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A NOTE ON OPERATORS CONSISTENT IN INVERTIBILITY

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Abstract. We generalize the notion of consistency in invertibility to Banach algebras and prove that the set of all elements consistent in invertibility is an upper semiregularity. In the case of bounded liner operators on a Hilbert space, we give a complete answer when the set of all CI operators will be a regularity. Analogous results are obtained for Fredholm consistent operators.
Keywords: Banach algebra; invertibility; semiregularity; Hilbert space.

1. Notations, motivations and preliminaries

For a closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ we use the symbol $P_{\mathcal{M}}$ to denote the orthogonal projection onto $\mathcal{M}$. For a given operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively, while $n(A) = \dim \mathcal{N}(A)$ and $d(A) = \dim \mathcal{R}(A)^{\perp}$
The notion of operators consistent in invertibility, CI for short, was introduced by Gong and Han in [7]. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is consistent in invertibility (CI) if for each $A \in \mathcal{B}(\mathcal{H})$, AT is invertible if and only if TA is invertible. A characterization of CI operators is given by the next Theorem:

Theorem 1.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is CI operator if and only if one of the three mutually exclusive cases hold:*

(i) T is invertible;

(ii) $\mathcal{R}(T)$ is not closed;

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(iii) $\mathcal{N}(T) \neq \{0\}$ *and* $\mathcal{R}(T) = \overline{\mathcal{R}(T)} \neq \mathcal{H}$.

It is easy to see that an operator $T \in \mathcal{B}(\mathcal{H})$ is not CI if and only if T is left invertible but not right invertible, or right invertible but not left invertible. The CI spectrum of $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$\sigma_{CI}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not } CI\}$$

Results concerning CI operators were obtained in [8, 9] and [1, 2, 10]. It is fairly easy to see that if A and B are CI operators, then AB is a CI operator, it would be of interest to determine whether the set of all CI operators is a regularity. We will prove that in general this is not the case.
The notion of consistency has been generalised, and explored in other cases, such as Fredholm consistency (FC) ([1, 2]). Using a characterization of FC operators used in [2] given in the following Theorem we will answer the same questions we did in the case of CI operators in $\mathcal{B}(\mathcal{H})$:

Theorem 1.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T if Fredholm consistent (FC) if and only if one of the following conditions is satisfied:*

(i) T is Fredholm,

(ii) $\mathcal{R}(T)$ is closed, $n(T) = d(T) = \infty$,

(iii) $\mathcal{R}(T)$ is not closed.

It is easy to see that an operator $T \in \mathcal{B}(\mathcal{H})$ is not Fredholms consistent if and only if T is left Fredholm, but not right Fredhlom, or it is right Fredholm, but not left Fredholm. Some other recent results on Fredholm operators can be found in

Let us now recall the definition of a regularity (upper semiregularity) in a Banach algebra:

Definition 1.1. [4] Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called a regularity if

(1) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$ then $a \in R \Leftrightarrow a^n \in R$,

(2) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1_{\mathcal{A}}$, then $ab \in R \Leftrightarrow a \in R$ and $b \in R$.

Definition 1.2. [5] Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called an upper semiregularity if

(1) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$ then $a \in R \Rightarrow a^n \in R$,

(2) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac+bd = 1_{\mathcal{A}}$, and $a, b \in R$, then $ab \in R$.

(3) R contains a neighborhood of the unit element $1_{\mathcal{A}}$.

Some important examples of regularities include sets of all invertible (left invertible, right invertible) operators, Fredholm (left Fredholm, right Fredholm) operators etc.

2. Consistency in invertibility

We introduce CI elements in Banach algebras in the same manner. Let $\mathcal{A}$ be a Banach algebra, and $\mathcal{A}^{-1}$ the group of all invertible elements. We say that $a \in \mathcal{A}$ is consistent in invertibility (CI) if for all $c \in \mathcal{A}$

$$ac \in \mathcal{A}^{-1} \Leftrightarrow ca \in \mathcal{A}^{-1}.$$

First we prove a lemma which gives a characterisation of CI elements similar to the characterisation of CI operators:

Lemma 2.1. *A Banach algebra element a is not CI if and only if $a \in \mathcal{A}_l^{-1} \setminus \mathcal{A}_r^{-1}$ or $a \in \mathcal{A}_r^{-1} \setminus \mathcal{A}_l^{-1}$.*

