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FIXED POINT THEOREMS FOR SUBSEQUENTIALLY MULTI-VALUED F_δ-CONTRACTIONS IN METRIC SPACES *

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Abstract. The aim of this paper is to prove common fixed point theorems for multi-valued contraction of Wordowski type, by using the concept of subsequential continuity in the setting of set valued context contractions with compatibility. We have also given an example and an application to integral inclusions of Fredholm type to support our results.
keywords: Subsequentially continuous; δ-compatible; F-contraction; Hardy Rogers contraction; integral inclusion.

1. Introduction

The multi-valued fixed point theory has many different applications, for example in integral or differential inclusions, economics, optimization, etc. The contraction principle due to Banach has been generalized in different directions and one of such generalizations is connected to Nadler [12], where he used the Hausdorff metric to prove the existence of a fixed point of multi valued mapping in metric space. Later, many authors have obtained some results in non linear analysis concerning the multivalued fixed point theory and its applications using two types of distances. One is the Hausdorff distance and another is the δ-distance which was defined by Fisher [8]. Although δ-distance is not a metric like the Hausdorff distance, it shares most of the properties of a metric and some results on δ-distance can be found in [1, 2, 3]. In this paper, we have used a Ćirić type F-contraction and Hardy-Rogers type F-contraction inequality introduced by Minak et al.[11](independently by Wardowski and Dung [17] as F-weak contraction and Cosentino and Vetro [7] respectively, using δ-distance to establish the existence of a strict coincidence and a common strict fixed point of a weakly compatible hybrid pair of maps which are

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strongly tangential. However, it is worth mentioning that the idea of F-contraction was initiated by Wardowski [16], and later, it became generalized by several authors in different directions. The examples are Minak et al. [11], Wardowski and Dung [17], Cosentino and Vetro [7].

2. Preliminaries

Let (X,d) be a metric space, $B(X)$ is the set of all non-empty bounded subsets of X. For all $A,B\in B(X)$, we define the two functions:$D,\delta : B(X)\times B(X)\to\mathbb{R}_+$ such that

$$D(A,B)=\inf\{d(a,b);a\in A,b\in B\},$$

$$\delta(A,B)=\sup\{d(a,b);a\in A,b\in B\}.$$

If A consists of a single point a, we write $\delta(A,B)=\delta(a,B)$ and $D(A,B)=D(a,B)$, also if $B=\{b\}$ is a singleton we write

$$\delta(A,B)=D(A,B)=d(a,b).$$

It is clear that δ satisfies the following properties:

$$\delta(A,B)=\delta(B,A)\geq 0,$$

$$\delta(A,B)\leq\delta(A,C)+\delta(C,B),$$

$$\delta(A,A)=\mathrm{diam}A,$$

$$\delta(A,B)=0 \text{ implies } A=B=\{a\},$$

for all $A,B,C\in B(X)$.
Notice that for all $a\in A$ and $b\in B$ we have

$$D(A,B)\leq d(a,b)\leq\delta(A,B),$$

where $A,B\in B(X)$.

Definition 2.1. [14] Two mappings $S:X\to B(X)$ and $f:X\to X$ are to be weakly commuting on X if $fSx\in B(X)$ and for all $x\in X$:

$$\delta(Sfx,fSx)\leq\max\{\delta(fx,Sx),diam(fSx)\}.$$

Definition 2.2. [10] A hybrid pair of mappings (f,S) of a metric space (X,d) is δ-compatible if

$$\lim_{n\to\infty}\delta(Sfx_n,fSx_n)=0,$$

whenever $\{x_n\}$ is a sequence in X such that $fSx_n\in B(X)$, $\lim_{n\to\infty}Sx_n=\{z\}$, and $\lim_{n\to\infty}fx_n=z$, for some $z\in X$.

Definition 2.3. [13] The pair of self mappings (f, g) on a metric space(X, d) is said to be reciprocally continuous if

$$\lim_{n\to\infty} fgx_n = ft$$

and

$$\lim_{n\to\infty} gfx_n = gt,$$

where $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X.

Later, Singh and Mishra [15] generalized the concept of reciprocal continuity to the setting of single and set-valued maps as follows.

