ such that $\xi(xy) < a \leq \xi(x)\wedge\xi(y)\wedge\frac{\Bbbk-k}{2}$. Then, $x_a \in \xi, y_a \in \xi$ but $\xi(xy) < a$ and $\xi(xy) + a + k < \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$, so $(xy)_a\overline{(\Bbbk, q_k)}\xi$. Thus, $(xy)_a\overline{\in \vee(\Bbbk, q_k)}\xi$, again a contradiction. Therefore, $\xi(xy) \geq \xi(x) \wedge \xi(y) \wedge \frac{\Bbbk-k}{2}$ for all $x, y \in S$. Now if there exists $x, y, z \in S$ such that $\xi(xyz) < \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$. Then, for $a \in (0, 1]$ such that $\xi(xyz) < a \leq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$, we have, $x_a \in \xi$ and $z_a \in \xi$ but $\xi(xyz) < a$ and $\xi(xyz) + a + k < \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$, so $(xyz)_a\overline{(\Bbbk, q_k)}\xi$. Thus,$(xyz)_a\overline{\in \vee(\Bbbk, q_k)}\xi$. Therefore, $\xi(xyz) \geq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$ for all $x, y, z \in S$.

(2)$\Longrightarrow$ (1): Let $y_a \in \xi$ for some $a \in (0,1]$. Then, $\xi(y) \geq a$. Now, $\xi(x) \geq \xi(y) \wedge \frac{\Bbbk-k}{2} \geq a \wedge \frac{\Bbbk-k}{2}$. If $a > \frac{\Bbbk-k}{2}$, then $\xi(x) \geq \frac{\Bbbk-k}{2}$ and $\xi(x)+a+k > \frac{\Bbbk-k}{2}+\frac{\Bbbk-k}{2}+k = \Bbbk$, it follows that $x_a(\Bbbk, q_k)\xi$. If $a \leq \frac{\Bbbk-k}{2}$, then $\xi(x) \geq a$ and so $x_a \in \xi$. Thus, $x_a \in \vee(\Bbbk, q_k)\xi$. Let $x_a \in \xi$ and $y_b \in \xi$ for some $a, b \in (0,1]$, then $\xi(x) \geq a$ and $\xi(y) \geq b$. Thus, $\xi(xy) \geq \xi(x) \wedge \xi(y) \wedge \frac{\Bbbk-k}{2} \geq a \wedge b \wedge \frac{\Bbbk-k}{2}$. If $a \wedge b > \frac{\Bbbk-k}{2}$, then $\xi(xy) \geq \frac{\Bbbk-k}{2}$ and $\xi(xy)+a \wedge b+k > \frac{\Bbbk-k}{2}+\frac{\Bbbk-k}{2}+k = \Bbbk$ and so $(xy)_{a\wedge b}(\Bbbk, q_k)\xi$. If $a \wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xy) \geq a \wedge b$ and hence, $(xy)_{a\wedge b} \in \xi$. Thus, $(xy)_{a\wedge b} \in \vee(\Bbbk, q_k)\xi$. Now let $x_a \in \xi$ and $z_b \in \xi$, then $\xi(x) \geq a$ and $\xi(z) \geq b$. Thus, $\xi(xyz) \geq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2} \geq a \wedge b \wedge \frac{\Bbbk-k}{2}$. If $a \wedge b > \frac{\Bbbk-k}{2}$, then $\xi(xyz) \geq \frac{\Bbbk-k}{2}$ and $\xi(xyz)+a \wedge b+k > \frac{\Bbbk-k}{2}+\frac{\Bbbk-k}{2}+k = \Bbbk$ and so $(xyz)_{a\wedge b}(\Bbbk, q_k)\xi$. If $a \wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xyz) \geq a \wedge b$ and hence, $(xyz)_{a\wedge b} \in \xi$. Thus, $(xyz)_{a\wedge b} \in \vee(\Bbbk, q_k)\xi$. Consequently, ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. $\square$

taking $\Bbbk = 1$, Theorem (3.2) leads to the result in [15].

Theorem 3.3. *If S is an ordered semigroup and ξ is fuzzy subset of S, then the following conditions are equivalent:*

(1) A fuzzy subset ξ of S is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.

(2) $U(\xi; a)(\neq \emptyset)$ is a bi-ideal of S for all $a \in (0, \frac{\Bbbk-k}{2}]$.

Proof. (1) $\Longrightarrow$ (2): Suppose that ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S and let $x, y, z \in S$ be such that $x, z \in U(\xi; a)$ for some $a \in (0, \frac{\Bbbk-k}{2}]$. Then $\xi(x) \geq a$ and $\xi(z) \geq a$ and by hypothesis

$$\begin{aligned} \xi(xyz) & \geq \xi(x) \wedge \xi(z) \wedge \tfrac{\Bbbk-k}{2} \\ & \geq a \wedge a \wedge \tfrac{\Bbbk-k}{2} = a. \end{aligned}$$

Hence, $xyz \in U(\xi; a)$. Also, by similar way, if $x, y \in S$ be such that $x, y \in U(\xi; a)$ for some $a \in (0, \frac{\Bbbk-k}{2}]$, then $xy \in U(\xi; a)$. Now let $x, y \in S$ be such that $y \in U(\xi; a)$ for some $a \in (0, \frac{\Bbbk-k}{2}]$. Then $\xi(y) \geq a$ and by hypothesis

$$\begin{aligned} \xi(x) & \geq \xi(y) \wedge \tfrac{\Bbbk-k}{2} \\ & \geq a \wedge \tfrac{\Bbbk-k}{2} = a. \end{aligned}$$

Hence $x \in U(\xi; a)$.
(2) $\Longrightarrow$ (1): Assume that $U(\xi; a)(\neq \emptyset)$ is a bi-ideal of S for all $a \in (0, \frac{\Bbbk-k}{2}]$. If there exists $x, y, z \in S$ such that $\xi(xyz) < \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$, then choose $a \in (0, \frac{\Bbbk-k}{2}]$ such that $\xi(xyz) < a \leq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$. Thus, $x, z \in U(\xi; t)$ but $xyz \notin U(\xi; a)$, a contradiction. Hence, $\xi(xyz) \geq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$ for all $x, y, z \in S$ and $0 \leq k < \Bbbk \leq 1$.
Let $x, y \in S$ be such that $\xi(x) < \xi(y) \wedge \frac{\Bbbk-k}{2}$. Choose $r \in (0, \frac{\Bbbk-k}{2}]$ such that $\xi(x) < r \leq \xi(y) \wedge \frac{\Bbbk-k}{2}$ then $\xi(y) \geq r$ implies that $y_r \in \xi$ but $x_r \overline{\in} \xi$. Now $\xi(x) + r + k < \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$, which implies that $x_r \overline{(\Bbbk, q_k)} \xi$, a contradiction. Hence, $\xi(x) \geq \xi(y) \wedge \frac{\Bbbk-k}{2}$.