Proof. Assume $a \in \mathcal{A}$ is not CI. Then there exists an element $c \in \mathcal{A}$ such that $ac \in \mathcal{A}^{-1}$ and $ca \notin \mathcal{A}^{-1}$, or $ca \in \mathcal{A}^{-1}$ and $ac \notin \mathcal{A}^{-1}$. If the first statement is correct, since $ac \in \mathcal{A}^{-1}$ we have that a must be right invertible. If a were left invertible as well, then c would be invertible, and ca would be invertible as well. From this contradiction we see that $a \in \mathcal{A}_r^{-1} \setminus \mathcal{A}_l^{-1}$. We analogously conclude that in the other case $a \in \mathcal{A}_l^{-1} \setminus \mathcal{A}_r^{-1}$. If $a \in \mathcal{A}_l^{-1} \setminus \mathcal{A}_r^{-1}$ we have that $a_l^{-1}a = 1_{\mathcal{A}}$ and $aa_l^{1} \notin \mathcal{A}^{-1}$ for an arbitrary left inverse of a, so a is not CI. We analogously conclude that a is not CI when $a \in \mathcal{A}_r^{-1} \setminus \mathcal{A}_l^{-1}$ as well. □

Theorem 2.1. *The set of all CI elements in $\mathcal{A}$ is an upper semiregularity.*

Proof. If a, b are commuting CI elements and $c \in \mathcal{A}$ arbitrary we have that

$$abc \text{ is invertible} \Leftrightarrow bca \text{ is invertible} \Leftrightarrow$$

$$\Leftrightarrow cab \text{ is invertible}$$

This stronger statement implies that conditions (1), and (2) of Definition 1.2 are satisfied.
Since invertible elements are CI, and we know that there exists an open neighborhood of $1_{\mathcal{A}}$ where all elements are invertible. We conclude that there exists an open neighborhood of $1_{\mathcal{A}}$ where all elements are CI. This completes the proof. □

As a corollary of the previous Theorem we have:

Corollary 2.1. *The set of all CI operators in $\mathcal{B}(\mathcal{H})$ is an upper semiregularity*

Since all invertible elements in a Banach algebra are CI have that $\sigma_{CI}(a) \subseteq \sigma(a)$, where

$$\sigma_{CI}(a) = \{\lambda \in \mathbb{C} : \ \lambda - a \text{ is not } CI\}.$$

Recall the following Theorem from [5]:

Theorem 2.2. *[5] Let $\mathcal{R} \subset \mathcal{A}$ be an upper semiregularity. Suppose that $\mathcal{R}$ satisfies the condition*

$$b \in \mathcal{R} \cap \mathcal{A}^{-1} \Rightarrow b^{-1} \in \mathcal{R}.$$

Then $\sigma_{\mathcal{R}}(f(a)) \subset f(\sigma_{\mathcal{R}}(a))$ for all $a \in \mathcal{A}$ and all locally non-constant functions f analytic on a neighborhood of $\sigma(a) \cup \sigma_{\mathcal{R}}(a)$.
Further, $\sigma_{\mathcal{R}}(f(a)) \subset f(\sigma_{\mathcal{R}}(a) \cup \sigma(a))$ for all functions f analytic on a neighborhood of $\sigma_{\mathcal{R}}(a) \cup \sigma(a)$.

Since $\sigma_{CI}(a) \subseteq \sigma(a)$ (and thus $\sigma_{CI}(a) \cup \sigma(a) = \sigma(a)$) we get that the following Theorem holds:

Theorem 2.3. *For every $a \in \mathcal{A}$ $\sigma_{CI}(f(a)) \subseteq f(\sigma_{CI}(a))$ for all locally non-constant functions f analytic on a neighborhood of $\sigma(a) \cup \sigma_{CI}(a) = \sigma(a)$, and $f(\sigma_{CI}(a)) \subseteq f(\sigma(a))$ for all functions f analytic on a neighborhood of $\sigma(a)$.*

It is now only natural to ask what further properties does the set of all bounded linear operators (Banach algebra elements) consistent in invertibility satisfy, and under which conditions it will be a regularity.
Remark: We from lemma 2.1 we see that

$$\sigma_{CI}(a) = (\sigma_l(a) \setminus \sigma_r(a)) \cup (\sigma_r(a) \setminus \sigma_l(a)).$$

In the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ this implies that the consistency spectrum of a bounded linear operator can be empty. For example, self-adjoint (normal) operators on Hilbert spaces will have an empty CI spectrum.