Definition 2.4. [15] Two maps $f : X \to X$ and $S : X \to B(X)$ are reciprocally continuous on X (resp. at $t \in X$) if and only if $fSx \in B(X)$ for each $x \in X$ (resp. $fSt \in B(X)$) and

$$\lim_{n\to\infty} fSx_n = fM, \ \lim_{n\to} Sfx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = M \in B(X)$, $\lim_{n\to\infty} fx_n = t \in M$

In 2009, Bouhadjera and Godet Thobie [5] introduced the concept of subcompatibility and subsequential continuity as follows:
Two self-mappings f and g on a metric space (X, d) are said to be subcompatible if there exists a sequence $\{x_n\}$ such that:

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \text{ and } \lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

for some $t \in X$.
The pair (f, g) of self mappings is said to be subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} fgx_n = fz$, $\lim_{n\to\infty} gfx_n = gz$.

Definition 2.5. [4] Let $f : X \to X$ and $S : X \to CB(X)$ two single and set-valued mappings respectively, the hybrid pair (f, S) is to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = M \in CB(X) \quad \text{and} \quad \lim_{n\to\infty} fx_n = z \in M,$$

for some $z \in X$ and $\lim_{n\to\infty} fSx_n = fM$, $\lim_{n\to\infty} Sfx_n = Sz$.

Notice that continuity or reciprocal continuity implies subsequential continuity, but the converse may be not.

Example 2.1. Let $X = [0, 1]$ and d the euclidian metric, we define f, S by

$$fx = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ \frac{1}{4}, & \frac{1}{2} < x \leq 1 \end{cases} \qquad Sx = \begin{cases} [0, x], & 0 \leq x \leq 1 \\ [x - \frac{1}{2}, x], & \frac{1}{2} < x \leq 1 \end{cases}$$

We consider a sequence $\{x_n\}$ such that for each $n \geq 1$ we have: $x_n = \frac{1}{2} - \frac{1}{n}$, clearly that $\lim\limits_{n\to\infty} fx_n = \dfrac{1}{2} \in [0, \dfrac{1}{2}]$ and $\lim\limits_{n\to\infty} Sx_n = [0, \dfrac{1}{2}] \in B(X)$, also we have:

$$\lim_{n\to\infty} fSx_n = \lim_{n\to\infty} [\frac{1}{2} + \frac{1}{n}, 1] = [\frac{1}{2}, 1] = f([0, \frac{1}{2}]),$$

and

$$\lim_{n\to\infty} Sfx_n = \lim_{n\to\infty} [\frac{1}{n}, \frac{1}{2} + \frac{1}{n}] = [0, \frac{1}{2}] = S(\frac{1}{2}),$$

then (f, S) is subsequentially continuous.
On the other hand, consider a sequence $\{y_n\}$ which defined for all $n \geq 1$ by: $y_n = 1 + \frac{1}{n}$, we have

$$\lim_{n\to\infty} fx_n = \frac{1}{2} \in [0, 1], \quad \text{and} \quad \lim_{n\to\infty} Sx_n = [0, 1] \in B(X),$$

however

$$\lim_{n\to\infty} fSx_n = \lim_{n\to\infty} f([\frac{1}{n}, 1 + \frac{1}{n}]) \neq f([0, 1]),$$

then f and S are never reciprocally continuous.

Let $\mathcal{F}$ be the set of all functions $F : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

(F_1) : F is strictly increasing,

(F_2) : for each sequence $\{\alpha_n\}$ in X, $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$,

(F_3) : there exists $k \in (0, 1)$ satisfying $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Example 2.2. Let $F_i : (0, +\infty) \to \mathbb{R}$, $i \in \{1, 2, 3\}$, defined by

1. $F_1(t) = \ln t$,
2. $F_2(t) = t + \ln t$,
3. $F_3(t) = -\frac{1}{\sqrt{t}}$.

Then $F_i \in \mathcal{F}$, for each $i \in \{1, 2, 3\}$.

Definition 2.6. [16] Let (X, d) be a metric space and $T : X \to X$ be a mapping. For $F \in \mathcal{F}$, we say T is F-contraction, if there exists $\tau > 0$ such that for $x, y \in X$, $d(Tx, Ty) > 0$ implies $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$.