Table 3.1: Hasse diagram for $:\leq \{(a_1, a_2)\}$

$\cdot$	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_1
a_3	a_1	a_1	a_2	a_1
a_4	a_1	a_1	a_2	a_2

By similar way, $\xi(xy) \geq \xi(x) \wedge \xi(y) \wedge \frac{\Bbbk - k}{2}$ for $x, y \in S$. Therefore, ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. $\square$

Example 3.1. Consider the ordered semigroup $S = \{a_1, a_2, a_3, a_4\}$ with the following multiplication and order relation

$$\leq := \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_1, a_2)\}.$$

The covering relation $\preceq := \{(a_1, a_2)\}$ is represented by table (3.1)

Then $\{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_3, a_4\}$ and $\{a_1, a_2, a_3, a_4\}$ are bi-ideals of S. Define a fuzzy subset ξ of S as follows:

S	a_1	a_2	a_3	a_4
$\xi(x)$	0.70	0.20	0.30	0.60

Then

$a \in (0, 1]$	$U(\xi; a)$
$0 < a \leq 0.20$	S
$0.20 < a \leq 0.30$	$\{a_1, a_3, a_4\}$
$0.30 < a \leq 0.60$	$\{a_1, a_4\}$
$0.60 < a \leq 1$	$\emptyset$

Therefore, using Theorem (3.3), ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S for $a \in (0, \frac{\Bbbk - k}{2}]$ with $\Bbbk = 0.8$ and $k = 0.4$.

If S is an ordered semigroup and ξ is a fuzzy subset of S, then define a set ξ_0 of S as follows:

$$\xi_0 = \{x \in S | \xi(x) > 0\}.$$

Proposition 3.1. *If ξ is a nonzero $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, then the subset ξ_0 of S is a bi-ideal of S.*

Proof. Let ξ be an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. If $x, y \in S$ such that $x \leq y$ and $y \in \xi_0$, then $\xi(y) > 0$. Since ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, therefore

$$\xi(x) \geq \xi(y) \wedge \frac{\Bbbk - k}{2} > 0,$$

Thus $\xi(x) > 0$ and so $x \in \xi_0$. Let $x, y \in \xi_0$. Then, $\xi(x) > 0$ and $\xi(y) > 0$. Now,

$$\begin{aligned} \xi(xy) \quad & \geq \xi(x) \wedge \xi(y) \wedge \tfrac{\Bbbk - k}{2} \\ & > 0, (\xi(x) > 0 and \xi(y) > 0). \end{aligned}$$

Thus $xy \in \xi_0$. For $x, z \in \xi_0$ we have

$$\begin{aligned} \xi(xyz) \quad & \geq \xi(x) \wedge \xi(z) \wedge \tfrac{\Bbbk - k}{2} \\ & > 0. \end{aligned}$$

so $xyz \in \xi_0$. Consequently ξ_0 is a bi-ideal of S. □

Lemma 3.1. A non-empty subset A of S is a bi-ideal if and only if the characteristic function $\mathbb{C}_A$ of A is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.

Proof. The proof follows from Theorem (3.3). □

If $\{\xi_i\}_{i \in I}$ is an indexed family of fuzzy subsets of an ordered semigroup S, then the intersection $\bigcap\limits_{i \in I} \xi_i$ of ξ_i is defined as

$$\left(\bigcap_{i \in I} \xi_i \right)(x) = \{\xi_{i_1}(x) \wedge \xi_{i_2}(x) \wedge \xi_{i_3}(x) \wedge ... \mid i_i \in I\} = \bigwedge_{i \in I} (\xi_i(x)).$$

Proposition 3.2. *If $\{\xi_i : i \in I\}$ is a family of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of an ordered semigroup S. Then $\bigcap\limits_{i \in I} \xi_i$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.*

Proof. Let $\{\xi_i\}_{i \in I}$ be a family of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of S. Let $x, y, z \in S$. Then,

$$\begin{aligned} \left(\bigcap_{i \in I} \xi_i \right)((xyz) \quad & = \bigwedge_{i \in I} \xi_i((xyz) \geq \bigwedge_{i \in I} (\xi_i(x) \wedge \xi_i(z) \wedge \tfrac{\Bbbk - k}{2}) \\ & = \left(\bigwedge_{i \in I} \left(\xi_i(x) \wedge \tfrac{\Bbbk - k}{2}\right) \wedge \bigwedge_{i \in I} \left(\xi_i(z) \wedge \tfrac{\Bbbk - k}{2}\right) \right) \\ & = \left(\bigcap_{i \in I} \xi_i \right)(x) \wedge \left(\bigcap_{i \in I} \xi_i \right)(z) \wedge \tfrac{\Bbbk - k}{2}. \end{aligned}$$

The remaining conditions for $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S can be proved in a similar way. Thus $\bigcap\limits_{i \in I} \xi_i$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. □

For $\Bbbk = 1$, the Proposition (3.1,3.2), and Lemma (3.1) leads to [Proposition 3.8, Proposition 3.10, Lemma 3.9, [15]] respectively.

4. Upper and lower parts of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals

In this section, we first define the $(\Bbbk, q_k)$-upper/lower parts of an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. Then by using the properties of bi-ideals, we characterize regular and intra-regular ordered semigroups in terms of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals.

Definition 4.1. *Let ξ_1 and ξ_2 be a fuzzy subsets of S. Then the fuzzy subsets $\left(\overline{\xi_1}\right)_k^{\Bbbk}$, $(\xi_1 \,(\wedge)_k^{\Bbbk}\, \xi_2)^-$, $(\xi_1 \,(\vee)_k^{\Bbbk}\, \xi_2)^-$, $(\xi_1 \,(\circ)_k^{\Bbbk}\, \xi_2)^-$, $\left(\xi_1{}^+\right)_k^{\Bbbk}$, $(\xi_1 \,(\wedge)_k^{\Bbbk}\, \xi_2)^+$, $(\xi_1 \,(\vee)_k^{\Bbbk}\, \xi_2)^+$ and $(\xi_1 \,(\circ)_k^{\Bbbk}\, \xi_2)^+$ of S are defined as follows:*

$$\begin{aligned}
\left(\overline{\xi_1}\right)_k^{\Bbbk} \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1)_k^{\Bbbk}(x) = \xi_1(x) \wedge \tfrac{\Bbbk - k}{2}, \\
(\xi_1 \,(\wedge)_k^{\Bbbk}\, \xi_2)^- \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 \,(\wedge)_k^{\Bbbk}\, \xi_2)(x) = (\xi_1 \wedge \xi_2)(x) \wedge \tfrac{\Bbbk - k}{2}, \\
(\xi_1 \,(\vee)_k^{\Bbbk}\, \xi_2)^- \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 \,(\vee)_k^{\Bbbk}\, \xi_2)(x) = (\xi_1 \vee \xi_2)(x) \wedge \tfrac{\Bbbk - k}{2}, \\
(\xi_1 \,(\circ)_k^{\Bbbk}\, \xi_2)^- \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 \,(\circ)_k^{\Bbbk}\, \xi_2)(x) = (\xi_1 \circ \xi_2)(x) \wedge \tfrac{\Bbbk - k}{2},
\end{aligned}$$

and

$$\begin{aligned}
\left(\xi_1{}^+\right)_k^{\Bbbk} \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1)_k^{\Bbbk}(x) = \xi_1(x) \vee \tfrac{\Bbbk - k}{2}, \\
(\xi_1 \,(\wedge)_k^{\Bbbk}\, \xi_2)^+ \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 \,(\wedge)_k^{\Bbbk}\, \xi_2)(x) = (\xi_1 \wedge \xi_2)(x) \vee \tfrac{\Bbbk - k}{2}, \\
(\xi_1 \,(\vee)_k^{\Bbbk}\, \xi_2)^+ \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 \,(\vee)_k^{\Bbbk}\, \xi_2)(x) = (\xi_1 \vee \xi_2)(x) \vee \tfrac{\Bbbk - k}{2}, \\
(\xi_1 \,(\circ)_k^{\Bbbk}\, \xi_2)^+ \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 \,(\circ)_k^{\Bbbk}\, \xi_2)(x) = (\xi_1 \circ \xi_2)(x) \vee \tfrac{\Bbbk - k}{2},
\end{aligned}$$

for all $x \in S$.