It would be natural to check whether the CI spectrum is closed, and from the following example we will see that this is generally not the case.

Example 1 Define the operator T on $\mathcal{B}(l^2 \oplus l^2)$ by

$$T = 2S \oplus (I - S^*) : l^2 \oplus l^2 \to l^2 \oplus l^2$$

where S is the right shift operator on l^2. Let $(\lambda_n)_n$ be a sequence of complex numbers such that

$$\lim_{n\to\infty} \lambda_n = 2, \ \lambda_n \in B(0,2) \setminus B(1,1),$$

where $B(\lambda, r)$ is the open ball with radius r and center λ. Recall that $S - \lambda I$ is right, but not left invertible for $|\lambda| < 1$, and $S - \lambda I$ is left, but not right invertible for $|\lambda| = 1$, and $S - \lambda I$ is invertible for $|\lambda| > 1$. We have that each $\lambda_n \in \sigma_{CI}(T)$ because $2S - \lambda_n I$ is right, but not left invertible, and $(1 - \lambda_n)I - S^*$ is invertible, T is left, but not right invertible. However, since $2S - 2I$ is not right invertible and $I - S^* - 2I = -(S^* + I)$ is not left invertible (as the Hilbert adjoint of an operator which is not right invertible), we see that $T - 2I$ is neither left nor right invertible, so $T - 2I$ is CI. We get that $\sigma_{CI}(T)$ is not closed.

It is easy to see that $T \in \mathcal{B}(\mathcal{H})$ is CI if and only T^n is CI for $n \geq 1$ so it is natural to investigate whether the set of all CI operators forms a regularity. The following examples will serve as motivation for the answer:

Example: 2. Let T and P_M be operators $\mathcal{B}(l^2)$ defined in the following way, $T = S^2$, where the S is the right shift operator on l^2 and P_M the orthogonal projection on the subspace

$$M = \{x = (x_1, x_2, \dots, x_n, \dots) \in l^2 : \ x_{2n-1} = x_{2n}, \ n \in \mathbb{N}\}.$$

Let $x = (x_1, x_2, \dots, x_n, \dots) \in l^2$ be arbitrary, then $(x_1, x_1, x_2, x_2, \dots, x_n, x_n, \dots)$ is an elements of M, so M is a non-trivial subspace of l^2. It is easy to verify that M is closed. It is easy to see that T commutes with P_M and $P_{M^\perp}$. We have that

$$2P_{M^\perp} + 2P_M - T = 2I - T,$$

which is invertible. For an $x \in l^2$ we have

$$(2P_M - T)x = (x_1 + x_2, x_1 + x_2, x_3 + x_4 - x_1, x_3 + x_4 - x_2, \dots).$$

Since $(1, 0, \dots, 0, \dots) \notin \mathcal{R}(2P_M - T)$ we have that $2P_M - T$ is not right invertible. Assume now that $(2P_M - T)x = 0$ for some $x = (x_1, x_2, \dots) \in l^2$. This means that

$$\begin{aligned} &x_1 + x_2 = 0, \\ &x_3 + x_4 - x_1 = 0, \\ &x_3 + x_4 - x_2 = 0, \\ &\vdots \end{aligned}$$

From the first three equations we get that $x_1 = x_2 = 0$, similarly we conclude that $x_3 = x_4 = 0$, and then $x_{2k-1} = x_{2k} = 0$, for $k \in \mathbb{N}$. It is easy to establish that $2P_M - T$ has closed range. This means that $2P_M - T$ is left, but not right invertible. It is easy to check that $(2I - T)^{-1}$ commutes with $P_{M^\perp}$ and $2P_M - T$. Finally we have the following:

$$(2I - T)^{-1} P_{M^\perp} + (2I - T)^{-1}(2P_M - T) = I,$$

and all the operators in question commute, $P_{M^\perp}$ is a CI operator since $\mathcal{N}(P_M) = \mathcal{R}(P_M)^\perp \neq \{0\}$, $2P_M - T$ is not a CI operator because he is left but not right invertible and

$$2P_{M^\perp}(2P_M - T) = (2P_M - T)(2P_{M^\perp}) = -2TP_{M^\perp}$$

is neither left nor right invertible, so it is a CI operator . This means that condition (2) in Definition (1.1) is not satisfied, so the set of all CI operators on l^2 is **not a regularity**.

