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FIXED-CIRCLE PROBLEM ON S-METRIC SPACES WITH A GEOMETRIC VIEWPOINT

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Abstract. Recently, a new geometric approach called the fixed-circle problem has been introduced to fixed-point theory. The problem has been studied using different techniques on metric spaces. In this paper, we consider the fixed-circle problem on S-metric spaces. We investigate existence and uniqueness conditions for fixed circles of self-mappings on an S-metric space. Some examples of self-mappings having fixed circles are also given.
Keywords: fixed-circle problem; self-mapping; S-metric space.

1. Introduction

The existence and uniqueness theorems of fixed points of self-mappings satisfying some contractive conditions have been extensively studied since the time of Stefan Banach (see [1, 2]). Many authors have investigated new fixed-point theorems on metric spaces or generalizations of metric spaces. For example, Sedghi, Shobe and Aliouche obtained Banach's contraction principle on S-metric spaces [12]. We studied some generalizations of Banach's contraction principle on an S-metric space [8] and investigated new fixed-point theorems for the following contractive condition (which is called Rhoades' condition [11]) (see [6, 14]):

$$(S25) \quad \mathcal{S}(Tx,Tx,Ty) < \max\{\mathcal{S}(x,x,y),\mathcal{S}(Tx,Tx,x),\mathcal{S}(Ty,Ty,y),\mathcal{S}(Ty,Ty,x),\mathcal{S}(Tx,Tx,y)\},$$

for each $x,y \in X$, $x \neq y$. We then gave the concept of diameter and obtained a new contractive condition using this notion as follows [6]:

$$(S25a) \quad \mathcal{S}(Tx,Tx,Ty) < diam\{U_x \cup U_y\},$$

for each $x,y \in X$ $(x \neq y)$, where $U_x = \{T^n x : n \in \mathbb{N}\}$, $U_y = \{T^n y : n \in \mathbb{N}\}$, $diam\{U_x\} < \infty$ and $diam\{U_y\} < \infty$.

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Although the existence of fixed points of functions has been studied on various metric spaces, there is no study on the existence of fixed circles. Therefore, the fixed-circle problem arises naturally. There are some examples of functions with a fixed circle on some special metric spaces. For example, let $\mathbb{C}$ be an S-metric space with the S-metric

$$\mathcal{S}(z,w,t)=\frac{|z-t|+|w-t|}{2},$$

for all $z,w,t\in\mathbb{C}$. Let the mapping T be defined as

$$Tz=\frac{1}{\overline{z}},$$

for all $z\in\mathbb{C}\backslash\{0\}$. The mapping T fixes the unit circle $C_{0,1}^{S}=\{x\in X:\mathcal{S}(x,x,0)=1\}$.

Recently, Özdemir, İskender and Özgür used new types of activation functions having a fixed circle for a complex valued neural network [5]. The usage of these types activation functions leads us to guarantee the existence of fixed points of the complex valued Hopfield neural network (see [5] for more details).

Hence it is important to investigate some fixed-circle theorems on various metric spaces. In [9], we obtained some fixed-circle theorems on metric spaces. We studied some existence theorems for fixed circles with a geometric interpretation and gave necessary conditions for the uniqueness of fixed circles. Also, we provided some examples of self-mappings with fixed circles. On the other hand, we proved new fixed-circle results and applied the obtained results to the discontinuity problem and discontinuous activation functions [10].

Motivated by the above studies, our aim in this paper is to obtain some fixed-circle theorems for self-mappings on S-metric spaces. In Section 2., we recall some necessary definitions, lemmas and basic facts. In Section 3., we introduce the notion of a fixed circle on an S-metric space and then obtain some existence and uniqueness theorems for self-mappings having fixed circles via different techniques. We investigate the case in which the number of fixed circles is infinitely many. Some examples of self-mappings with fixed circles are given with a geometric viewpoint. Using Mathematica (Wolfram Research, Inc., Mathematica, Trial Version, Champaign, IL (2016)), we draw some figures related to the given examples.

2. Preliminaries

Definition 2.1. [12] Let X be a nonempty set and $\mathcal{S}:X^3\to[0,\infty)$ be a function satisfying the following conditions for all $x,y,z,a\in X$.

1. $\mathcal{S}(x,y,z)=0$ if and only if $x=y=z$,
2. $\mathcal{S}(x,y,z)\leq\mathcal{S}(x,x,a)+\mathcal{S}(y,y,a)+\mathcal{S}(z,z,a)$.

