FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 2 (2020), 507–522
https://doi.org/10.22190/FUMI2002507S

ON STAR COLORING OF DEGREE SPLITTING OF COMB PRODUCT GRAPHS

Ulagammal Subramanian and Vernold Vivin Joseph

Abstract. A star coloring of a graph G is a proper vertex coloring in which every path on four vertices in G is not bi-colored. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G. Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) - \bigcup_{i=1}^{t} S_i$. The degree splitting graph $DS(G)$ is obtained by adding vertices $w_1, w_2, \ldots w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$. The comb product between two graphs G and H, denoted by $G \triangleright H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and grafting the i^{th} copy of H at the vertex o to the i^{th} vertex of G. In this paper, we give the exact value of star chromatic number of degree splitting of comb product of complete graph with complete graph, complete graph with path, complete graph with cycle, complete graph with star graph, cycle with complete graph, path with complete graph and cycle with path graph.

Keywords: Star coloring; degree splitting graph; comb product

1. Introduction

All graphs in this paper are finite, simple, connected and undirected graph in [4, 5, 10]. The concept of star chromatic number was introduced by Branko Grunbaum in 1973. A star coloring [1, 8, 9] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. Equivalently, in a star coloring, the induced subgraph formed by the vertices of any two colors has connected components that are star graph. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G.

Guillaume Fertin et al. [8] determined the star chromatic number of trees, cycles, complete bipartite graphs, outer planar graphs and 2-dimensional grids. They also

Received October 25, 2018; accepted May 19, 2019
2010 *Mathematics Subject Classification.* Primary 05C15 ; Secondary 05C75

investigated and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, graphs with bounded treewidth and cubic graphs and planar graphs with high - girth.

Albertson et al. [1] showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, even when G is a graph that is both planar and bipartite. Coleman et al. [6] proved that star coloring remains NP-hard problem even on bipartite graphs.

For a given graph $G = (V(G), E(G))$ with $V(G) = S_1 \cup S_2 \cup S_3 \cup \ldots S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) - \bigcup_{i=1}^{t} S_i$. The degree splitting graph [11, 12] of G, denoted by $DS(G)$, is obtained by adding vertices $w_1, w_2, \ldots w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$.

Comb product is also same as the hierarchical product graphs was first introduced by Barriére et al. [3] in 2009. Also, the exact value of metric dimension of hierarchical product graphs was obtained by Tavakoli et al. in [14]. Let G and H be two connected graphs. Let o be a vertex of H. The comb product between G and H, denoted by $G \triangleright H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and grafting the i^{th} copy of H at the vertex o to the i^{th} vertex of G. By the definition of comb product, we can say that $V(G \triangleright H) = \{(a, u) \mid a \in V(G), u \in V(H)\}$ and $(a, u)(b, v) \in E(G \triangleright H)$ whenever $a = b$ and $uv \in E(H)$, or $ab \in E(G)$ and $u = v = o$. Ridho Alfarisi et al. [2] determined the partition dimension of comb product of path and complete graph and in [7] they also determined the star partition dimension of comb product of cycle and complete graph. Saputro et al. showed the metric dimension of comb product of the connected graphs G and H in [13].