Example 3. Any complex matrix $T \in \mathbb{C}^{n \times n}$ is a CI operator since it is either invertible or $\{0\} \neq \mathcal{N}(T), \mathcal{R}(T) \neq \mathbb{C}^n$. This means that the set of all CI matrices coincides with $\mathbb{C}^{n \times n}$ (which is equivalent to saying $\sigma_{CI}(T) = \emptyset$ for all $T \in \mathbb{C}^{n \times n}$)

We can now characterize when the set of all CI operators on a Hilbert space will be a regularity

Theorem 2.4. *The set of all CI operators in $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a regularity of and only if $\mathcal{H}$ is finite dimensional.*

Proof. If $\mathcal{H}$ is finite dimensional, it is isomorphic to $\mathbb{C}^{n\times n}$ for some $n \in \mathbb{N}$. From the previous example we see that in this case the set of all CI operators will forms a regularity.
Conversely, assume that $\mathcal{H}$ is not finite dimensional. If $\mathcal{H}$ is separable, then it is isomorphic to l^2 so we can conclude from Example 2 that the set of all CI operators in $\mathcal{B}(\mathcal{H})$ is not a regularity. If $\mathcal{H}$ is not separable, then it contains a separable closed subspace $\mathcal{K}$. We have that $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$. We also know that $\mathcal{K}$ is isomorphic to l^2. From Example 2 we have a pair of commuting operators which do not satisfy condition 2. from Definition 1.1. Without loss of generality let us denote them by $2P_M^{\perp}$ and $2P_M - T$ as well. Then the operators

$$A = 2P_M^{\perp} \oplus 0, \ B = 2P_M - T \oplus I_{\mathcal{K}^{\perp}}$$

commute, and there exist operators C, D such that $AC + BD = I_{\mathcal{H}}$ which commute with A and B as well. Furthermore, A is a CI operator, B is not a CI operator, but their product is a CI operator. This is in contradiction with condition 2. of Definition 1.1, so the set of all CI operators is not a regularity. □

3. Fredholm consistency

As in the case of CI operators, the notion of Fredholm consistency gan be generalized to Banach algebras as well. In [6] T-Fredholm elements of a Banach algebra were introduced. If $T : \mathcal{A} \to \mathcal{B}$ is a bounded algebra homomorphism between complex Banach algebras $\mathcal{A}$ and $\mathcal{B}$ where $1_{\mathcal{A}} \neq 0_{\mathcal{A}}(1_{\mathcal{B}} \neq 0_{\mathcal{B}})$ we say that $a \in \mathcal{A}$ is T-Fredholm (left T-Fredholm, right T-Fredholm) if and only if $T(a) \in \mathcal{B}^{-1}(\mathcal{B}_l^{-1}, \mathcal{B}_r^{-1})$. We can now say that $a \in \mathcal{A}$ is T-Fredhom consistent (T-FC) if for each $c \in \mathcal{A}$

$$ac \text{ is } T-\text{Fredholm } \Leftrightarrow ca \text{ is } T-\text{Fredholm}.$$

In a matter analogous to Lemma 2.1 and Theorems 2.1 and 3.3 we get the following results:

Lemma 3.1. *A Banach algebra element a is not T-FC if and only if a is left T-Fredholm but not right T-Fredholm, or a is right T-Fredholm but not left T-Fredholm.*