Definition 2.7. [7] A self mapping T on a metric space (X, d) is a Hardy- Rogers type F-contraction if there exists $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that $d(Tx, Ty) > 0$ implies that

$$F(d(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \lambda d(x, Ty) + \mu d(y, Tx)),$$

for all $x, y \in X$, where, $\alpha + \beta + \gamma + 2\lambda = 1$, $\gamma \neq 1$, $\mu \geq 0$.

Definition 2.8. [11] A self mapping T on a metric space (X, d) is a Ćirić type F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that $d(Tx, Ty) > 0$ implies that

$$\tau + F(d(Tx, Ty)) \leq F(M(x, y)),$$

$\forall x, y \in X$. where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$

Notice that every F-contraction is a Ćirić type F-contraction or Hardy-Rogers type F-contraction but the reverse implication does not hold.

Definition 2.9. [7] A self mapping T on a metric space (X, d) is a Hardy- Rogers type F-contraction if there exists $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that $d(Tx, Ty) > 0$ implies that

$$F(d(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \lambda d(x, Ty) + L d(y, Tx)),$$

for all $x, y \in X$, where, $\alpha + \beta + \gamma + 2\lambda = 1$, $\gamma \neq 1$ and $L \geq 0$.

Definition 2.10. [11] A self mapping T on a metric space (X, d) is a Ćirić type F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that $d(Tx, Ty) > 0$ implies that

$$\tau + F(d(Tx, Ty)) \leq F(M(x, y)),$$

$\forall x, y \in X$, where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$.

Notice that every F-contraction is a Ćirić type F-contraction or Hardy-Rogers type F-contraction but the reverse implication does not hold.

Definition 2.11. [1] Let (X, d) be a metric space and $T : X \to B(X)$. we say that T is a generalized multivalued F-contraction, if there exists τ such that

$$\tau + F(\delta(Tx, Ty)) \leq F(M(x, y),$$

for all $x, y \in X$ with $\min\{d(x, y), \delta(Tx, Ty)\} > 0$, where $F \in \mathcal{F}$ and $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$.

3. Main results

Theorem 3.1. *Let $f, g : X \to X$ be single valued mappings and $S, T : X \to B(X)$ be multi-valued mappings of metric space (X, d). If the two pairs (f, S) and (g, T) are subsequentially continuous and δ-compatible. Then pairs (f, S) and (g, T) have a strict coincidence point. Moreover, f, g, S and T have a common strict fixed point provided there exists $\tau > 0$ such that for all x, y in X we have:*

$$\delta(Sx, Ty) > 0 \quad \text{implies} \quad \tau + F(\delta(Sx, Ty)) \leq F(R(x, y)), \tag{3.1}$$

where $F \in \mathcal{F}$ and

$$R(x, y) = \max\{d(fx, gy), D(fx, Sx), D(gy, Ty), \frac{1}{2}[D(fx, Ty) + D(gy, Sx)]\}$$

Proof. Since (f, S) is subsequentially continuous, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = M \in B(X), \quad \lim_{n\to\infty} fx_n = z \in M.$$

$$\lim_{n\to\infty} fSx_n = fM, \quad \lim_{n\to\infty} Sfx_n = Sz,$$

Also, the pair (f, S) is δ-compatible implies that

$$\lim_{n\to\infty} \delta(fSx_n, Sfx_n) = \delta(fM, Sz) = 0,$$

which gives that $fM = Sz = \{fz\}$, and so z is a coincidence point of f and S. Similarly, for the pair (g, T) there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ty_n = N \in B(X) \quad \text{and} \quad \lim_{n\to\infty} gy_n = t \in N$$

and

$$\lim_{n\to\infty} gTy_n = gN, \quad \lim_{n\to\infty} Tgy_n = Tt.$$

The pair (g, T) is δ-compatible, implies that

$$\lim_{n\to\infty} \delta(gTy_n, Tgy_n) = \delta(gN, Tt) = 0.$$

Then $gN = Tt$ and Tt is a singleton, i.e, $Tt = \{gt\}$ and t is strict coincidence point of g and T.
Now, we claim $fz = gt$, if not by using (3.1), $\delta(Sz, Tt) > 0$, if not $d(fz, gt) \leq \delta(Sz, Tt) = 0$, which a contradiction. So we have:

$$\tau + F(\delta(Sz, Tt)) \leq F(R(z, t)).$$

Since $Sz = \{fz\}$ and $Tt = \{gt\}$, then

$$D(fz, Sz) = D(gt, Tt) = 0,$$

$$D(fz, Tt) = d(fz, gt)$$

and $D(gt, Sz) = d(fz, gt)$. Hence

$$\begin{aligned} R(z, t) &= \max\{d(fz, gt), D(fz, Sz), D(gt, Tt), \frac{1}{2}(D(fz, Tt) + D(gt, Sz))\} \\ &= d(fz, gt). \end{aligned}$$

Subsisting in (3.1) we get

$$\tau + F(\delta(Sz, Tt)) \leq F(d(fz, gt)).$$

This yields

$$F(\delta(Sz, Tt)) < \tau + F(\delta(Sz, Tt)) \leq F(d(fz, gt)) = F(\delta(Sz, Tt)),$$

F is a strictly increasing function, implies that

$$\delta(Sz, Tt)) < \delta(Sz, Tt),$$

which is a contradiction. Then $fz = gt$ and so $Sz = Tt$,
Now we claim $z = fz$, if not by taking $x = x_n$ and $y = t$ in (3.1), $\delta(Sx_n, Tt) > 0$, otherwise letting $n \to \infty$, we get

$$d(z, fz) = d(z, gt) \leq \delta(M, Tt) = 0,$$

which contradicts that $z \neq fz$, and so we have

$$\tau + F(\delta(Sx_n, Tt)) \leq F(\max\{d(fx_n, gt), D(fx_n, Sx_n),$$

$$D(gt, Tt), \frac{1}{2}(D(fx_n, Tt) + D(gt, Sx_n))\}).$$

Letting $n \to \infty$, we get:

$$F(d(z, fz) < \tau + F(\delta(M, Tt) \leq F(d(z, fz)),$$

which is a contradiction. Hence z is a fixed point for f and S.
We will show $z = t$, if not by taking $x = x_n$ and $y = y_n$ in (3.1), $\delta(Sx_n, Ty_n) > 0$, if not letting $n \to \infty$, we obtain

$$d(z, t) \leq \delta(M, N) = 0,$$

which is a contradiction, so we have:

$$\tau + F(\delta(Sx_n, Ty_n)) \leq F(\max\{d(fx_n, gy_n), D(fx_n, Sx_n),$$

$$D(gy_n, Ty_n), \frac{1}{2}(D(fx_n, Ty_n) + D(gy_n, Sx_n))\}).$$

Letting $n \to \infty$, we get

$$F(d(z, t) < \tau + F(\delta(M, N) \leq F(d(z, t)),$$

which is a contradiction. Hence $z = t$ and consequently z is a common fixed point for f, g, S and T.
For the uniqueness, suppose there is another fixed point w and using (3.1), $\delta(Sz, Tw) > 0$, if not $d(z, t) \leq \delta(Sz, Tt) = 0$, which is a contradiction, then we have:

$$d(z, w) < \tau + F(\delta(Sz, Tw)) \leq F(d(z, w)),$$

which is a contradiction. Then z is unique. □

If $f = g$ and $S = T$ we obtain the following corollary:

Corollary 3.1. *Let $f : X \to X$ be a single valued mapping and $S : X \to B(X)$ be a multi-valued mapping of metric space (X, d). Suppose that the pair (f, S) is subsequentially continuous, as well is δ-compatible and there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all x, y in X we have:*

$$\delta(Sx, Sy) > 0 \ \textit{ implies } \ \tau + F(\delta(Sx, Sy)) \leq F(M(x, y)),$$

where $F \in \mathcal{F}$ and

$$R(x, y) = \max\{d(fx, fy), D(fx, Sx), D(fy, Sy), \frac{1}{2}[D(fx, Sy) + D(fy, Sx)]\}.$$

Therefore, f and T have a strict common fixed point.