Lemma 4.1. Let ξ_1 and ξ_2 be fuzzy subsets of S. Then the following hold:
(i) $\left(\xi_1 \,(\wedge)_k^{\Bbbk}\, \xi_2\right)^- = \left(\left(\overline{\xi_1}\right)_k^{\Bbbk} \wedge \left(\overline{\xi_2}\right)_k^{\Bbbk}\right)$,
(ii) $\left(\xi_1 \,(\vee)_k^{\Bbbk}\, \xi_2\right)^- = \left(\left(\overline{\xi_1}\right)_k^{\Bbbk} \vee \left(\overline{\xi_2}\right)_k^{\Bbbk}\right)$,
(iii) $\left(\xi_1 \,(\circ)_k^{\Bbbk}\, \xi_2\right)^- = \left(\left(\overline{\xi_1}\right)_k^{\Bbbk} \circ \left(\overline{\xi_2}\right)_k^{\Bbbk}\right)$.

Proof. (i) Let $x \in S$ and ξ_1 and ξ_2 be fuzzy subsets of an ordered semigroup S, then

$$\begin{aligned}
\left(\xi_1 \,(\wedge)_k^{\Bbbk}\, \xi_2\right)^- \quad & = (\xi_1 \,(\wedge)_k^{\Bbbk}\, \xi_2)(x) = (\xi_1 \wedge \xi_2)(x) \wedge \tfrac{\Bbbk - k}{2} \\
& = \xi_1(x) \wedge \xi_2(x) \wedge \tfrac{\Bbbk - k}{2} \\
& = \xi_1(x) \wedge \xi_2(x) \wedge \tfrac{\Bbbk - k}{2} \wedge \tfrac{\Bbbk - k}{2} \\
& = \left\{\xi_1(x) \wedge \tfrac{\Bbbk - k}{2}\right\} \wedge \left\{\xi_2(x) \wedge \tfrac{\Bbbk - k}{2}\right\} \\
& = (\xi_1)_k^{\Bbbk}(x) \wedge (\xi_2)_k^{\Bbbk}(x) = \left((\xi_1)_k^{\Bbbk} \wedge (\xi_2)_k^{\Bbbk}\right)(x).
\end{aligned}$$

The proof of part (ii) and (iii) is similar to the proof of part (i). $\square$

Lemma 4.2. *Let ξ_1 and ξ_2 be fuzzy subsets of S. Then the following hold:*
(i) $\left(\xi_1 \left(\wedge\right)_k^{\Bbbk} \xi_2\right)^+ = \left(\left({\xi_1}^+\right)_k^{\Bbbk} \wedge \left({\xi_2}^+\right)_k^{\Bbbk}\right)$,
(ii) $\left(\xi_1 \left(\vee\right)_k^{\Bbbk} \xi_2\right)^+ = \left(\left({\xi_1}^+\right)_k^{\Bbbk} \vee \left({\xi_2}^+\right)_k^{\Bbbk}\right)$,
(iii) $\left(\xi_1 \left(\circ\right)_k^{\Bbbk} \xi_2\right)^+ \succeq \left(\left({\xi_1}^+\right)_k^{\Bbbk} \circ \left({\xi_2}^+\right)_k^{\Bbbk}\right)$ *if* $A_x = \emptyset$ *and* $\left(\xi_1 \left(\circ\right)_k^{\Bbbk} \xi_2\right)^+$
$= \left(\left({\xi_1}^+\right)_k^{\Bbbk} \circ \left({\xi_2}^+\right)_k^{\Bbbk}\right)$ *if* $A_x \neq \emptyset$.

Proof. The proof follows from Lemma (4.1). □

Let A be a non-empty subset of S, then the upper and lower parts of the characteristic function $\mathbb{C}_A$ are defined as follows:

$$\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk} : S \longrightarrow [0,1] | x \longmapsto \left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) = \begin{cases} \frac{\Bbbk - k}{2} & if x \in A \\ 0 & otherwise. \end{cases}$$

$$\left({\mathbb{C}_A}^+\right)_k^{\Bbbk} : S \longrightarrow [0,1] | x \longmapsto \left({\mathbb{C}_A}^+\right)_k^{\Bbbk}(x) = \begin{cases} 1 & if x \in A \\ \frac{\Bbbk - k}{2} & otherwise. \end{cases}$$

Lemma 4.3. *Let A and B be non-empty subset of S. Then the following hold:*
(1) $\left(\mathbb{C}_A \left(\wedge\right)_k^{\Bbbk} \mathbb{C}_B\right)^- = \left(\overline{\mathbb{C}}_{A\cap B}\right)_k^{\Bbbk}$,
(2) $\left(\mathbb{C}_A \left(\vee\right)_k^{\Bbbk} \mathbb{C}_B\right)^- = \left(\overline{\mathbb{C}}_{A\cup B}\right)_k^{\Bbbk}$,
(3) $\left(\mathbb{C}_A \left(\circ\right)_k^{\Bbbk} \mathbb{C}_B\right)^- = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}$.

Proof. (1) Let $x \in S$, if $x \in A \cap B$, then $\left(\overline{\mathbb{C}}_{A\cap B}\right)_k^{\Bbbk}(x) = \frac{\Bbbk - k}{2}$. Also, since $x \in A \cap B$, implies that $x \in A$ and $x \in B$. Therefore, $\mathbb{C}_A(x) = 1$ and $\mathbb{C}_B(x) = 1$. Hence,

$$\begin{aligned} \left(\mathbb{C}_A \left(\wedge\right)_k^{\Bbbk} \mathbb{C}_B\right)^- &= \left(\mathbb{C}_A \left(\wedge\right)_k^{\Bbbk} \mathbb{C}_B\right)(x) = \mathbb{C}_A(x) \wedge \mathbb{C}_B(x) \wedge \tfrac{\Bbbk - k}{2} \\ &= 1 \wedge 1 \wedge \tfrac{\Bbbk - k}{2} = \tfrac{\Bbbk - k}{2}. \end{aligned}$$