Proof. Assume $a \in \mathcal{A}$ is not $T - FC$. Then there exists an element $c \in \mathcal{A}$ such that $T(ac) \in \mathcal{B}^{-1}$ and $T(ca) \notin \mathcal{B}^{-1}$, or $T(ca) \in \mathcal{B}^{-1}$ and $T(ac) \notin \mathcal{B}^{-1}$. If the first statement is correct, since $T(ac) = T(a)T(c) \in \mathcal{B}^{-1}$ we have that $T(a)$ must be right invertible. If $T(a)$ were left invertible as well, then $T(c)$ would be invertible, and $T(ca)$ would be invertible as well. From this contradiction we see that $T(a) \in \mathcal{B}_r^{-1} \setminus \mathcal{B}_l^{-1}$, which means that a is right T-Fredholm but not left T-Fredholm. We analogously conclude that in the other case $T(a) \in \mathcal{B}_l^{-1} \setminus \mathcal{B}_r^{-1}$. If

a is left T-Fredholm but not right T-Fredholm we have that $T(a) \in \mathcal{B}_l^{-1} \setminus \mathcal{B}_r^{-1}$ we have that $T(a)_l^{-1}T(a) = 1_{\mathcal{B}}$ and $T(a)T(a)_l^{-1} \notin \mathcal{B}^{-1}$ for an arbitrary left inverse of $T(a)$, so a is not $T - FC$. We analogously conclude that a is not $T - FC$ when a is left T-Fredholm but not right T-Fredholm. □

Corollary 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras such that $1_{\mathcal{A}} \neq 0_{\mathcal{A}}(1_{\mathcal{B}} \neq 0_{\mathcal{B}})$, and $T : \mathcal{A} \to \mathcal{B}$ a bounded algebra homomorphism. Then, $a \in \mathcal{A}$ is $T - FC$ if and only if $T(a)$ is CI.*

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras such that $1_{\mathcal{A}} \neq 0_{\mathcal{A}}(1_{\mathcal{B}} \neq 0_{\mathcal{B}})$, and $T : \mathcal{A} \to \mathcal{B}$ a bounded algebra homomorphism. The set of all T-Fredholm consistent elements is an upper semiregularity.*

Proof. Let $a, b \in \mathcal{A}$ be commuting T-Fredholm consistent elements and $c \in \mathcal{A}$ arbitrary. We have that

$$abc \text{ is T-Fredholm} \Leftrightarrow bca \text{ is T-Fredholm} \Leftrightarrow$$

$$\Leftrightarrow cab \text{ T-Fredholm.}$$

Since invertible elements are $T - FC$, and we know that there exists an open neighborhood of $1_{\mathcal{A}}$ where all elements are invertible. We conclude that there exists an open neighborhood of $1_{\mathcal{A}}$ where all elements are $T - FC$. This completes the proof. □

Corollary 3.2. *The set of all Fredholm consistent operators in $\mathcal{B}(\mathcal{H})$ is an upper semiregularity.*

Since invertible elements of a Banach algebra are T-FC we see that a Theorem analogous to Theorem 2.3 will hold for the T-FC spectrum as well where

$$\sigma_{TFC}(a) = \{\lambda \in \mathbb{C} : \ a - \lambda \text{ is not } T - FC\}$$

Theorem 3.2. *For every $a \in \mathcal{A}$ $\sigma_{TFC}(f(a)) \subseteq f(\sigma_{TFC}(a))$ for all locally nonconstant functions f analytic on a neighborhood of $\sigma(a) \cup \sigma_{TFC}(a) = \sigma(a)$, and $f(\sigma_{CI}(a)) \subseteq f(\sigma(a))$ for all functions f analytic on a neighborhood of $\sigma(a)$.*

Again, in the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and when we observe Fredholm operators, self-adjoint operators have an empty FC spectrum. The following examples will show that the set of all Fredholm consistent operators in $\mathcal{B}(\mathcal{H})$ is not generally a regularity, and that the FC spectrum is generally not closed:

Example 4. Let $A \in \mathcal{B}(l^2)$ be defined in the following way:

$$Ax = (x_1, 0, x_2, 0, x_3, 0, \dots),\ x = (x_1, x_2, x_3, \dots) \in l^2.$$

In other words, $Ae_n = e_{2n-1}$ where e_n is the n-th vector in the standard orthonormal basis. It is easy to see that A is left invertible, but not right invertible and $d(A) = \infty$. This means that A is left Fredholm but not right Fredholm so A is not Fredholm consistent. On the other hand for

$$(I - A)x = (0, x_2, x_3 - x_2, x_4, x_5 - x_3, x_6, \dots),\ x = (x_1, x_2, x_3, \dots) \in l^2$$

we have

$$\mathcal{N}(I - A) = \mathcal{R}(I - A)^{\perp} = \{x \in l^2 : x_n = 0, n \geq 2\}.$$