Then S is called an S-metric on X and the pair $(X, \mathcal{S})$ is called an S-metric space.

The following lemma can be considered as the symmetry condition and it will be used in the proofs of some theorems.

Lemma 2.1. *[12] Let $(X, \mathcal{S})$ be an S-metric space. Then we have*

$$\mathcal{S}(x, x, y) = \mathcal{S}(y, y, x).$$

The relationships between a metric and an S-metric was given in what follows.

Lemma 2.2. *[4] Let (X, d) be a metric space. Then the following properties are satisfied*:

1. *$\mathcal{S}_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.*
2. *$x_n \to x$ in (X, d) if and only if $x_n \to x$ in $(X, \mathcal{S}_d)$.*
3. *$\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in $(X, \mathcal{S}_d)$.*
4. *(X, d) is complete if and only if $(X, \mathcal{S}_d)$ is complete.*

The metric $\mathcal{S}_d$ was called an S-metric generated by d [7]. We know some examples of an S-metric which are not generated by any metric (see [4, 7, 14] for more details).

On the other hand, Gupta claimed that every S-metric on X defines a metric d_S on X as follows:

$$d_S(x, y) = S(x, x, y) + S(y, y, x), \tag{2.1}$$

for all $x, y \in X$ [3]. However, the function $d_S(x, y)$ defined in (2.1) does not always define a metric because the triangle inequality is not satisfied for all elements of X everywhere (see [7] for more details).

The notions of an open ball, a closed ball and diameter were introduced on S-metric spaces as the following definitions.

Definition 2.2. [12] Let $(X, \mathcal{S})$ be an S-metric space. The open ball $B_S(x_0, r)$ and closed ball $B_S[x_0, r]$ with a center x_0 and a radius r are defined by

$$B_S(x_0, r) = \{x \in X : \mathcal{S}(x, x, x_0) < r\}$$

and

$$B_S[x_0, r] = \{x \in X : \mathcal{S}(x, x, x_0) \leq r\},$$

for $r > 0$ and $x_0 \in X$.

Definition 2.3. [6] Let $(X, \mathcal{S})$ be an S-metric space and A be a nonempty subset of X. The diameter of A is defined by

$$diam\{A\} = sup\{\mathcal{S}(x, x, y) : x, y \in A\}.$$

If A is S-bounded, then we will write $diam\{A\} < \infty$.

Now we define the notion of a circle on an S-metric space.

Definition 2.4. Let $(X, \mathcal{S})$ be an S-metric space and $x_0 \in X$, $r \in (0, \infty)$. We define the circle centered at x_0 with the radius r as

$$C^S_{x_0,r} = \{x \in X : \mathcal{S}(x, x, x_0) = r\}.$$

3. Some Fixed-Circle Theorems on S-Metric Spaces

In this section, we introduce the notion of a fixed circle on an S-metric space. Then we investigate some existence and uniqueness theorems for self-mappings having fixed circles.

Definition 3.1. Let $(X, \mathcal{S})$ be an S-metric space, $C^S_{x_0,r}$ be a circle on X and $T : X \to X$ be a self-mapping. If $Tx = x$ for all $x \in C^S_{x_0,r}$ then the circle $C^S_{x_0,r}$ is said to be a fixed circle of T.

3.1. The existence of fixed circles

We obtain some existence theorems for fixed circles of self-mappings.

Theorem 3.1. *Let $(X, \mathcal{S})$ be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Let us define the mapping*

$$\varphi : X \to [0, \infty), \varphi(x) = \mathcal{S}(x, x, x_0), \tag{3.1}$$

for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying

$$\mathcal{S}(x, x, Tx) \leq \varphi(x) + \varphi(Tx) - 2r \tag{3.2}$$

and

$$\mathcal{S}(x, x, Tx) + \mathcal{S}(Tx, Tx, x_0) \leq r, \tag{3.3}$$

for all $x \in C^S_{x_0,r}$, then $C^S_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C^S_{x_0,r}$. Then using the conditions (3.2), (3.3), Lemma 2.1 and the triangle inequality, we get

$$\begin{aligned}
\mathcal{S}(x,x,Tx) &\leq \varphi(x)+\varphi(Tx)-2r \\
&= \mathcal{S}(x,x,x_0)+\mathcal{S}(Tx,Tx,x_0)-2r \\
&\leq \mathcal{S}(x,x,Tx)+\mathcal{S}(x,x,Tx)+\mathcal{S}(Tx,Tx,x_0)+\mathcal{S}(Tx,Tx,x_0)-2r \\
&= 2\mathcal{S}(x,x,Tx)+2\mathcal{S}(Tx,Tx,x_0)-2r \\
&\leq 2r-2r=0
\end{aligned}$$

and so

$$\mathcal{S}(x,x,Tx)=0,$$

which implies $Tx = x$. Consequently, $C^S_{x_0,r}$ is a fixed circle of T. □

Remark 3.1. 1) Notice that the condition (3.2) guarantees that Tx is not in the interior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Similarly, the condition (3.3) guarantees that Tx is not the exterior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Hence $Tx \in C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$ and so we get $T(C^S_{x_0,r}) \subset C^S_{x_0,r}$.