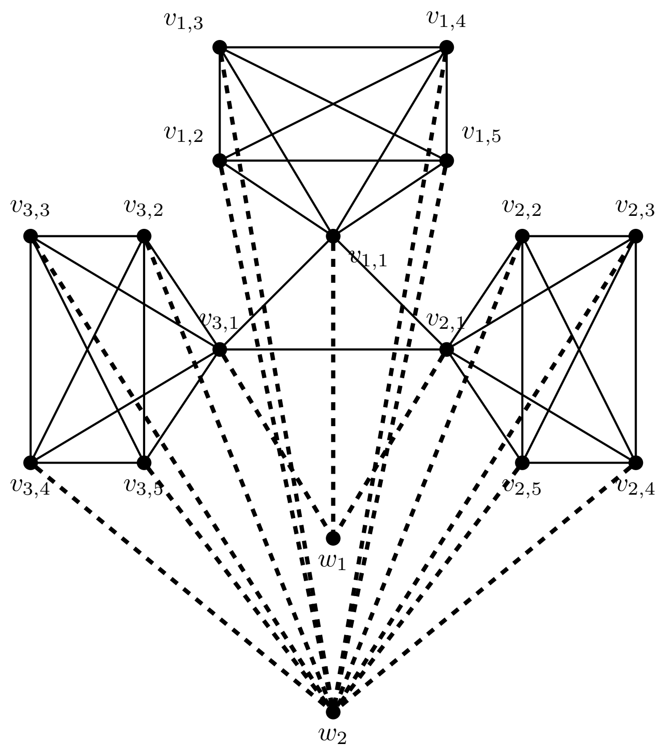

Figure 1: $DS(K_3 \triangleright K_5)$

In this paper, we have given the exact value of star chromatic number of degree splitting graph of comb product of complete graph with complete graph, complete graph with path, complete graph with cycle, complete graph with star graph, cycle with complete graph, path with complete graph and cycle with path graph denoted by $DS(K_m \triangleright K_n)$, $DS(K_m \triangleright P_n)$, $DS(K_m \triangleright C_n)$, $DS(K_m \triangleright K_{1,n})$, $DS(C_m \triangleright K_n)$, $DS(P_m \triangleright K_n)$ and $DS(C_m \triangleright P_n)$ respectively.

In order to prove our results, we shall make use of the following theorem by Guillaume et al. [8].

Theorem 1.1. *[8] If C_n is a cycle with $n \geq 3$ vertices, then*

$$\chi_s(C_n) = \begin{cases} 4, & \textit{when} \quad n = 5 \\ 3, & \textit{otherwise.} \end{cases}$$

Proof. The proof of the theorem can be found in [8]. □

2. Main Results

In the following subsections, we will find the star chromatic number of degree splitting graph of comb product of complete with complete graph, complete with path, complete with cycle, complete with star graph, comb product of cycle with complete, path with complete and cycle with path graph denoted by $DS\left(K_m \triangleright K_n\right)$, $DS\left(K_m \triangleright C_n\right)$, $DS\left(K_m \triangleright K_{1,n}\right)$, $DS\left(K_m \triangleright P_n\right)$, $DS\left(C_m \triangleright K_n\right)$, $DS\left(P_m \triangleright K_n\right)$ and $DS\left(C_m \triangleright P_n\right)$ respectively. Figure 1 shows an example of degree splitting of comb product $\left(K_3 \triangleright K_5\right)$.

2.1. Star Coloring of Degree Splitting of $\left(K_m \triangleright K_n\right)$

The comb product between K_m and K_n, denoted by $K_m \triangleright K_n$ has vertex set

$$V\left(K_m \triangleright K_n\right) = \left\{v_{i,j} : 1 \le i \le m, 1 \le j \le n\right\}$$

and edge set

$$\begin{aligned} E\left(K_m \triangleright K_n\right) &= \left\{v_{i,1}v_{i+k,1} : 1 \le i \le m, 1 \le k \le m-i\right\} \\ & \bigcup \left\{v_{i,j}v_{i,j+k} : 1 \le i \le m, 1 \le j \le n, 1 \le k \le n-j\right\}. \end{aligned}$$

Thus

$$\left|V\left(K_m \triangleright K_n\right)\right| = mn$$

and

$$\left|E\left(K_m \triangleright K_n\right)\right| = \frac{mn(n-1) + m(m-1)}{2}.$$

Theorem 2.1. *Let K_m and K_n be two complete graphs of order $m, n \ge 3$ and $m \le n$, then*

$$\chi_s\left(DS\left(K_m \triangleright K_n\right)\right) = m + n.$$

Proof. We have,

$$V\left(K_m \triangleright K_n\right) = \left\{v_{i,j} : 1 \le i \le m, 1 \le j \le n\right\} = S_1 \cup S_2$$

where

$$S_1 = \left\{v_{i,1} : 1 \le i \le m\right\}$$

and

$$S_2 = \left\{v_{i,j} : 1 \le i \le m, 2 \le j \le n\right\}.$$

To obtain $DS\left(K_m \triangleright K_n\right)$ from $K_m \triangleright K_n$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus, we get $V\left(DS\left(K_m \triangleright K_n\right)\right) = V\left(K_m \triangleright K_n\right) \cup \left\{w_1, w_2\right\}$. First we find the upper bound for $\chi_s\left(DS\left(K_m \triangleright K_n\right)\right)$.