If S and T are single valued maps, we get the following corollary:

Corollary 3.2. *Let (X, d) be a metric space and let $f, g, S, T : X \to X$ be self mappings if the hybrid pair (f, S) is subsequentially continuous as well as compatible. Then f and S have a coincidence point. Moreover, f and S have a common fixed point provided there exists $\tau > 0$ such that for all x, y in X we have:*

$$d(Sx, Ty) > 0 \ \textit{ implies } \ \tau + F(d(Tx, Ty)) \leq F(R(x, y)),$$

where $F \in \mathcal{F}$ and

$$R(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Sx)]\}.$$

Now we shall state and prove our second main result using Hardy-Rogers type F-contractions [7] to establish strict coincidence and common strict fixed point of two hybrid pairs of self maps.

Theorem 3.2. *Let $f, g : X \to X$ be single valued mappings and $S, T : X \to B(X)$ be multi-valued mappings of metric space (X, d) such that the pairs (f, S) and (g, T) are subsequentially continuous as well as δ-compatible. Then, the pairs (f, S) and (g, T) have a strict coincidence point. Moreover, f, g, S and T have a common strict fixed point provided there exists $\tau > 0$ such that for all x, y in X we have: $\delta(Sx, Ty) > 0$ implies*

$$\tau + F(\delta(Sx, Ty)) \leq F\{\alpha d(fx, gy) + \beta d(fx, Sx)$$

$$+\gamma d(gy, Ty) + \lambda d(fx, Ty) + L d(gy, Sx)\}, \tag{3.2}$$

for all $x, y \in X$ with $\delta(Sx, Ty) > 0$, where $F \in \mathcal{F}$, $\alpha + \beta + \gamma + \lambda + L < 1$ and $L \geq 0$.

Proof. As in proof of Theorem 3.1, (f, S) is subsequentially continuous, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = M \in B(X), \quad \lim_{n\to\infty} fx_n = z \in M$$

and

$$\lim_{n\to\infty} fSx_n = fM, \quad \lim_{n\to\infty} Sfx_n = Sz,$$

again, the pair (f,S) is δ-compatible we get

$$\lim_{n\to\infty} \delta(fSx_n, Sfx_n) = \delta(fM, Sz) = 0,$$

which implies that $fM = Sz = \{fz\}$, and so z is a coincidence point of f and S. Similarly, for g and T there is a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ty_n = N \in B(X) \quad \text{and} \quad \lim_{n\to\infty} gy_n = t \in N$$

and

$$\lim_{n\to\infty} gTy_n = gN, \quad \lim_{n\to\infty} Tgy_n = Tt.$$

The pair (g,T) is δ-compatible, implies that

$$\lim_{n\to\infty} \delta(gTy_n, Tgy_n) = \delta(gN, Tt) = 0.$$

then $gN = Tt$ and Tt is a singleton,i.e, $Tt = \{gt\}$ and t is a strict coincidence point of B and T.
We show $fz = gt$, if not so $\delta(Sz,Tt) > 0$, by using (3.2) we get

$$\begin{aligned} F(\delta(Sz,Tt)) &< \tau + F(\delta(Sz,Tt)) \\ &\leq F((\alpha+\lambda+L)d(fz,gt)) \\ &\leq F(d(fz,gt)) = F(\delta(Sz,Tt)). \end{aligned}$$

Since F is increasing, we get

$$\delta(Sz,Tt) < \delta(Sz,Tt),$$

which is a contradiction. Hence $fz = gt$.
Now we claim $z = fz$, if not by taking $x = x_n$ and $y = t$ in (3.2), $\delta(Sx_n,Tt) > 0$, otherwise letting $n \to \infty$, we get

$$d(z,fz) = d(z,gt) \leq \delta(M,Tt) = 0,$$

which is a contradiction. Then using (3.2) we get

$$\begin{aligned} \tau + F(\delta(Sx_n,Tt)) \leq F\{(&\alpha d(fx_n,gt) + \beta d(fx_n,Sx_n) \\ &+\gamma D(gt,Tt) + \lambda d(fx_n,Tt) + LDd(gt,Sx_n)\}. \end{aligned}$$