2) If an S-metric is generated by any metric d, then Theorem 3.1 can be used on the corresponding metric space.

3) The converse statement of Theorem 3.1 is also true.

Now we give an example of a self-mapping with a fixed circle.

Example 3.1. Let $X = \mathbb{R}$ and the function $\mathcal{S} : X^3 \to [0,\infty)$ be defined by

$$\mathcal{S}(x,y,z)=|x-z|+|y-z|,$$

for all $x,y,z \in \mathbb{R}$ [13]. Then $(X,\mathcal{S})$ is called the usual S-metric space. This S-metric is generated by the usual metric on $\mathbb{R}$. Let us consider the circle $C^S_{0,2}$ and define the self-mapping $T_1 : \mathbb{R} \to \mathbb{R}$ as

$$T_1x=\left\{\begin{array}{ccc} x & if & x\in\{-1,1\} \\ 10 & & otherwise \end{array}\right.,$$

for all $x \in \mathbb{R}$. Then the self-mapping T_1 satisfies the conditions (3.2) and (3.3). Hence $C^S_{0,2} = \{-1,1\}$ is a fixed circle of T_1.

Notice that $C^S_{\frac{9}{2},11} = \{-1,10\}$ is another fixed circle of T_1 and so the fixed circle is not unique for a giving self-mapping.

On the other hand, if we consider the usual metric d on $\mathbb{R}$ then we obtain $C_{0,2} = \{-2,2\}$. The circle $C_{0,2}$ is not a fixed circle of T_1.

Example 3.2. Let $X = \mathbb{R}^2$ and let the function $\mathcal{S} : X^3 \to [0,\infty)$ be defined by

$$\mathcal{S}(x,y,z)=\sum_{i=1}^{2}\left(|x_i-z_i|+|x_i+z_i-2y_i|\right),$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then it can be easily seen that $\mathcal{S}$ is an S-metric on $\mathbb{R}^2$, which is not generated by any metric, and the pair $(\mathbb{R}^2, \mathcal{S})$ is an S-metric space.

Let us consider the unit circle $C_{0,1}^S$ and define the self-mapping $T_2 : \mathbb{R} \to \mathbb{R}$ as

$$T_2 x = \begin{cases} x & if \quad x \in C_{0,1}^S \\ (1,0) & \quad otherwise \end{cases},$$

for all $x \in \mathbb{R}^2$. Then the self-mapping T_2 satisfies the conditions (3.2) and (3.3). Therefore $C_{0,1}^S$ is a fixed circle of T_2 as shown in Figure 3.1.

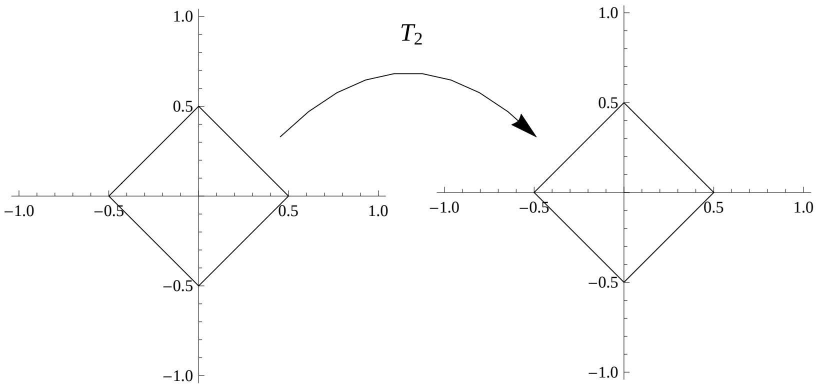

FIG. 3.1: The fixed circle of T_2.

In the following example, we give an example of a self-mapping which satisfies the condition (3.2) and does not satisfy the condition (3.3).