Clearly, $m + n$ colors are needed at least to star color $DS\left(K_m \triangleright K_n\right)$. We now distinguish n as three cases: For every $1 \le i \le m$, Case(i): When $n \equiv 3 (\mathrm{mod}\ 3)$.

$$\sigma\left(v_{i,3k-2}\right) = i + j - 1, for \;\; 1 \leq k \leq \frac{n}{3}$$

$$\sigma\left(v_{i,3k-1}\right) = i + j - 1, for \;\; 1 \leq k \leq \frac{n}{3}$$

$$\sigma\left(v_{i,3k}\right) = i + j - 1, for \;\; 1 \leq k \leq \frac{n}{3}$$

and

$$\sigma(w_1) = \sigma(w_2) = m + n$$

Case(ii): When $n \equiv 1 (\mathrm{mod}\ 3)$.

$$\sigma\left(v_{i,3k-2}\right) = i + j - 1, for \;\; 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma\left(v_{i,3k-1}\right) = i + j - 1, for \;\; 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

$$\sigma\left(v_{i,3k}\right) = i + j - 1, for \;\; 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma(w_1) = \sigma(w_2) = m + n$$

Case(iii): When $n \equiv 2 (\mathrm{mod}\ 3)$.

$$\sigma\left(v_{i,3k-2}\right) = i + j - 1, for \;\; 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma\left(v_{i,3k-1}\right) = i + j - 1, for \;\; 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma\left(v_{i,3k}\right) = i + j - 1, for \;\; 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma(w_1) = \sigma(w_2) = m + n.$$

Thus, the upper bound for the star coloring of $(DS\left(K_m \triangleright K_n\right)) \leq m + n$.

Now, we prove the lower bound for $\chi_s\left(DS\left(K_m \triangleright K_n\right)\right)$.

Suppose $\chi_s\left(DS\left(K_m \triangleright K_n\right)\right) < m + n$. Let $\chi_s\left(DS\left(K_m \triangleright K_n\right)\right) = m + n - 1$, then there exists a bicolored path P_4. Since $\{v_{1,i}\}$ induce a clique of order n (say K_n). If we assign the same n colors to the second copy of K_n, then we get a path on four vertices between these clique which is bicolored, a contradiction for proper star coloring. Thus, $\chi_s\left(DS\left(K_m \triangleright K_n\right)\right) = m+n-1$ color is impossible. Therefore, the lower bound of $\chi_s\left(DS\left(K_m \triangleright K_n\right)\right) \geq m + n$. Thus we get the lower and upper bound of $\chi_s\left(DS\left(K_m \triangleright K_n\right)\right)$. Hence $\chi_s\left(DS\left(K_m \triangleright K_n\right)\right) = m + n$. This completes the proof of the theorem. $\square$

2.2. Star Coloring of Degree Splitting of $(K_m \triangleright C_n)$

A graph $K_m \triangleright C_n$ has vertex set

$$V(K_m \triangleright C_n) = \{v_{i,j} : 1 \le i \le m, 1 \le j \le n\}$$

and edge set

$$\begin{aligned} |E(K_m \triangleright C_n)| &= \{v_{i,1}v_{i+k,1} : 1 \le i \le m, 1 \le k \le m-i\} \\ &\bigcup \{v_{i,j}v_{i,j+1} : 1 \le i \le m-1, 1 \le j \le n-1\} \bigcup \{v_{m,1}v_{1,1}\}. \end{aligned}$$

Thus

$$|V(K_m \triangleright C_n)| = mn$$

and

$$|E(K_m \triangleright C_n)| = \frac{m(m-1)+2mn}{2}.$$

Theorem 2.2. *Let K_m and C_n be two connected graphs of order $m \ge 4$ and $n \ge 5$, then*

$$\chi_s(DS(K_m \triangleright C_n)) = m+1.$$

Proof. We have

$$V(K_m \triangleright C_n) = \{v_{i,j} : 1 \le i \le m, 1 \le j \le n\} = S_1 \cup S_2$$

where

$$S_1 = \{v_{i,1} : 1 \le i \le m\}$$

and

$$S_2 = \{v_{i,j} : 1 \le i \le m, 2 \le j \le n\}.$$

To obtain $DS(K_m \triangleright C_n)$ from $K_m \triangleright C_n$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus we get

$$V(DS(K_m \triangleright C_n)) = \{v_{i,j} : 1 \le i \le m, 1 \le j \le n\} \bigcup \{w_1, w_2\}.$$

We first prove the lower bound for the star chromatic number of degree splitting of comb product of complete graph with cycle. For this, we show that any coloring with m colors will give us at least one bicolored path of length 4. Since each $\{v_{i,1} : 1 \le i \le m\}$ is adjacent to w_1, it gives a complete graph of order $m+1$. Thus, no coloring that uses m colors can be a star coloring. Therefore, the lower bound of star chromatic number is $\chi_s(DS(K_m \triangleright C_n)) \ge m+1$.