Now if $x \notin A \cap B$, then $\left(\overline{\mathbb{C}}_{A\cap B}\right)_k^{\Bbbk}(x) = 0$. Assume that $x \notin A$, then $\left(\mathbb{C}_A \left(\wedge\right)_k^{\Bbbk} \mathbb{C}_B\right)^- = \left(\mathbb{C}_A \left(\wedge\right)_k^{\Bbbk} \mathbb{C}_B\right)(x) = \mathbb{C}_A(x) \wedge \mathbb{C}_B(x) \wedge \frac{\Bbbk - k}{2} = 0 \wedge \mathbb{C}_B(x) \wedge \frac{\Bbbk - k}{2} = 0$.
Thus $\left(\mathbb{C}_A \left(\wedge\right)_k^{\Bbbk} \mathbb{C}_B\right)^- = \left(\overline{\mathbb{C}}_{A\cap B}\right)_k^{\Bbbk}$. The proof of part (2) follows from part (1).
(3) Assume $x \in S$, if $x \in (AB]$, then $\left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}(x) = \frac{\Bbbk - k}{2}$ and $x \leq yz$ for some $y \in A$ and $z \in B$. Hence $(y, z) \in A_x$, so

$$\begin{aligned} \left(\mathbb{C}_A \left(\circ\right)_k^{\Bbbk} \mathbb{C}_B\right)(x) &= \left(\mathbb{C}_A \circ \mathbb{C}_B\right)(x) \wedge \tfrac{\Bbbk - k}{2} \\ &= \left\{ \bigvee_{(a,b)\in A_x} \left(\mathbb{C}_A(a) \wedge \mathbb{C}_B(b)\right) \right\} \wedge \tfrac{\Bbbk - k}{2} \\ &\geq \mathbb{C}_A(y) \wedge \mathbb{C}_B(z) \wedge \tfrac{\Bbbk - k}{2} \\ &= 1 \wedge 1 \wedge \tfrac{\Bbbk - k}{2} = \tfrac{\Bbbk - k}{2} = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}(x). \end{aligned}$$

conversely, since $\mathbb{C}_A \circ \mathbb{C}_B)(x) \leq 1$ for all $x \in S$. Therefore, $(\mathbb{C}_A \,(\circ)_k^{\Bbbk}\, \mathbb{C}_B)(x) = (\mathbb{C}_A \circ \mathbb{C}_B)(x) \wedge \frac{\Bbbk - k}{2} \leq 1 \wedge \frac{\Bbbk - k}{2} = \frac{\Bbbk - k}{2} = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}(x)$. Hence, $(\mathbb{C}_A \,(\circ)_k^{\Bbbk}\, \mathbb{C}_B)^- = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}$ hold for $x \in (AB]$. By similar way, the require result also hold for $x \notin (AB]$. Consequently, $(\mathbb{C}_A \,(\circ)_k^{\Bbbk}\, \mathbb{C}_B)^- = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}$. □

Lemma 4.4. A non-empty subset A of an ordered semigroup S is a bi-ideal of S if and only if the lower part $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}$ of the characteristic function $\mathbb{C}_A$ of A is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.

Proof. If A is a bi-ideal of S, then by Theorem 2.2 and Lemma (3.1), $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.
Conversely, suppose that $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. Let $x, y \in S$ such that $x \leq y$. If $y \in A$, then $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(y) = \frac{\Bbbk - k}{2}$. Since $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, and $x \leq y$, we have, $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) \geq \left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(y) \wedge \frac{\Bbbk - k}{2} = \frac{\Bbbk - k}{2}$. It follows that $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) = \frac{\Bbbk - k}{2}$ and so $x \in A$. Let $x, y \in A$. Then, $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) = \frac{\Bbbk - k}{2}$ and $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(y) = \frac{\Bbbk - k}{2}$. Now,

$$\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(xy) \geq \left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) \wedge \left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(y) \wedge \frac{\Bbbk - k}{2} = \frac{\Bbbk - k}{2}.$$

Hence, $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(xy) = \frac{\Bbbk - k}{2}$ and so $xy \in A$. Now, let $x, z \in A$ and $y \in S$. Then, $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) = \frac{\Bbbk - k}{2}$ and $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(z) = \frac{\Bbbk - k}{2}$, therefore we have,

$$\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(xyz) \geq \left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) \wedge \left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(z) \wedge \frac{\Bbbk - k}{2} = \frac{\Bbbk - k}{2}.$$

Hence, $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(xyz) = \frac{\Bbbk - k}{2}$ and so $xyz \in A$. Therefore, A is a bi-ideal of S. □

Lemma 4.5. *A non-empty subset A of an ordered semigroup S is a left (resp. right)-ideal of S if and only if the lower part $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}$ of the characteristic function $\mathbb{C}_A$ of A is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy left (resp. right)-ideal of S.*

Proof. The proof follows from Lemma (4.4). □

In the following proposition, we show that if ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, then $\left(\overline{\xi}\right)_k^{\Bbbk}$ is a fuzzy bi-ideal of S.

Proposition 4.1. *If ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, then $\left(\overline{\xi}\right)_k^{\Bbbk}$ is a fuzzy bi-ideal of S.*

Proof. Let $x, y \in S$, $x \leq y$. Since ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S and $x \leq y$, we have $\xi(x) \geq \xi(y) \wedge \frac{\Bbbk-k}{2}$. It follows that $\xi(x) \wedge \frac{\Bbbk-k}{2} \geq \xi(y) \wedge \frac{\Bbbk-k}{2}$, and hence $\left(\overline{\xi}\right)_k^{\Bbbk}(x) \geq \left(\overline{\xi}\right)_k^{\Bbbk}(y)$. For $x, y \in S$, we have

$$\begin{aligned}
\xi(xy) &\geq \xi(x) \wedge \xi(y) \wedge \tfrac{\Bbbk-k}{2} \\
\xi(xy) \wedge \frac{\Bbbk-k}{2} &\geq \xi(x) \wedge \xi(y) \wedge \tfrac{\Bbbk-k}{2} \wedge \tfrac{\Bbbk-k}{2} \\
&= \left(\xi(x) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \left(\xi(y) \wedge \tfrac{\Bbbk-k}{2}\right),
\end{aligned}$$

and so $\left(\overline{\xi}\right)_k^{\Bbbk}(xy) \geq \left(\overline{\xi}\right)_k^{\Bbbk}(x) \wedge \left(\overline{\xi}\right)_k^{\Bbbk}(y)$.
Now for $x, y, z \in S$, we have

$$\begin{aligned}
\xi(xyz) &\geq \xi(x) \wedge \xi(z) \wedge \tfrac{\Bbbk-k}{2} \\
\xi(xy) \wedge \frac{\Bbbk-k}{2} &\geq \xi(x) \wedge \xi(z) \wedge \tfrac{\Bbbk-k}{2} \wedge \tfrac{\Bbbk-k}{2} \\
&= \left(\xi(x) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \left(\xi(z) \wedge \tfrac{\Bbbk-k}{2}\right),
\end{aligned}$$

so $\left(\overline{\xi}\right)_k^{\Bbbk}(xyz) \geq \left(\overline{\xi}\right)_k^{\Bbbk}(x) \wedge \left(\overline{\xi}\right)_k^{\Bbbk}(z)$. Consequently, $\left(\overline{\xi}\right)_k^{\Bbbk}$ is a fuzzy bi-ideal of S. $\square$

Numerous classes of ordered semigroups like regular, left (right) simple, completely regular and intra-regular ordered semigroups provide in-depth knowledge of semigroup theory. In the following, we characterize regular, left and right simple and completely regular ordered semigroups in terms of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy left (resp. right) and $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals.

Lemma 4.6. [12] An ordered semigroup S is completely regular if and only if for every $A \subseteq S, A \subseteq (A^2SA^2]$ or for $a \in S$, we have $a \in (a^2Sa^2]$.

Theorem 4.1. *If S is an ordered semigroup, then the following conditions are equivalent:*
(1) S is completely regular.
(2) For every $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal ξ of S, $\left(\overline{\xi}\right)_k^{\Bbbk}(a) = \left(\overline{\xi}\right)_k^{\Bbbk}\left(a^2\right)$ for all $a \in S$.