Example 3.3. Let $X = \mathbb{R}$ and the function $\mathcal{S} : X^3 \to [0, \infty)$ be defined by

$$\mathcal{S}(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [7]. Then $\mathcal{S}$ is an S-metric which is not generated by any metric and $(X, \mathcal{S})$ is an S-metric space. Let us consider the circle $C_{0,3}^S$ and define the self-mapping $T_3 : \mathbb{R} \to \mathbb{R}$ as

$$T_3 x = \begin{cases} -\frac{7}{2} & if \quad x = -\frac{3}{2} \\ \frac{7}{2} & if \quad x = \frac{3}{2} \\ 7 & \quad otherwise \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T_3 satisfies the condition (3.2) but does not satisfy the condition (3.3). Clearly T_3 does not fix the circle $C^S_{0,3}$.

In the following example, we give an example of a self-mapping which satisfies the condition (3.3) and does not satisfy the condition (3.2).

Example 3.4. Let $(X, \mathcal{S})$ be an S-metric space, $C^S_{x_0,r}$ be a circle on X and the self-mapping $T_4 : X \to X$ be defined as

$$T_4 x = x_0,$$

for all $x \in X$. Then the self-mapping T_4 satisfies the condition (3.3) but does not satisfy the condition (3.2). Clearly T_4 does not fix the circle $C^S_{x_0,r}$.

Now we give another existence theorem for fixed circles.

Theorem 3.2. *Let $(X, \mathcal{S})$ be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Let the mapping φ be defined as (3.1). If there exists a self-mapping $T : X \to X$ satisfying*

$$\mathcal{S}(x, x, Tx) \leq \varphi(x) - \varphi(Tx) \tag{3.4}$$

and

$$h\mathcal{S}(x, x, Tx) + \mathcal{S}(Tx, Tx, x_0) \geq r, \tag{3.5}$$

for all $x \in C^S_{x_0,r}$ and some $h \in [0, 1)$, then $C^S_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C^S_{x_0,r}$. On the contrary, assume that $x \neq Tx$. Then using the conditions (3.4) and (3.5), we obtain

$$\begin{aligned} \mathcal{S}(x, x, Tx) &\leq \varphi(x) - \varphi(Tx) \\ &= \mathcal{S}(x, x, x_0) - \mathcal{S}(Tx, Tx, x_0) \\ &= r - \mathcal{S}(Tx, Tx, x_0) \\ &\leq h\mathcal{S}(x, x, Tx) + \mathcal{S}(Tx, Tx, x_0) - \mathcal{S}(Tx, Tx, x_0) \\ &= h\mathcal{S}(x, x, Tx), \end{aligned}$$

which is a contradiction since $h \in [0, 1)$. Hence we get $Tx = x$ and $C^S_{x_0,r}$ is a fixed circle of T. □

Remark 3.2. 1) Notice that the condition (3.4) guarantees that Tx is not in the exterior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Similarly, the condition (3.5) shows that Tx can lie on either the exterior or the interior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Hence Tx should lie on the interior of the circle $C^S_{x_0,r}$.

2) If an S-metric is generated by any metric d, then Theorem 3.2 can be used on the corresponding metric space.

3) The converse statement of Theorem 3.2 is also true.

Now we give some examples of self-mappings which have a fixed-circle.

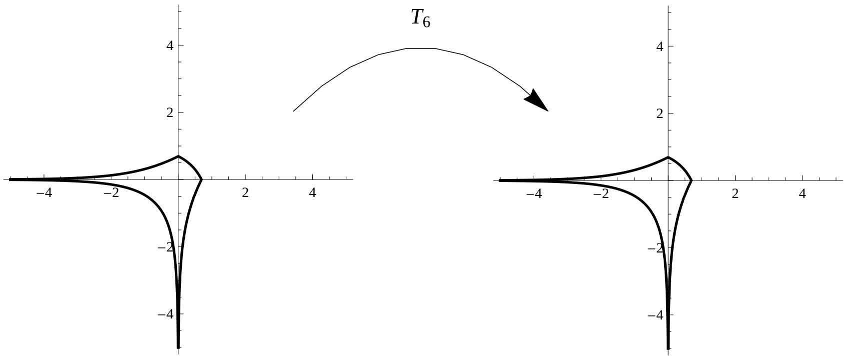

FIG. 3.2: The fixed circle of T_6.