Now, we prove the upper bound for the star chromatic number of degree splitting of $(K_m \triangleright C_n)$. Since the complete graph has the chromatic number m. We assign the m colors to the mn vertices of the graph $K_m \triangleright C_n$ alternatively and we assign $\sigma(w_1) = \sigma(w_2) = m+1$. Thus the upper bound of the $\chi_s(DS(K_m \triangleright C_n)) \le m+1$.

Thus we get the lower and upper bound of the $\chi_s(DS(K_m \triangleright C_n))$. Therefore,

$$\chi_s(DS(K_m \triangleright C_n)) = m+1.$$

This concludes the proof of the theorem. □

2.3. Star Coloring of Degree Splitting of $(K_m \triangleright P_n)$

A graph $K_m \triangleright P_n$ has vertex set

$$V\left(K_m \triangleright P_n\right) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} E\left(K_m \triangleright P_n\right) &= \{v_{i,1}v_{i+k,1} : 1 \leq i \leq m, 1 \leq k \leq m-i\} \\ &\quad \bigcup \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}. \end{aligned}$$

Thus

$$|V\left(K_m \triangleright P_n\right)| = mn$$

and

$$|E\left(K_m \triangleright P_n\right)| = \frac{m(m-1)+2m(n-1)}{2}.$$

Theorem 2.3. *Let K_m be a complete graph of order $m \geq 3$ and P_n be a path graph of order $n \geq 3$ then,*

$$\chi_s\left(DS\left(K_m \triangleright P_n\right)\right) = m+1.$$

Proof. We have

$$V\left(K_m \triangleright P_n\right) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \{v_{i,1} : 1 \leq i \leq m\},$$

$$S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n-1\}$$

and

$$S_3 = \{v_{i,n} : 1 \leq i \leq m\}.$$

To obtain $DS\left(K_m \triangleright P_n\right)$ from $K_m \triangleright P_n$, we add three vertices w_1, w_2 and w_3 corresponding to S_1, S_2 and S_3 respectively. Thus, $V\left(DS\left(K_m \triangleright P_n\right)\right) = V\left(K_m \triangleright P_n\right) \cup \{w_1, w_2, w_3\}$. Now, we assign the following coloring pattern:

For every $1 \leq i \leq m$

For $n \equiv 1 (\text{mod } 3)$

$$\sigma\left(v_{i,3k-2}\right) = i+j-1 (\text{mod } m),\ for\ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma\left(v_{i,3k-1}\right) = i+j-1 (\text{mod } m),\ for\ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

$$\sigma\left(v_{i,3k}\right) = i+j-1 (\text{mod } m),\ for\ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = m + 1.$$

For $n \equiv 2 (\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1 (\text{mod } m), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k-1}) = i + j - 1 (\text{mod } m), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k}) = i + j - 1 (\text{mod } m), \ for \ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = m + 1.$$

For $n \equiv 3 (\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1 (\text{mod } m), \ for \ \ 1 \leq k \leq \frac{n}{3}$$

$$\sigma(v_{i,3k-1}) = i + j - 1 (\text{mod } m), \ for \ \ 1 \leq k \leq \frac{n}{3}$$

$$\sigma(v_{i,3k}) = i + j - 1 (\text{mod } m), \ for \ \ 1 \leq k \leq \frac{n}{3}$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = m + 1.$$

Thus the upper bound of star coloring of degree splitting of $(K_m \triangleright P_n) \leq m + 1$.