Proof. (1)$\Longrightarrow$ (2): Suppose S is completely regular and $a \in S$, by Lemma (4.6), $a \in (a^2Sa^2]$. Then there exists $x \in S$; such that $a \leq a^2xa^2$. Since ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, therefore

$$\begin{aligned}
\xi(a) &\geq \xi(a^2xa^2) \wedge \tfrac{\Bbbk-k}{2} \\
&\geq \left(\xi(a^2) \wedge \xi(a^2) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2} \\
&= \left(\xi(a^2 = a.a) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2} \\
&\geq \left(\xi(a) \wedge \xi(a) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2} \\
&= \left(\xi(a) \wedge \tfrac{\Bbbk-k}{2}\right)
\end{aligned}$$

it follows that $\left(\overline{\xi}\right)_k^{\Bbbk}(a) = \xi(a) \wedge \frac{\Bbbk - k}{2} \geq \left(\xi(a^2) \wedge \frac{\Bbbk - k}{2}\right) = \left(\overline{\xi}\right)_k^{\Bbbk}\left(a^2\right) \geq \xi(a) \wedge \frac{\Bbbk - k}{2}$. Hence $\left(\overline{\xi}\right)_k^{\Bbbk}(a) = \left(\overline{\xi}\right)_k^{\Bbbk}\left(a^2\right)$ for all $a \in S$.
(2)$\Longrightarrow$ (1): Let $a \in S$, consider the bi-ideal $B\left(a^2\right) = (a^2 \cup a^4 \cup a^2 S a^2]$ of S generated by a^2, then by Lemma (3.1), $\mathbb{C}_{B(a^2)}$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. By (2) $\left(\overline{\mathbb{C}}_{B(a^2)}\right)_k^{\Bbbk}(a) = \left(\overline{\mathbb{C}}_{B(a^2)}\right)_k^{\Bbbk}\left(a^2\right)$. Since $a^2 \in B\left(a^2\right)$, so $\left(\overline{\mathbb{C}}_{B(a^2)}\right)_k^{\Bbbk}\left(a^2\right) = \frac{\Bbbk - k}{2}$. Thus $\left(\overline{\mathbb{C}}_{B(a^2)}\right)_k^{\Bbbk}(a) = \frac{\Bbbk - k}{2}$. Hence $a \in B\left(a^2\right)$ it implies that $a \leq a^2$ or $a \leq a^4$ or $a \leq a^2 x a^2$ for some $x \in S$. If $a \leq a^2$, then $a \leq a^2 = a.a \leq a^2.a^2 = a.a.a^2 \leq a^2 a a^2 \in a^2 S a^2$ and $a \in (a^2 S a^2]$. Similarly, if $a \leq a^4$ or $a \leq a^2 x a^2$ we get $a \in (r^2 S s^2]$ for some $r, s \in S$. Thus S is completely regular. □

An equivalence relation ρ on S is called congruence if $(x, y) \in \rho$ implies $(xz, yz) \in \rho$ and $(zx, zy) \in \rho$ for every $z \in S$. A congruence ρ on S is called semi lattice congruence [12] if $\left(x, x^2\right) \in \rho$ and $(xy, yx) \in \rho$. An ordered semigroup S is called a semi lattice of left and right simple semigroups if there exists a semi lattice congruence ρ on S such that the ρ-class $(x)_\rho$ of S containing x is a left and right simple subsemigroup of S for every $x \in S$, or equivalently, there exists a semilattice Y and a family $\{S_i : i \in Y\}$ of left and right simple subsemigroups of S such that

$$S_i \cap S_j = \emptyset i \neq j, S = \bigcup_{i \in Y} S_i, S_i S_j \subseteq S_{ij} \forall i, j \in Y.$$

A subset P of S is called semiprime [7], if for every $a \in S$ such that $a^2 \in P$, we have $a \in P$, or equivalently, for each subset A of S, such that $A^2 \subseteq P$, implies that have $A \subseteq P$.
Let $\mathbb{N}$ be the equivalence relation on S which is denoted by

$$\mathbb{N} = \{(a, b) \in S \times S \mid N(x) = N(y)\}.$$

Lemma 4.7. [12] Let S be an ordered semigroup, then the following conditions are equivalent:
(i) $(x)_{\mathbb{N}}$ is a left (resp. right) simple subsemigroup of S, for every $x \in S$.
(ii) Every left (resp. right) ideal of S is a right (resp. left) ideal of S and semiprime.

Lemma 4.8. [12] An ordered semigroup S is a semilattice of left and right simple semigroups if and only if for all bi-ideals A and B of S, we have $(A^2] = A$ and $(B^2] = B$.

Theorem 4.2. *An ordered semigroup S is a semilattice of left and right simple semigroups if and only if for every $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal ξ of S, then for all $a, b \in S$,*
(i) $\left(\overline{\xi}\right)_k^{\Bbbk}(a) = \left(\overline{\xi}\right)_k^{\Bbbk}\left(a^2\right)$.
(ii) $\left(\overline{\xi}\right)_k^{\Bbbk}(ab) = \left(\overline{\xi}\right)_k^{\Bbbk}(ba)$.

Proof. If ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal, then by hypothesis, there exists a semilattice Y and a family $\{\xi_i : i \in Y\}$of left and right simple subsemigroups of S such that $S_i \cap S_j = \emptyset$, $i \neq j, S = \bigcup_{i \in Y} S_i, S_iS_j \subseteq S_{ij}$ for all $i, j \in Y$. (i): Let $a \in S$, then there exists Y such that $a \in S_i$, as S_i is left and right simple, thus $(S_ia] = S_i$ and $(aS_i] = S_i$. Therefore, $S_i = (aS_i] = (a((S_ia]] = (aS_ia]$. Since $a \in (aS_ia]$ so there exists $x \in S_i$ such that $a \leq axa$. Since $x \in S_i$, $x \leq aya$ for some $y \in S_i$. Therefore, $a \leq axa \leq a(aya)a = a^2ya^2$ implies that $a \in (a^2Sa^2]$. Hence by Lemma (4.6) and Theorem (4.1), $\left(\overline{\xi}\right)_k^{\Bbbk}(a) = \left(\overline{\xi}\right)_k^{\Bbbk}\left(a^2\right)$.
(ii) Now to prove $\left(\overline{\xi}\right)_k^{\Bbbk}(ab) = \left(\overline{\xi}\right)_k^{\Bbbk}(ba)$, let $a, b \in S$, then by (i), $\left(\overline{\xi}\right)_k^{\Bbbk}(ab) = \left(\overline{\xi}\right)_k^{\Bbbk}\left((ab)^2\right) = \left(\overline{\xi}\right)_k^{\Bbbk}\left((ab)^4\right)$. Also, by Lemma (4.8),

$$\begin{aligned}
(ab)^4 \quad &= (ab)^2\,(ab)^2 = (ab)(ab)(ab)(ab)\\
&= (aba)(babab) \in B(aba)B(babab)\\
&\subseteq (B(aba)B(babab)] = (B(babab)B(aba)]\\
&= (B(babab)\,(B(aba))^2] = (B(babab)B(aba)B(aba)]\\
&\subseteq ((babab)S(aba)(aba)] \subseteq (bababSaba].
\end{aligned}$$