Example 3.5. Let $X = \mathbb{R}$ and $(X, \mathcal{S})$ be the usual S-metric space. Let us consider the circle $C_{1,2}^S = \{0, 2\}$ and define the self-mapping $T_5 : \mathbb{R} \to \mathbb{R}$ as

$$T_5 x = \begin{cases} e^x - 1 & if \quad x = 0 \\ 2x - 2 & if \quad x = 2 \\ 3 & otherwise \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T_5 satisfies the conditions (3.4) and (3.5). Hence $C_{1,2}^S$ is a fixed circle of T_5.

On the other hand, if we consider the usual metric d on $\mathbb{R}$ then we have $C_{1,2} = \{-1, 3\}$. The circle $C_{1,2}$ is not a fixed circle of T_5. But $C_{1,1} = \{0, 2\}$ is a fixed circle of T_5 on (X, d).

Example 3.6. Let $X = \mathbb{R}^2$ and let the function $\mathcal{S} : X^3 \to [0, \infty)$ be defined by

$$\mathcal{S}(x, y, z) = \sum_{i=1}^{2} \left(|e^{x_i} - e^{z_i}| + |e^{x_i} + e^{z_i} - 2e^{y_i}|\right),$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then it can be easily checked that $\mathcal{S}$ is an S-metric on $\mathbb{R}^2$, which is not generated by any metric, and the pair $\left(\mathbb{R}^2, \mathcal{S}\right)$ is an S-metric space.

Let us consider the circle $C_{x_0,r}^S$ centered at $x_0 = (0, 0)$ with the radius $r = 2$ and define the self-mapping $T_6 : \mathbb{R} \to \mathbb{R}$ as

$$T_6 x = \begin{cases} x & if \quad x \in C_{0,2}^S \\ (\ln 2, 0) & otherwise \end{cases},$$

for all $x \in \mathbb{R}^2$. Then the self-mapping T_6 satisfies the conditions (3.4) and (3.5). Therefore $C^S_{0,2}$ is the fixed circle of T_6 as shown in Figure 3.2.

In the following example, we give an example of a self-mapping which satisfies the condition (3.4) and does not satisfy the condition (3.5).

Example 3.7. Let $(X, \mathcal{S})$ be an S-metric space and $C^S_{x_0,r}$ be a circle on X. If we consider the self-mapping $T_4 x = x_0$, then the self-mapping T_4 satisfies the condition (3.4) but does not satisfy the condition (3.5). It can be easily seen that T_4 does not fix a circle $C^S_{x_0,r}$.

In the following example, we give an example of a self-mapping which satisfies the condition (3.5) and does not satisfy the condition (3.4).

Example 3.8. Let $X = \mathbb{R}$ and $(X, \mathcal{S})$ be an S-metric space with an S-metric defined as in Example 3.3. Let us consider the unit circle $C^S_{0,1}$ and define the self-mapping $T_7 : \mathbb{R} \to \mathbb{R}$ as

$$T_7 x = 1,$$

for all $x \in \mathbb{R}$. Then the self-mapping T_7 satisfies the condition (3.5) but does not satisfy the condition (3.4). It can be easily shown that T_7 does not fix the unit circle $C^S_{0,1}$.

Let $I_X : X \to X$ be the identity map defined as $I_X(x) = x$ for all $x \in X$. Notice that the identity map satisfies the conditions (3.2) and (3.3) (resp. (3.4) and (3.5)) in Theorem 3.1 (resp. Theorem 3.2) for any circle. Now we determine a condition which excludes the I_X from Theorem 3.1 and Theorem 3.2. For this purpose, we give the following theorem.

Theorem 3.3. *Let $(X, \mathcal{S})$ be an S-metric space, $T : X \to X$ be a self mapping having a fixed circle $C^S_{x_0,r}$ and the mapping φ be defined as (3.1). The self-mapping T satisfies the condition*

$$(I_S) \quad \mathcal{S}(x, x, Tx) \leq \frac{\varphi(x) - \varphi(Tx)}{h},$$

for all $x \in X$ and some $h > 2$ if and only if $T = I_X$.

Proof. Let $x \in X$ be an arbitrary element. Then using the inequality (I_S), Lemma 2.1 and triangle inequality, we obtain

$$\begin{aligned} h\mathcal{S}(x, x, Tx) &\leq \varphi(x) - \varphi(Tx) \\ &= \mathcal{S}(x, x, x_0) - \mathcal{S}(Tx, Tx, x_0) \\ &\leq 2\mathcal{S}(x, x, Tx) + \mathcal{S}(Tx, Tx, x_0) - \mathcal{S}(Tx, Tx, x_0) \\ &= 2\mathcal{S}(x, x, Tx) \end{aligned}$$

and so

$$(h - 2)\mathcal{S}(x, x, Tx) \leq 0.$$

Since $h > 2$ it should be $\mathcal{S}(x, x, Tx) = 0$ and so $Tx = x$. Consequently, we obtain $T = I_X$.