Now, we prove the lower bound of $\chi_s(DS(K_m \triangleright P_n))$. Suppose the lower bound of the

$$\chi_s(DS(K_m \triangleright P_n)) < m + 1.$$

That is

$$\chi_s(DS(K_m \triangleright P_n)) = m.$$

We must assign m colors for $\{v_{i,1}, 1 \leq i \leq m\}$ for proper star coloring. Since each $\{v_{i,1}\}$ is adjacent to w_1, it gives a complete graph of order $m + 1$. Therefore $\chi_s(DS(K_m \triangleright P_n))$ with m colors is impossible. Therefore $\chi_s(DS(K_m \triangleright P_n)) \geq m + 1$. Hence, $\chi_s((DS(K_m \triangleright P_n)) = m + 1$. This concludes the proof of the theorem. □

2.4. Star Coloring of Degree Splitting of $(K_m \triangleright K_{1,n})$

A graph $K_m \triangleright K_{1,n}$ has a vertex set

$$V\left(K_m \triangleright K_{1,n}\right) = \{v_{i,j} : 1 \le i \le m, 1 \le j \le n\}$$

and edge set

$$\begin{aligned} E\left(K_m \triangleright K_{1,n}\right) &= \{v_{i,1}v_{i+k,1} : 1 \le i \le m-1, 1 \le k \le m-i\} \\ &\bigcup \{v_{i,1}v_{i,j+1} : 1 \le i \le m, 1 \le j \le n\}. \end{aligned}$$

Thus

$$|V\left(K_m \triangleright K_{1,n}\right)| = mn$$

and

$$|E\left(K_m \triangleright K_{1,n}\right)| = \frac{m(m-1)+2mn}{2}.$$

Theorem 2.4. *Let K_m be a complete graph of order m, $(m \ge 3)$ and $K_{1,n}$ be a star graph with n + 1 vertices $(n \ge 2)$ then*

$$\chi_s\left(DS\left(K_m \triangleright K_{1,n}\right)\right) = m+1.$$

Proof. We have

$$V\left(K_m \triangleright K_{1,n}\right) = \{v_{i,j} : 1 \le i \le m, 1 \le j \le n\} = S_1 \cup S_2.$$

To obtain $DS(K_m \triangleright K_{1,n})$ from $(K_m \triangleright K_{1,n})$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively, where

$$S_1 = \{v_{i,1} : 1 \le i \le m\}$$

and

$$S_2 = \{v_{i,j} : 1 \le i \le m, 2 \le j \le n\}.$$

Thus we get

$$V\left(DS\left(K_m \triangleright K_{1,n}\right)\right) = V\left(K_m \triangleright K_{1,n}\right) \cup \{w_1, w_2\}.$$

Now we assign the coloring pattern as follows:

For every $1 \le i \le m$, assign i to $\sigma(v_{i,1})$, and

For $2 \le j \le n$ assign

$$\sigma(v_{i,j}) = \begin{cases} 2 \ if \ i \equiv 1 (\text{mod } 3) \\ 3 \ if \ i \equiv 2 (\text{mod } 3) \\ 1 \ if \ i \equiv 3 (\text{mod } 3) \end{cases}$$

alternatively

and

$$\sigma(w_1) = \sigma(w_2) = m + 1.$$

Thus $\chi_s((DS(K_m \triangleright K_{1,n})) \geq m + 1$.

Now, we prove the lower bound of $\chi_s(DS(K_m \triangleright K_{1,n}))$. Suppose the lower bound of the $\chi_s(DS(K_m \triangleright K_{1,n})) < m + 1$. That is $\chi_s(DS(K_m \triangleright K_{1,n})) = m$. We must assign m colors for $\{v_{i,1} : 1 \leq i \leq m\}$ for proper star coloring. Since each $\{v_{i,1}\}$ is adjacent to w_1, it gives a complete graph of order $m + 1$. Therefore $\chi_s(DS(K_m \triangleright K_{1,n}))$ with m color is impossible. Therefore $\chi_s(DS(K_m \triangleright K_{1,n})) \geq m + 1$. Thus we get the lower and upper bound of $\chi_s(DS(K_m \triangleright K_{1,n}))$. Hence, $\chi_s(DS(K_m \triangleright K_{1,n})) = m + 1$. It concludes the proof of the theorem. □

2.5. Star Coloring of Degree Splitting of $(C_m \triangleright K_n)$

A graph $C_m \triangleright K_n$ has vertex set

$$V(C_m \triangleright K_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} E(C_m \triangleright K_n) &= \{v_{i,1}v_{i+1,1} : 1 \leq i \leq m-1\} \\ &\bigcup \{v_{m,1}v_{1,1}\} \bigcup \{v_{i,j}v_{i,j+k} : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq n-j\}. \end{aligned}$$