Taking $n \to \infty$, we get

$$\tau + F(\delta(M,Tt)) \leq F((\alpha+\lambda+L)d(z,fz))$$

then

$$F(d(z,fz)) < \tau + F(\delta(M,Tt)) \leq F(d(z,fz)),$$

which is a contradiction. Hence $z = fz$.
We will show $z = t$, if not by taking $x = x_n$ and $y = y_n$ in (3.2), $\delta(Sx_n, Ty_n) > 0$, if not letting $n \to \infty$, we get:

$$d(z,t) \leq \delta(M,N) = 0,$$

which is a contradiction, using (3.2 we get:

$$\tau + F(\delta(Sx_n, Ty_n)) \leq F(\alpha d(fx_n, gy_n) + \beta D(fx_n, Sx_n)$$
$$+\gamma D(gy_n, Ty_n) + \lambda D(fx_n, Ty_n) + L(gy_n, Sx_n))).$$

Letting $n \to \infty$, we get

$$F(d(z,t) < \tau + F(\delta(M,N) \leq F((\alpha + \lambda + L)d(z,t))$$
$$\leq F(d(z,t)),$$

which is a contradiction. Hence $z = t$ and consequently z is a common fixed point for f, g, S and T.
For the uniqueness, suppose there is another fixed point w and using (3.2) we get:

$$d(z,w) < \tau + F(\delta(Sz,Tw)) \leq F(\alpha d(z,w) + \beta d(z,Sz)$$
$$+\gamma d(w,Tw) + \lambda d(z,Tw) + Ld(w,Sz))$$
$$\leq F((\alpha + \lambda + L)d(z,w))$$
$$\leq F(d(z,w)),$$

which is a contradiction. Then z is unique. □

Example 3.1. Let $X = [0,4]$, $d(x,y) = |x-y|$ and f, g, S and T defined by

$$fx = gx = \begin{cases} \frac{x+2}{2}, & 0 \leq x \leq 2 \\ 1, & 2 < x \leq 4 \end{cases} \qquad Tx = Sx = \begin{cases} \{2\}, & 0 \leq x \leq 2 \\ [\frac{3}{2}, 2], & 2 < x \leq 4 \end{cases}$$

Consider a sequence $\{x_n\}$ for all $n \geq 1$ such that $x_n = 2 - \frac{1}{n}$, it is clear that

$$\lim_{n\to\infty} fx_n = 2 \in \{2\}$$

and

$$\lim_{n\to\infty} Sx_n = \{2\},$$

which implies that the pair (f, S) is subsequentially continuous. On other hand, we have

$$\lim n \to \infty \delta(fSx_n, Sfx_n) = \delta(\{2\}, \{2\}) = 0,$$

so (f, S) is δ-compatible.
For the inequality (3.1), we discuss the following cases:

1. For $x, y \in [0,1]$, we have: $\delta(Sx, Sy) = 0$, so (3.1) is satisfied for all x, y.

2. For $x \in [0,1]$ and $y \in (1,5]$, we have:

$$\delta(Sx, Sy) = \frac{1}{2} \leq e^{-\frac{1}{2}} \leq e^{-\frac{1}{2}} D(fy, Sy).$$

3. For $x, y \in (1,5]$ we have

$$\delta(Sx, Ty) = \frac{1}{2} \leq e^{-\frac{1}{2}} = e^{-\frac{1}{2}} D(fx, Sx).$$

4. For $x \in (1,5]$ and $y \in [0,1]$ we have

$$\delta(Sx, Sy) = \frac{1}{2} \leq e^{-\frac{1}{2}} = e^{-\frac{1}{2}} D(fx, Sx).$$

Then f and S satisfy(3.1), therefore 2 is the unique common strict fixed point of f and S.