Conversely, it is clear that the identity map I_X satisfies the condition (I_S). $\square$

Remark 3.3. 1) If a self-mapping T, which has a fixed circle, satisfies the conditions (3.2) and (3.3) (resp. (3.4) and (3.5)) in Theorem 3.1 (resp. Theorem 3.2) but does not satisfy the condition (I_S) in Theorem 3.3 then the self-mapping T cannot be an identity map.

2) If an S-metric is generated by any metric d, then Theorem 3.3 can be used on the corresponding metric space.

3.2. The uniqueness of fixed circles

We investigate the uniqueness conditions of fixed circles given in the existence theorems. For any given circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$ on X, we notice that there exists at least one self-mapping T of X such that T fixes the circles $C^S_{x_0,r}$, $C^S_{x_1,\rho}$. Indeed let us define the mappings $\varphi_1, \varphi_2 : X \to [0, \infty)$ as

$$\varphi_1(x) = \mathcal{S}(x, x, x_0)$$

and

$$\varphi_2(x) = \mathcal{S}(x, x, x_1),$$

for all $x \in X$. If we define the self-mapping $T_8 : X \to X$ as

$$T_8x = \begin{cases} x & if \quad x \in C^S_{x_0,r} \cup C^S_{x_1,\rho} \\ \alpha & \qquad otherwise \end{cases},$$

for all $x \in X$, where α is a constant satisfying $S(\alpha, \alpha, x_0) \neq r$ and $S(\alpha, \alpha, x_1) \neq \rho$, it can be easily seen that the self-mapping $T_8 : X \to X$ satisfies the conditions (3.2) and (3.3) in Theorem 3.1 (resp. (3.4) and (3.5) in Theorem 3.2) for the circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$ using the mappings φ_1 and φ_2, respectively. Hence T_8 fixes both of the circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$. In this way, the number of fixed circles can be extended to any positive integer n using the same arguments.

In the following example, the self-mapping T_9 has two fixed circle.

Example 3.9. Let $X = \mathbb{R}$ and $(X, \mathcal{S})$ be an S-metric space with the S-metric defined in Example 3.3. Let us consider the circles $C^S_{0,2}$, $C^S_{0,4}$ and define the self-mapping $T_9 : \mathbb{R} \to \mathbb{R}$ as

$$T_9x = \begin{cases} x & if \quad x \in \{-2, -1, 1, 2\} \\ \alpha & \qquad otherwise \end{cases},$$

for all $x \in X$ where $\alpha \in X$. Then the conditions (3.2) and (3.3) are satisfied by T_9 for the circles $C^S_{0,2}$ and $C^S_{0,4}$, respectively. Consequently, $C^S_{0,2}$ and $C^S_{0,4}$ are the fixed circles of T_9.

Now we investigate the uniqueness conditions for the fixed circles in Theorem 3.1 using Rhoades' contractive condition on S-metric spaces.

Theorem 3.4. *Let $(X, \mathcal{S})$ be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Let $T : X \to X$ be a self-mapping satisfying the conditions (3.2) and (3.3) given in Theorem 3.1. If the contractive condition*

$$\begin{aligned} \mathcal{S}(Tx,Tx,Ty) < \max\{&\mathcal{S}(x,x,y), \mathcal{S}(Tx,Tx,x), \mathcal{S}(Ty,Ty,y), \\ &\mathcal{S}(Ty,Ty,x), \mathcal{S}(Tx,Tx,y)\}, \end{aligned} \tag{3.6}$$

is satisfied for all $x \in C^S_{x_0,r}$, $y \in X \setminus C^S_{x_0,r}$ by T, then $C^S_{x_0,r}$ is a unique fixed circle of T.

Proof. Suppose that there exist two fixed circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$ of the self-mapping T, that is, T satisfies the conditions (3.2) and (3.3) for each circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$. Let $x \in C^S_{x_0,r}$ and $y \in C^S_{x_1,\rho}$ be arbitrary points with $x \neq y$. Using the contractive condition (3.6), we obtain

$$\begin{aligned} \mathcal{S}(x,x,y) &= \mathcal{S}(Tx,Tx,Ty) < \max\{\mathcal{S}(x,x,y), \mathcal{S}(Tx,Tx,x), \mathcal{S}(Ty,Ty,y), \\ &\quad \mathcal{S}(Ty,Ty,x), \mathcal{S}(Tx,Tx,y)\} \\ &= \mathcal{S}(x,x,y), \end{aligned}$$

which is a contradiction. Hence it should be $x = y$. Consequently, $C^S_{x_0,r}$ is the unique fixed circle of T. □

The following example shows that the circle $C^S_{x_0,r}$ is not necessarily unique in Theorem 3.2.