Thus

$$|V(C_m \triangleright K_n)| = mn$$

and

$$|E(C_m \triangleright K_n)| = \frac{mn^2 - mn + 2m}{2}.$$

Theorem 2.5. *Let C_m and K_n be two connected graphs of order $m > n$ and $m > 3, n \geq 3$, then*

$$\chi_s(DS(C_m \triangleright K_n)) = m + 1.$$

Proof. We have

$$V(C_m \triangleright K_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2$$

where

$$S_1 = \{v_{i,1} : 1 \leq i \leq m\}$$

and

$$S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}.$$

To obtain $DS(C_m \triangleright K_n)$ from $C_m \triangleright K_n$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus we get $V(DS(C_m \triangleright K_n)) = V(C_m \triangleright K_n) \cup \{w_1, w_2\}$. First we find the upper bound for $\chi_s(DS(C_m \triangleright K_n))$.

We define the coloring pattern as follows:

For every $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$\sigma\left(v_{i,j}\right) = i + j - 1(\text{mod } m)$$

and also

$$\sigma\left(w_1\right) = \sigma\left(w_2\right) = m + 1.$$

Thus the upper bound for star chromatic number of $(DS\left(C_m \triangleright K_n\right)) \leq\ m + 1$.

Now, we prove the lower bound for $\chi_s((DS\left(C_m \triangleright K_n\right))$.

Suppose the lower bound of $\chi_s((DS\left(C_m \triangleright K_n\right)) < m{+}1$. Let $\chi_s((DS\left(C_m \triangleright K_n\right)) = m$, then there exist a bicolored path P_4. Since $\{v_{1,i}\}$ induce a clique of order n. If we assign the same n colors to the second copy of the clique, then we get a path on four vertices between these cliques which is bicolored, a contradiction for proper star coloring. Thus we obtain $\chi_s((DS\left(C_m \triangleright K_n\right)) = m$ color is impossible. It concludes that the lower bound is $\chi_s((DS\left(C_m \triangleright K_n\right)) \geq m + 1$. Therefore, $\chi_s((DS\left(C_m \triangleright K_n\right)) = m + 1$. Hence the proof of the theorem. □

2.6. Star Coloring of Degree Splitting of $(P_m \triangleright K_n)$

A graph $P_m \triangleright K_n$ has a vertex set

$$V\left(P_m \triangleright K_n\right) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} E\left(P_m \triangleright K_n\right) &= \{v_{i,1}v_{i+1,1} : 1 \leq i \leq m-1\} \\ &\bigcup \left\{v_{i,j}v_{i,j+k} : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq n-j\right\}. \end{aligned}$$

Thus

$$\left|V\left(P_m \triangleright K_n\right)\right| = mn$$

and

$$\left|E\left(P_m \triangleright K_n\right)\right| = \frac{mn(n-1) + 2(m-1)}{2}.$$

Theorem 2.6. *Let P_m be a path graph of order $m \geq 4$ and K_n be a complete graph with $n \geq 2$, then*

$$\chi_s\left(DS\left(P_m \triangleright K_n\right)\right) = n + 2.$$

Proof. We have

$$V\left(P_m \triangleright K_n\right) = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\} = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \left\{v_{1,1}, v_{m,1}\right\},$$

$$S_2 = \{v_{i,1} : 2 \leq i \leq m-1\}$$

and

$$S_3 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}.$$

To obtain $DS(P_m \triangleright K_n)$ from $P_m \triangleright K_n$, we add three vertices w_1, w_2 and w_3 corresponding to S_1, S_2 and S_3 respectively. Thus we get $V(DS(P_m \triangleright K_n)) = \{v_{i,j} : 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{w_1, w_2, w_3\}$. First we find the upper bound for $\chi_s(DS(P_m \triangleright K_n))$. For every $1 \leq i \leq m$,

For $n \equiv 1(\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1(\text{mod } n+1), \; for \;\; 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k-1}) = i + j - 1(\text{mod } n+1), \; for \;\; 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil - 1$$

$$\sigma(v_{i,3k}) \;\; = i + j - 1(\text{mod } n+1), \; for \;\; 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil - 1$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = n + 2.$$

For $n \equiv 2(\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1(\text{mod } n+1), \; for \;\; 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k-1}) = i + j - 1(\text{mod } n+1), \; for \;\; 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k}) = i + j - 1(\text{mod } n+1), \; for \;\; 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil - 1$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = n + 2.$$

For $n \equiv 3(\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1(\text{mod } n+1), \; for \;\; 1 \leq k \leq \frac{n}{3}$$

$$\sigma(v_{i,3k-1}) = i + j - 1(\text{mod } n+1), \; for \;\; 1 \leq k \leq \frac{n}{3}$$

$$\sigma(v_{i,3k}) \;\; = i + j - 1(\text{mod } n+1), \; for \;\; 1 \leq k \leq \frac{n}{3}$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = n + 2.$$

Thus $\chi_s(DS(P_m \triangleright K_n) \leq n + 2$.