Hence $(ab)^4 \leq (babab)z(aba)$ for some $z \in S$. As ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, hence

$$\begin{aligned}
\xi((ab)^4) \quad &\geq \xi\left((babab)z(aba)\right) \wedge \tfrac{\Bbbk-k}{2}\\
&= \xi((ba)(babza)(ba)) \wedge \tfrac{\Bbbk-k}{2}\\
&\geq \left(\xi((ba) \wedge (ba)) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2}\\
&= \xi(ba) \wedge \tfrac{\Bbbk-k}{2}\\
\xi((ab)^4) \wedge \frac{\Bbbk-k}{2} \quad &\geq \left(\xi(ba) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2}
\end{aligned}$$

implies that $\left(\overline{\xi}\right)_k^{\Bbbk}\left((ab)^4\right) \geq \left(\overline{\xi}\right)_k^{\Bbbk}((ba))$. Thus

$$\begin{aligned}
\left(\overline{\xi}\right)_k^{\Bbbk}(ab) \quad &= \left(\overline{\xi}\right)_k^{\Bbbk}\left((ab)^2\right)\\
&= \left(\overline{\xi}\right)_k^{\Bbbk}\left((ab)^4\right) \geq \left(\overline{\xi}\right)_k^{\Bbbk}((ba))
\end{aligned}$$

leads to $\left(\overline{\xi}\right)_k^{\Bbbk}(ab) \geq \left(\overline{\xi}\right)_k^{\Bbbk}((ba))$. In a similar way $\left(\overline{\xi}\right)_k^{\Bbbk}(ba) \geq \left(\overline{\xi}\right)_k^{\Bbbk}((ab))$ can be shown. Hence $\left(\overline{\xi}\right)_k^{\Bbbk}(ab) = \left(\overline{\xi}\right)_k^{\Bbbk}(ba)$.
Conversely, we know that $\mathbb{N}$ is a semilattice of left and right simple semigroups, so by Lemma (4.7), it is enough to prove that every left (resp. right) ideal of S is an ideal of S. Let L be a left ideal of S and let $a \in L$ and $t \in S$. Since L is a left ideal of S, by Lemma (4.4), $\left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy left ideal of S. Hence $\left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}(at) = \left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}(ta)$. As $ta \in SL \subseteq L$ it implies that $\left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}(ta) = \frac{\Bbbk-k}{2}$. So $at \in L$ that is $LS \subseteq L$. Thus, L is right ideal. if $a^2 \in L$, by hypothesis

$\left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}\left(a^2\right)=\frac{\Bbbk-k}{2}=\left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}(a)$. Thus $a\in L$, so L is semiprime. Similarly, we can prove that right ideal R is left ideal of S and semiprime. $\square$

Proposition 4.2. *If $\{\xi_i : i\in I\}$ is a family of $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideals of an ordered semigroup S. Then, $\bigcap\limits_{i\in I}\bigcap\limits_{i\in I}\left(\overline{\xi_i}\right)_k^{\Bbbk}$ is an $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal of S.*

Corollary 4.1. *Let S be an ordered semigroup and f_1 and f_2 be fuzzy subsets of S. Then, $\left(f_1\,(\wedge)_k^{\Bbbk}\,f_2\right)^-$ is an $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal of S.*

Definition 4.2. *An $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal ξ of S is called idempotent if $\xi\,(\wedge)_k^{\Bbbk}\,\xi=\xi$.*

Theorem 4.3. *If ξ is an $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal of S, then $\left(\xi\,(\circ)_k^{\Bbbk}\,\xi\right)^-\preceq\left(\overline{\xi}\right)_k^{\Bbbk}$.*

Proof. Suppose that ξ is an $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal of S and $a\in S$. If $A_a=\emptyset$, then $\left(\xi\,(\circ)_k^{\Bbbk}\,\xi\right)^-(a)=(\xi\circ\xi)(a)\wedge\frac{\Bbbk-k}{2}=0\wedge\frac{\Bbbk-k}{2}=0\leq\left(\overline{\xi}\right)_k^{\Bbbk}(a)$. Thus, $\left(\xi\,(\circ)_k^{\Bbbk}\,\xi\right)^-\preceq\left(\overline{\xi}\right)_k^{\Bbbk}$ hold in this case. Now let $A_a\neq\emptyset$, then

$$\begin{aligned}\left(\xi\,(\circ)_k^{\Bbbk}\,\xi\right)^-(a) &= (\xi\circ\xi)(a)\wedge\tfrac{\Bbbk-k}{2}\\ &=\left\{\bigvee\nolimits_{y,z\in A}(\xi(y)\wedge\xi(z))\right\}\wedge\tfrac{\Bbbk-k}{2}\\ &\leq\left\{\bigvee\nolimits_{y,z\in A}\xi(yz)\right\}\wedge\tfrac{\Bbbk-k}{2}\\ &\leq\left\{\bigvee\nolimits_{y,z\in A}\xi(a)\right\}\wedge\tfrac{\Bbbk-k}{2}=\xi(a)\wedge\tfrac{\Bbbk-k}{2}=\left(\overline{\xi}\right)_k^{\Bbbk}(a).\end{aligned}$$

Hence, $\left(\xi\,(\circ)_k^{\Bbbk}\,\xi\right)^-\preceq\left(\overline{\xi}\right)_k^{\Bbbk}$. $\square$

Lemma 4.9. *Every $(\in,\in\vee(\Bbbk,q_k))$-fuzzy one-sided ideal of S is an $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal of S.*

Proof. The proof is straightforward. $\square$

If S is an ordered semigroup, then we define the fuzzy subsets "1" and "0" as follows:

$$\begin{aligned}1 &: S\longrightarrow[0,1]|x\longrightarrow 1(x)=1,\\ 0 &: S\longrightarrow[0,1]|x\longrightarrow 0(x)=0,\end{aligned}$$

for all $x\in S$.

Lemma 4.10. *Let S be an ordered semigroup and f_1 and f_2 be fuzzy subsets of S. Then, $(f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \preceq (1 \,(\circ)_k^{\Bbbk}\, f_2)^-$(resp. $(f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \preceq (f_1 \,(\circ)_k^{\Bbbk}\, 1)^-$).*

Proof. The proof is straightforward. $\square$

Lemma 4.11. *Let S be an ordered semigroup and f an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. Then, $(f \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f)^- \preceq \left(\overline{f}\right)_k^{\Bbbk}$.*

Proof. Let $a \in S$. If $A_a = \emptyset$, then

$$(f \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f)^-(a) = (f \circ 1 \circ f)(a) \wedge \frac{\Bbbk - k}{2} = 0 \wedge \frac{\Bbbk - k}{2} = 0 \leq \left(\overline{f}\right)_k^{\Bbbk}(a).$$

Let $A_a \neq \emptyset$, then

$$\begin{aligned}
(f \circ^k 1 \circ^k f)^-(a) &= (f \circ 1 \circ f)(a) \wedge \tfrac{\Bbbk - k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} (f(y) \wedge (1 \circ f)(z))\right] \wedge \tfrac{\Bbbk - k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} \left(f(y) \wedge \left\{\bigvee_{(p,q)\in A_z} \bigvee_{(p,q)\in A_z} (1(p) \wedge f(q))\right\}\right)\right] \wedge \tfrac{\Bbbk - k}{2} \\
&= \bigvee\nolimits_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \bigvee_{(p,q)\in A_z} (f(y) \wedge 1(p) \wedge f(q)) \wedge \tfrac{\Bbbk - k}{2} \\
&= \bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \bigvee_{(p,q)\in A_z} (f(y) \wedge f(q)) \wedge \tfrac{\Bbbk - k}{2} \\
&= \bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \bigvee_{(p,q)\in A_z} \left(\left(f(y) \wedge \tfrac{\Bbbk - k}{2}\right) \wedge \left(f(q) \wedge \tfrac{\Bbbk - k}{2}\right)\right) \wedge \tfrac{\Bbbk - k}{2}.
\end{aligned}$$