Example 3.10. Let $(X, \mathcal{S})$ be an S-metric space and $C_{x_1,r_1}, \cdots, C_{x_n,r_n}$ be any circles on X. Let us define the self-mapping $T_{10} : X \to X$ as

$$T_{10}x = \begin{cases} x & if \quad x \in \bigcup\limits_{i=1}^{n} C_{x_i,r_i} \\ x_0 & otherwise \end{cases},$$

for all $x \in X$, where x_0 is a constant in X. Then it can be easily checked that the conditions (3.4) and (3.5) are satisfied by T_{10} for the circles $C_{x_1,r_1}, \cdots, C_{x_n,r_n}$, respectively. Consequently, the circles $C_{x_1,r_1}, \cdots, C_{x_n,r_n}$ are fixed circles of T_{10}. Notice that these circles do not have to be disjoint.

Now we give the following uniqueness theorem for the fixed circles in Theorem 3.2 using the notion of diameter on S-metric spaces.

Theorem 3.5. *Let $(X, \mathcal{S})$ be an S-metric space, $C^S_{x_0,r}$ be any circle on X, $U_x = \{T^n x : n \in \mathbb{N}\}$, $U_y = \{T^n y : n \in \mathbb{N}\}$, $diam\{U_x\} < \infty$ and $diam\{U_y\} < \infty$. Let*

$T : X \to X$ be a self-mapping satisfying the conditions (3.4) and (3.5) given in Theorem 3.2. If the contractive condition

$$\mathcal{S}(Tx, Tx, Ty) < diam\{U_x \cup U_y\}, \tag{3.7}$$

is satisfied for all $x \in C^S_{x_0,r}$, $y \in X \setminus C^S_{x_0,r}$ by T, then $C^S_{x_0,r}$ is the unique fixed circle of T.

Proof. Assume that there exist two fixed circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$ of the self-mapping T, that is, T satisfies the conditions (3.4) and (3.5) for each circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$. Let $x \in C^S_{x_0,r}$ and $y \in C^S_{x_1,\rho}$ be arbitrary points with $x \neq y$. Using the contractive condition (3.7), we obtain

$$\mathcal{S}(x, x, y) = \mathcal{S}(Tx, Tx, Ty) < diam\{U_x \cup U_y\} = \mathcal{S}(x, x, y),$$

which is a contradiction. Hence it should be $x = y$. Consequently, $C^S_{x_0,r}$ is the unique fixed circle of T. □

3.3. Infinity of fixed circles

We give a new approach to obtain fixed-circle results. To do this, let us denote by $R_S(x, y)$ the right side of the inequality ($S25$). Using the number $R_S(x, y)$, we obtain the following theorem. This theorem generates many (finite or infinite) fixed circles for a given self-mapping.

Theorem 3.6. *Let $(X, \mathcal{S})$ be an S-metric space, $T : X \to X$ be a self-mapping and $r = \inf\{\mathcal{S}(Tx, Tx, x) : Tx \neq x\}$. If there exists a point $x_0 \in X$ satisfying*

$$\mathcal{S}(x, x, Tx) < R_S(x, x_0) \tag{3.8}$$

for all $x \in X$ when $\mathcal{S}(Tx, Tx, x) > 0$ and

$$\mathcal{S}(Tx, Tx, x_0) = r \tag{3.9}$$

for all $x \in C^S_{x_0,r}$, then $C^S_{x_0,r}$ is a fixed circle of T. The self-mapping T also fixes the closed ball $B_S[x_0, r]$.