4. Application to integral inclusions

In this subsection, we shall apply the obtained results to assert the existence of solution for a system of integral inclusions.
Let us consider the following integral inclusion systems.

$$x_i(t) \in f(t) + \int_0^1 K_i(t, s, x_i(s))ds, i = 1, 2 \tag{4.1}$$

where f is a continuous function on $[0,1]$, i,e., $f \in C([0,1])$ and $K : [0,1] \times [0,1] \times \mathbb{R} \to CB(\mathbb{R})$ is a set valued function.
Clearly $X = C([0,1])$ with convergence uniform metrics $d_\infty(x, y) = \sup_{x \in X} |x(t) - y(t)|$ is a complete metric space. Assume that

1. the function $K_i : (t, s) \mapsto K(t, s, x_1(s))$ is continuous on $[0,1] \times (0,1]$ for all $x \in C((0,1])$.
2. For all $x_i \in X$ and $k_i \in K_i$ $(i = 1, 2)$, there exists a function $\varphi : [0,1] \times [0,1] \to [0, +\infty)$ such that

$$|k_1(t, s, x_1(s)) - k_2(t, s, x_2(s)) \leq \varphi(t, s)|x_1 - x_2|.$$

3. There exists $\tau > 0$ such that

$$\sup_{t \in [0,1]} \int_0^1 \varphi(t, s)ds \leq e^{-\tau}.$$

4. There exist two sequences $\{x_n\}, \{y_n\}$ and two elements x, y in X such that

$$\lim_{n\to\infty} Sx_n = M \in B(X),$$

$$\lim_{n\to\infty} x_n = x \in M$$

and

$$\lim_{n\to\infty} Ty_n = N \in B(X),$$
$$\lim_{n\to\infty} y_n = y \in N.$$

Theorem 4.1. *Under the assumptions* $(1)-(4)$ *the system of integral inclusions (4.1) has a solution in* $C((0,1]) \times C([0,1])$.

Proof. Define two set valued mapping:

$$Sx_1(t) = \{z \in X, z(t) \in f(t) + \int_0^1 K_1(t,s,x_1(s))ds\},$$

$$Tx_2(t) = \{z \in X, z(t) \in f(t) + \int_0^1 K_2(t,s,x_2(s))ds\}.$$

The system (4.1) has a solution if and only if S and T have a common fixed point.
Denote I_X the identity operator on X.
From condition (4), the two pairs (I_X, S) and $(I_{X,T})$ are subsequentially continuous as well as δ-compatible.

For the contractive condition (3.1), let $x_1, x_2 \in C([0,1])$ and $z_1 \in Sx_1$, then there exists $k_1 \in K_1$ such that

$$z_1(t) = \int_0^1 k_1(s,t)ds,$$

for $z_2 \in f(t) + \int_0^1 K_2(t,s,x_2(t))ds$, i.e., $z_2(t) = f(t) + \int_0^1 k_2(t,s)ds$, we have

$$|z_1 - z_2| \le \int_0 |k_1(t,s) - k_2(t,s)|ds$$
$$\le \int_0^1 |x_1 - x_2|\varphi(t,s)ds.$$

Since $K_i, i = 1,2$ are bounded, so we have

$$\sup_{z_i \in X} |z_1 - z_2 \le \|x_1 - x_2\|_\infty \int_0^1 \varphi(t,s)ds,$$

which implies that

$$\delta(Sx_1, Tx_2) \le e^{-\tau} d(x_1, x_2)$$
$$\le e^{-\tau} \max\{d(x_1,x_2), d(x_1,Sx_1), d(x_2,Tx_2), \frac{1}{2}(d(x_1,Tx_2) + d(x_2,Sx_1))\}.$$

taking logarithm of two sides we get

$$\ln(\delta(Sx_1,Tx_2)) \le -\tau + \ln\Big(\max\{d(x_1,x_2), d(x_1,Sx_1), d(x_2,Tx_2), \frac{1}{2}(d(x_1,Tx_2) + d(x_2,Sx_1))\}\Big).$$

Hence all hypotheses of Theorem 3.1 satisfied with $F(t) = \ln t$ and $f = g = I_X$, therefore the system (4.1) has a solution. □

Conclusion. We have established common fixed point theorems for two hybrid pairs contraction of Wordowski type using δ-distance without exploiting the notion of continuity or reciprocal continuity, weak reciprocal continuity. Since F-contraction is a proper generalization of ordinary contraction, our results generalize, extend and improve the results of Wordowski [16] and others existing in literature, for instance Acar et al. [1], Ćirić [6], Cosentino et al. [7], Hardy Rogers [9] and Minak et al.[11] without using the completeness of space or subspace, and the containment requirement of range space.

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