Since $a \leq yz \leq y(pq) = ypq$ and f is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, so we have,

$$\begin{aligned}
f(a) &\geq f(ypq) \wedge \tfrac{\Bbbk - k}{2} \geq \left(f(y) \wedge f(q) \wedge \tfrac{\Bbbk - k}{2}\right) \wedge \tfrac{\Bbbk - k}{2} \\
&= \left\{\left(f(y) \wedge \tfrac{\Bbbk - k}{2}\right) \wedge \left(f(q) \wedge \tfrac{\Bbbk - k}{2}\right)\right\} \wedge \tfrac{\Bbbk - k}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \bigvee_{(p,q)\in A_z} \left(\left(f(y) \wedge \frac{\Bbbk - k}{2}\right) \wedge \left(f(q) \wedge \frac{\Bbbk - k}{2}\right)\right) \wedge \frac{\Bbbk - k}{2} \\
&\leq \bigvee_{(y,pq)\in A_a} \bigvee_{(y,pq)\in A_a} \left(\left(f(y) \wedge \frac{\Bbbk - k}{2}\right) \wedge \left(f(q) \wedge \frac{\Bbbk - k}{2}\right)\right) \wedge \frac{\Bbbk - k}{2} \\
&\leq \bigvee_{(y,pq)\in A_a} \bigvee_{(y,pq)\in A_a} f(a) \wedge \frac{\Bbbk - k}{2} = \left(\overline{f}\right)_k^{\Bbbk}(a).
\end{aligned}$$

$\square$

Lemma 4.12. [7] Let S be an ordered semigroup. Then the following are equivalent:
(i) S is regular,
(ii) $B = (BSB]$ for all bi-ideals B of S,
(iii) $B(a) = (B(a)SB(a)]$ for every $a \in S$.

Theorem 4.4. *If S is an ordered semigroup and f is a fuzzy subset of S, then the following conditions are equivalent:*
(1) S is regular.
(2) For every $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal f of S, $(f \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f)^- = \left(\overline{f}\right)_k^{\Bbbk}$.

Proof. (1) $\Longrightarrow$ (2): Let S is regular and f is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. Assume $a \in S$. then, there exists $x \in S$ such that $a \leq axa \leq ax(axa) = a(xaxa)$. So $(a, xaxa) \in A_a$, and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}
(f \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f)^-(a) &= (f \circ 1 \circ f)(a) \wedge \tfrac{\Bbbk - k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} (f(y) \wedge (1 \circ f)(z)) \right] \wedge \tfrac{\Bbbk - k}{2} \\
&\geq \left(f(a) \wedge (1 \circ f)(xaxa) \wedge \tfrac{\Bbbk - k}{2} \right) \\
&= \left[f(a) \wedge \bigvee_{(p,q)\in A_{xaxa}} \bigvee_{(p,q)\in A_{xaxa}} \{1(p) \wedge f(q)\} \right] \wedge \tfrac{\Bbbk - k}{2} \\
&\geq (f(a) \wedge \{1(xax) \wedge f(a)\}) \wedge \tfrac{\Bbbk - k}{2} \\
&= (f(a) \wedge \{1 \wedge f(a)\}) \wedge \tfrac{\Bbbk - k}{2} \\
&= (f(a) \wedge f(a)) \wedge \tfrac{\Bbbk - k}{2} = f(a) \wedge \tfrac{\Bbbk - k}{2} \\
&= \left(\overline{f}\right)_k^{\Bbbk}(a).
\end{aligned}$$

On the other hand, by Lemma (4.11), we have, $(f \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f)^-(a) \leq \left(\overline{f}\right)_k^{\Bbbk}(a)$. Therefore, $(f \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f)^-(a) = \left(\overline{f}\right)_k^{\Bbbk}(a)$.
(2) $\Longrightarrow$ (1): Suppose that $(f \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f)^- = \left(\overline{f}\right)_k^{\Bbbk}$ for every $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal f of S. To prove that S is regular, by Lemma (4.12), it is enough to prove that

$$A = (ASA] \forall \, bi - ideals \, A \, of \, S.$$

Let $x \in A$. Since A is a bi-ideal of S, by Lemma (4.4), $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. By hypothesis, $(\mathbb{C}_A \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, \mathbb{C}_A)^-(x) = \left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x)$. Since $x \in A$, we have $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) = \frac{\Bbbk - k}{2}$. Thus, $(\mathbb{C}_A \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, \mathbb{C}_A)^-(x) = \frac{\Bbbk - k}{2}$. But, by Lemma (4.3), we have $(\mathbb{C}_A \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, \mathbb{C}_A)^- = \left(\overline{\mathbb{C}}_{(ASA]}\right)_k^{\Bbbk}$, and $\left(\overline{\mathbb{C}}_{(ASA]}\right)_k^{\Bbbk}(x) = \frac{\Bbbk - k}{2}$, hence we have, $x \in (ASA]$ and so $A \subseteq (ASA]$. On the other hand, since A is a bi-ideal of S, we have $(ASA] \subseteq (A] = A$. Hence, $A = (ASA]$. Therefore, S is regular. $\square$

Lemma 4.13. *Let f_1 and f_2 be $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of S. Then $(f_1 \,(\circ)_k^{\Bbbk}\, f_2)^-$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.*

Proof. The proof is straightforward. $\square$

Lemma 4.14. *Let S be an ordered semigroup. Then the following are equivalent:*
(i) S is both regular and intra-regular,
(ii) $A = (A^2]$ for every bi-ideals A of S,
(iii) $A \cap B = (AB] \cap (BA]$ for all bi-ideals A, B of S.

Theorem 4.5. *Let S be an ordered semigroup. Then the following are equivalent:*
(i) S is both regular and intra-regular,
(ii) $(f \,(\circ)_k^{\Bbbk}\, f)^- = \left(\overline{f}\right)_k^{\Bbbk}$ for every $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals f of S,
(iii) $(f_1 \,(\wedge)_k^{\Bbbk}\, f_2)^- = \Big((f_1 \,(\circ)_k^{\Bbbk}\, f_2) \wedge_k^{\Bbbk} (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-\Big)^-$ for all $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals f_1 and f_2 of S.

Proof. (i)$\Longrightarrow$(ii). Let F be an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S and $a \in S$. Since S is regular and intra-regular, there exist $x, y, z \in S$ such that $a \leq axa \leq axaxa$ and $a \leq ya^2z$. Then, $a \leq axaxa \leq ax(ya^2z)xa = (axya)(azxa)$ and $(axya, azxa) \in A_a$. Thus,

$$
\begin{aligned}
(f \,(\circ)_k^{\Bbbk}\, f)^-(a) &= (f \circ f)(a) \wedge \tfrac{\Bbbk-k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} (f(y) \wedge f(z))\right] \wedge \tfrac{\Bbbk-k}{2} \\
&\geq \{(f(axya) \wedge f(azxa))\} \wedge \tfrac{\Bbbk-k}{2} \\
&\geq \left\{\left(f(a) \wedge f(a) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \left(f(a) \wedge f(a) \wedge \tfrac{\Bbbk-k}{2}\right)\right\} \wedge \tfrac{\Bbbk-k}{2} \\
&= \left(f(a) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2} = \left(f(a) \wedge \tfrac{\Bbbk-k}{2}\right) = \left(\overline{f}\right)_k^{\Bbbk}(a).
\end{aligned}
$$