Proof. Let $x \in C^S_{x_0,r}$ and $Tx \neq x$. Then using the inequality (3.8) and Lemma 2.1, we get

$$\begin{aligned} &\mathcal{S}(x, x, Tx) < R_S(x, x_0) \\ &= \max\left\{ \begin{array}{c} \mathcal{S}(x, x, x_0), \mathcal{S}(Tx, Tx, x), \mathcal{S}(Tx_0, Tx_0, x_0), \\ \mathcal{S}(Tx_0, Tx_0, x), \mathcal{S}(Tx, Tx, x_0) \end{array} \right\}. \end{aligned} \tag{3.10}$$

At first, using the inequality (3.10) and Lemma 2.1, we show $Tx_0 = x_0$. Suppose that $Tx_0 \neq x_0$. For $x = x_0$, we obtain

$$\begin{aligned} \mathcal{S}(x_0, x_0, Tx_0) &< R_S(x_0, x_0) \\ &= \max\left\{ \begin{array}{c} \mathcal{S}(x_0, x_0, x_0), \mathcal{S}(Tx_0, Tx_0, x_0), \mathcal{S}(Tx_0, Tx_0, x_0), \\ \mathcal{S}(Tx_0, Tx_0, x_0), \mathcal{S}(Tx_0, Tx_0, x_0) \end{array} \right\} \\ &= \mathcal{S}(Tx_0, Tx_0, x_0) = \mathcal{S}(x_0, x_0, Tx_0), \end{aligned}$$

a contradiction. It should be $Tx_0 = x_0$. Then by the inequality (3.10), the condition (3.9), definition of r and Lemma 2.1, we have

$$\begin{aligned} \mathcal{S}(x,x,Tx) &< \max\left\{ \begin{array}{c} \mathcal{S}(x,x,x_0), \mathcal{S}(Tx,Tx,x), \mathcal{S}(x_0,x_0,x_0), \\ \mathcal{S}(x_0,x_0,x), \mathcal{S}(Tx,Tx,x_0) \end{array} \right\} \\ &= \max\{r, \mathcal{S}(Tx,Tx,x)\} = \mathcal{S}(Tx,Tx,x) = \mathcal{S}(x,x,Tx), \end{aligned}$$

a contradiction. Therefore we get $Tx = x$, that is, $C^S_{x_0,r}$ is a fixed circle of T.

Finally we prove that T fixes the closed ball $B_S[x_0, r]$. To do this, we show that T fixes any circle $C^S_{x_0,\rho}$ with $\rho < r$. Let $x \in C^S_{x_0,\rho}$ and $Tx \neq x$. From the similar arguments used in the above, we have $Tx = x$. □

We give the following example.

Example 3.11. Let $X = \mathbb{R}$ be the usual S-metric space. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \left\{ \begin{array}{ccc} x & if & |x| < 3 \\ x+2 & if & |x| \geq 3 \end{array} \right. ,$$

for all $x \in \mathbb{R}$. The self-mapping T satisfies the conditions of Theorem 3.6 with $x_0 = 0$. Indeed, we get

$$\mathcal{S}(x,x,Tx) = 2|x - Tx| = 4 > 0,$$

for all $x \in \mathbb{R}$ such that $|x| \geq 3$. Then we have

$$\begin{aligned} R_S(x,0) &= \max\{\mathcal{S}(x,x,0), \mathcal{S}(Tx,Tx,x), \mathcal{S}(0,0,0), \mathcal{S}(0,0,x), \mathcal{S}(Tx,Tx,0)\} \\ &= \max\{2|x|, 4, 0, 2|x|, 2|x+2|\} \\ &= \max\{2|x|, 2|x+2|\} \end{aligned}$$

and so

$$\mathcal{S}(x,x,Tx) < R_S(x,0).$$

Therefore the condition (3.8) is satisfied. We also obtain

$$r = \min\{\mathcal{S}(Tx,Tx,x) : Tx \neq x\} = 4.$$

It can be easily seen that the condition (3.9) is satisfied by T. Consequently, T fixes the circle $C^S_{0,4} = \{x \in \mathbb{R} : |x| = 2\}$ and the closed ball $B_S[0,4] = \{x \in \mathbb{R} : |x| \leq 2\}$.

Remark 3.4. 1) Notice that the condition (3.9) guarantees that $Tx \in C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$ and so $T(C^S_{x_0,r}) \subset C^S_{x_0,r}$.

2) The self-mapping T defined in Example 3.11 has other fixed circles. Theorem 3.6 gives us some of these circles.

3) A self-mapping T can fix infinitely many circles (see Example 3.11).

The converse statement is not always true as seen in the following example.

Example 3.12. Let $x_0 \in X$ be any point. If we define the self-mapping $T : X \to X$ as

$$Tx = \left\{ \begin{array}{ccc} x & if & x \in B_S[x_0,\mu] \\ x_0 & if & x \notin B_S[x_0,\mu] \end{array} \right. ,$$

for all $x \in X$ with $\mu > 0$, then T does not satisfies the condition (3.8), but T fixes every circle $C^S_{x_0,\rho}$ with $\rho \leq \mu$.

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