Now, we prove that $\chi_s(DS\,(P_m \triangleright K_n) \geq n+2$. Suppose $\chi_s(DS\,(P_m \triangleright K_n) < n+2$. That is $\chi_s(DS\,(P_m \triangleright K_n) = n+1$. Since $\{v_{1,i}\}$ induce a clique of order n. If we assign the same n colors to the second copy of the clique, then we get a path on four vertices between these cliques which is bicolored, a contradiction for proper star coloring. Thus $\chi_s(DS\,(P_m \triangleright K_n) \geq n+1$. Therefore, $\chi_s(DS\,(P_m \triangleright K_n) = n+2$. Hence, there is a another proof to the theorem. □

2.7. Star Coloring of Degree Splitting of $(C_m \triangleright P_n)$

A graph $C_m \triangleright P_n$ has vertex set

$$V\,(C_m \triangleright P_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} E\,(C_m \triangleright P_n) \quad &= \quad \{v_{i,1}v_{i+1,1} : 1 \leq i \leq m-1\} \\ &\bigcup \{v_{m,1}v_{1,1}\} \bigcup \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}. \end{aligned}$$

Thus

$$|V\,(C_m \triangleright P_n)| = mn$$

and

$$|E\,(C_m \triangleright P_n)| = m + m(n-1).$$

Theorem 2.7. *Let C_m be a cycle of length $m \geq 3$ and P_n be a path of length $n \geq 3$ then,*

$$\chi_s\,(DS\,(C_m \triangleright P_n)) = \begin{cases} 4, if\ m = 3k, k \geq 1 \\ 5, otherwise \end{cases}.$$

Proof. We have

$$V\,(C_m \triangleright P_n) = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\} = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \{v_{i,1} : 1 \leq i \leq m\},$$

$$S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n-1\}$$

and

$$S_3 = \{v_{i,n} : 1 \leq i \leq m\}.$$

To obtain $DS\,(C_m \triangleright P_n)$ from $C_m \triangleright P_n$, we add three vertices w_1, w_2 and w_3 corresponding to S_1, S_2 and S_3 respectively. Thus we get $V\,(DS\,(C_m \triangleright P_n)) = V\,(C_m \triangleright P_n) \cup \{w_1, w_2, w_3\}$. First we find the upper bound for $\chi_s\,(DS\,(C_m \triangleright P_n))$.

The star chromatic number is defined as follows:

Case(i):

If $m = 3k$, $k \geq 1$

For $n \equiv 1 (\mathrm{mod}\ 3)$

$$\sigma\left(v_{i,3k-2}\right) = i (\mathrm{mod}\ 3),\ for\ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma\left(v_{i,3k-1}\right) = i + 1 (\mathrm{mod}\ 3),\ for\ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

$$\sigma\left(v_{i,3k}\right) = i + 2 (\mathrm{mod}\ 3),\ for\ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma\left(w_1\right) = \sigma\left(w_2\right) = \sigma\left(w_3\right) = 4.$$

For $n \equiv 2 (\mathrm{mod}\ 3)$

$$\sigma\left(v_{i,3k-2}\right) = i (\mathrm{mod}\ 3),\ for\ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma\left(v_{i,3k-1}\right) = i + 1 (\mathrm{mod}\ 3),\ for\ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma\left(v_{i,3k}\right) = i + 2 (\mathrm{mod}\ 3),\ for\ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma\left(w_1\right) = \sigma\left(w_2\right) = \sigma\left(w_3\right) = 4.$$

For $n \equiv 3 (\mathrm{mod}\ 3)$

$$\sigma\left(v_{i,3k-2}\right) = i (\mathrm{mod}\ 3),\ for\ \ 1 \leq k \leq \frac{n}{3}$$

$$\sigma\left(v_{i,3k-1}\right) = i + 1 (\mathrm{mod}\ 3)\ for\ \ 1 \leq k \leq \frac{n}{3}$$

$$\sigma\left(v_{i,3k}\right) = i + 2 (\mathrm{mod}\ 3),\ for\ \ 1 \leq k \leq \frac{n}{3}$$

and

$$\sigma\left(w_1\right) = \sigma\left(w_2\right) = \sigma\left(w_3\right) = 4.$$

Thus, $\chi_s\left(DS\left(C_m \triangleright P_n\right)\right) = 4$ if $m = 3k$, $k \geq 1$.