On the other hand, by Theorem (4.3), $(f \,(\circ)_k^{\Bbbk}\, f)^-(a) \leq \left(\overline{f}\right)_k^{\Bbbk}(a)$.
(ii)$\Longrightarrow$(iii). Let f_1 and f_2 be $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of S. Then, by Corollary (4.1), $(f_1 \,(\wedge)_k^{\Bbbk}\, f_2)^-$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. By (ii),

$$
(f_1 \,(\wedge)_k^{\Bbbk}\, f_2)^- = \Big((f_1 \,(\wedge)_k^{\Bbbk}\, f_2)^- \,(\circ)_k^{\Bbbk}\, (f_1 \,(\wedge)_k^{\Bbbk}\, f_2)^-\Big)^- \preceq (f_1 \,(\circ)_k^{\Bbbk}\, f_2)^-.
$$

In a similar way, one can prove that, $(f_1 \,(\wedge)_k^{\Bbbk}\, f_2)^- \preceq (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-$.
Thus, $(f_1 \,(\wedge)_k^{\Bbbk}\, f_2)^- \preceq ((f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \,(\wedge)_k^{\Bbbk}\, (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-)^-$. Moreover, $(f_1 \,(\circ)_k^{\Bbbk}\, f_2)^-$ and $(f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-$ are $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of S by Corollary (4.1), and hence,
$(f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \,(\wedge)_k^{\Bbbk}\, (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. Using (ii),

we have,

$$
\begin{aligned}
&((f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \,(\wedge)_k^{\Bbbk}\, (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-)^- \\
&= \Big(((f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \,(\wedge)_k^{\Bbbk}\, (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-) \,(\circ)_k^{\Bbbk}\, ((f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \,(\wedge)_k^{\Bbbk}\, (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-)\Big)^- \\
&\preceq \Big((f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \,(\circ)_k^{\Bbbk}\, (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-\Big)^- = (f_1 \,(\circ)_k^{\Bbbk}\, (f_2 \,(\circ)_k^{\Bbbk}\, f_2) \,(\circ)_k^{\Bbbk}\, f_1)^- \\
&= (f_1 \,(\circ)_k^{\Bbbk}\, f_2 \,(\circ)_k^{\Bbbk}\, f_1)^- as (f_2 \,(\circ)_k^{\Bbbk}\, f_2)^- = \left(\overline{f_2}\right)_k^{\Bbbk} by (i) above \\
&\preceq (f_1 \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f_1)^- = \left(\overline{f}\right)_k^{\Bbbk} as (f_1 \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f_1)^- = \left(\overline{f}\right)_k^{\Bbbk} by theorem (4.4)
\end{aligned}
$$

In a similar way, one can prove that, $((f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \,(\wedge)_k^{\Bbbk}\, (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-)^- \preceq \left(\overline{f_2}\right)_k^{\Bbbk}$. Consequently, $((f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \,(\wedge)_k^{\Bbbk}\, (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-)^- \preceq \left(\overline{f_1}\right)_k^{\Bbbk} \wedge \left(\overline{f_2}\right)_k^{\Bbbk} = (f_1 \,(\wedge)_k^{\Bbbk}\, f_2)^-$. Therefore, we get $(f_1 \,(\wedge)_k^{\Bbbk}\, f_2)^- = ((f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \,(\wedge)_k^{\Bbbk}\, (f_2 \,(\circ)_k^{\Bbbk}\, f_1)^-)^-$.
(iii)$\Longrightarrow$(i). To prove that S is both regular and intra-regular, by Lemma (4.14), it is enough to prove that $A \cap B = (AB] \cap (BA]$ for all bi-ideals A and B of S. Let $x \in A \cap B$. Then, $x \in A$ and $x \in B$. By Lemma (4.4), $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}$ and $\left(\overline{\mathbb{C}}_B\right)_k^{\Bbbk}$ are $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of S. Using (iii), we have

$$
\begin{aligned}
((\mathbb{C}_A \,(\circ)_k^{\Bbbk}\, \mathbb{C}_B)^- \,(\wedge)_k^{\Bbbk}\, (\mathbb{C}_B \,(\circ)_k^{\Bbbk}\, \mathbb{C}_A)^-)^-(x) &= (\mathbb{C}_A \,(\wedge)_k^{\Bbbk}\, \mathbb{C}_B)^-(x) \\
&= \left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) \wedge \left(\overline{\mathbb{C}}_B\right)_k^{\Bbbk}(x).
\end{aligned}
$$

Since $x \in A$ and $x \in B$, we have $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) = \frac{\Bbbk-k}{2}$ and $\left(\overline{\mathbb{C}}_B\right)_k^{\Bbbk}(x) = \frac{\Bbbk-k}{2}$. Thus, $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) \wedge \left(\overline{\mathbb{C}}_B\right)_k^{\Bbbk}(x) = \frac{\Bbbk-k}{2} \wedge \frac{\Bbbk-k}{2} = \frac{\Bbbk-k}{2}$. It follows that

$$
((\mathbb{C}_A \,(\circ)_k^{\Bbbk}\, \mathbb{C}_B)^- \,(\wedge)_k^{\Bbbk}\, (\mathbb{C}_B \,(\circ)_k^{\Bbbk}\, \mathbb{C}_A)^-)^-(x) = \frac{\Bbbk - k}{2}.
$$

By Lemma (4.3) , we have $((\mathbb{C}_A \,(\circ)_k^{\Bbbk}\, \mathbb{C}_B)^- \,(\wedge)_k^{\Bbbk}\, (\mathbb{C}_B \,(\circ)_k^{\Bbbk}\, \mathbb{C}_A)^-)^- = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk} \wedge \left(\overline{\mathbb{C}}_{(BA]}\right)_k^{\Bbbk} = \left(\overline{\mathbb{C}}_{(AB] \cap (BA]}\right)_k^{\Bbbk}$. Thus, $\left(\overline{\mathbb{C}}_{(AB] \cap (BA]}\right)_k^{\Bbbk}(x) = \frac{\Bbbk-k}{2}$ and $x \in (AB] \cap (BA]$. Moreover, if $x \in (AB] \cap (BA]$, then,

$$
\begin{aligned}
\frac{\Bbbk - k}{2} &= \left(\overline{\mathbb{C}}_{(AB] \cap (BA]}\right)_k^{\Bbbk}(x) \\
&= \Big(\left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk} \wedge \left(\overline{\mathbb{C}}_{(BA]}\right)_k^{\Bbbk}\Big)(x) \\
&= ((\mathbb{C}_A \,(\circ)_k^{\Bbbk}\, \mathbb{C}_B)^- \,(\wedge)_k^{\Bbbk}\, (\mathbb{C}_B \,(\circ)_k^{\Bbbk}\, \mathbb{C}_A)^-)^-(x) \\
&= (\mathbb{C}_A \,(\wedge)_k^{\Bbbk}\, \mathbb{C}_B)^-(x) (by (iii)) \\
&= \left(\overline{\mathbb{C}}_{A \cap B}\right)_k^{\Bbbk}(x).
\end{aligned}
$$

Thus, $\left(\overline{\mathbb{C}}_{A \cap B}\right)_k^{\Bbbk}(x) = \frac{\Bbbk-k}{2}$ and $x \in A \cap B$. Therefore, $A \cap B = (AB] \cap (BA]$, consequently, S is both regular and intra-regular. □

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