Case(ii)(a): When $m = 3k + 1$, $k \in N$.

We color the $3k$ vertices of C_m by $\sigma\left(v_{i,1}\right) = i(\ \mathrm{mod}\ 3)$ and we assign the remains of one uncolored vertex by 4.

Also, we assign

$$\sigma\left(v_{i,j}\right) = i + j - 1 (\mathrm{mod}\ 3).$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = 5.$$

Case(ii)(b): When $m = 3(k-1)+2$, $k \in N$, here $m = 5$ is not included. That is $m = 3(k-1)+5$, $k \geq 2$.

We color the $3(k-1)$ vertices of C_m by 1, 2 and 3 and for the remaining five vertices assign the color 4, 1, 2, 3, 4.

Also, we assign

$$\sigma(v_{i,j}) = i + j - 1 (\text{mod } 3).$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = 5.$$

Thus, $\chi_s(DS(C_m \triangleright P_n)) = 5$.

When $m = 5$, then $\chi_s(DS(C_m \triangleright P_n)) = 5$. Hence, the theorem is proved. □

REFERENCES

1. M. O. Albertson, G. G. Chappell, H. A. Kiestead, A. Kündgen and R. Ramamurthi: *Coloring with no 2-colored P_4's.* Electron. J. Comb. **11** (2004), Paper # R26.
2. R. Alfarisi and D. Darmaji: *On the star parition dimension of comb product of cycle and complete graph.* In: International Conference on Mathematics: Education, Theory and Application, IOP series Journal of physics: Conference series, **855** 2017, 012005.
3. L. Barriére, F. Comellas, C. Dafió and M. A. Fiol: *The hierarchical product of graphs.* Discrete Appl. Math. **157** (2009), 39–48.
4. J. A. Bondy and U. S. R. Murty: *Graph theory with applications.* MacMillan, London, 1976.
5. J. Clark and D. A. Holton: *A first look at graph theory.* World Scinetific, 1969.
6. T. F. Coleman and J. Moré: *Estimation of sparse Hessian matrices and graph coloring problems.* Math. Program. **28**(3) (1984), 243–270.
7. D. Darmaji and R. Alfarisi: *On the parition dimension of comb product of path and complete graph.* In: International Conference on Mathematics: Pure, Applied and Computation, AIP Conf.Proc.1867, 020038-1-020038-7.
8. G. Fertin, A. Raspaud and B. Reed: *On Star coloring of graphs.* J. Graph theory. **47**(3) (2004), 163–182.
9. B. Grünbaum: *Acyclic colorings of planar graphs.* Israel J. Math. **14** (1973), 390–408.
10. F. Harary: *Graph theory.* Narosa Publishing home, New Delhi 1969.
11. R. Ponraj and S. Somasundaram: *On the degree splitting graph of a graph.* Natl. Acad. Sci. Lett. **27** (7-8) (2004), 275–278.
12. E. Sampathkumar and H. B. Walikar: *On splitting graph of a graph.* Journal of Karnatak University Science, **25 & 26** (Combined) (1980-81), 13–16.

13. S. W. Saputro, N. Mardiana and I. A. Purwasih: *The metric dimension of comb product graphs*. In: Graph theory conference in honor of Egawa's 60th birthday, 2013, pp. 10–14.

14. M. Tavakoli, F. Rahbarnia and A. R. Ashrafi: *Distribution of some graph invariants over hierarchical product of graphs*. Appl. Math. Comput. **220** (2013), 405–413.

Ulagammal Subramanian
Department of Mathematics
University College of Engineering Nagercoil
(A Constituent College of Anna University, Chennai)
Konam, Nagercoil- 629 004
Tamil Nadu
India
ulagammal2877@gmail.com

Vernold Vivin Joseph
Department of Mathematics
University College of Engineering Nagercoil
(A Constituent College of Anna University, Chennai)
Konam, Nagercoil- 629 004
Tamil Nadu
India
vernoldvivin@yahoo.in