The last part in the equation follows from the fact that for

$$(I - A^*)x = (0, x_2 - x_3, x_3 - x_5, x_4 - x_7, x_5 - x_9, \dots)$$

we have that $x \in \mathcal{N}(I - A^*)$ if

$$x_n = x_{2n-1} = x_{4n-3} = ...$$

so

$$\mathcal{N}(I - A^*) = \{x \in l^2 : x_n = 0, n \geq 2\}.$$

We see $n(I - A) = d(I - A) = 1$ which means that $I - A$ is Fredholm, and thus FC. Now we define an operator $T \in \mathcal{B}(l^2 \oplus l^2)$ as

$$T = A \oplus I_{l^2}.$$

We have that T is also not Fredholm consistent and that

$$I_{l^2 \oplus l^2} - T = (I_{l^2} - A) \oplus 0$$

so $n(I_{l^2 \oplus l^2} - T) = d(I_{l^2 \oplus l^2} - T) = \infty$ which means that $I - T$ is Fredholm consistent in $\mathcal{B}(l^2 \oplus l^2)$. For $(I_{l^2 \oplus l^2} - T)T$ we also have that $n((I_{l^2 \oplus l^2} - T)T) = d((I_{l^2 \oplus l^2} - T)T) = \infty$ so this operator is Fredholm consistent in $\mathcal{B}(l^2 \oplus l^2)$ as well. Finally, since $(I_{l^2 \oplus l^2} - T) + T = I_{l^2 \oplus l^2}$, and $I_{l^2 \oplus l^2} - T$ and T trivially commute we see that the condition 2. from Definition 1.1 isn't satisfied from which we conclude that the set of all Fredholm consistent operators in $\mathcal{B}(l^2 \oplus l^2)$ is not a regularity.

Example 5. Let $\mathcal{H}$ be separable Hilbert space. Then $\mathcal{H}$ can be represented as an orthogonal direct sum of closed infinite dimensional subspaces M_n, $n \in \mathbb{N}$ ($\mathcal{H} = \bigoplus_{n=1}^{\infty} M_n$). To see that such subspaces exists we can do the following. Since $\mathcal{H}$ is separable,let M_1 be a closed infinite dimensional subspace of $\mathcal{H}$ with infinite codimension. We have that $M_1^{\perp}$ is also a separable infinite dimensional Hilbert space. Let M_2 be the closed subspace of $M_1^{\perp}$ isomorphic to the subspace M_1. Continuing this process we construct the subspaces M_n, $n \in \mathbb{N}$. Let $(\lambda_n)_n$ be a sequence of complex numbers that converges to 0. For each $n \in \mathbb{N}$ there exists a bounded linear operator $T_n \in \mathcal{B}(\mathcal{M}_n)$ such that $T_n, T_n - \lambda_m$, $m \in \mathbb{N} \setminus \{n\}$ are invertible and $n\,(T_n - \lambda_n) = \infty$ and $\mathcal{R}\,(T_n - \lambda_n) = M_n$. This means that $\lambda_n \in \sigma_{FC}\,(T_n)$ and $0, \lambda_m \notin \sigma_{FC}\,(T_n)\,,\ m \in \mathbb{N} \setminus \{n\}$. Furthermore we can select these operators in such a way that the family of operators T_n is uniformly bounded. We have that $T = \bigoplus_{n=1}^{\infty} T_n$ is a invertible bounded linear operator on $\mathcal{H}$ such that

$$n\,(T - \lambda_n) = \infty,\ \mathcal{R}\,(T - \lambda_n) = \mathcal{H},\ n \in \mathbb{N}.$$

This means that $\lambda_n \in \sigma_{FC}\,(T)\,,\ n \in \mathbb{N}$, but $0 \notin \sigma_{FC}\,(T)$. We conclude that $\sigma_{FC}\,(T)$ is not closed. To see that the operators T_n indeed exists we can construct them now. For each $n \in \mathbb{N}$ there exists $r_n > 0$ such that $\lambda_m \notin B(\lambda_n, r_n)$ for $m \neq n$. It follows that $|r_n| < |\lambda_n|$ and that $0 \notin B(\lambda_n, r_n)$. Furthermore, for each $n \in \mathbb{N}$ there exists a subspace $\mathcal{K}_n$ such that $\mathcal{M}_n = \mathcal{K}_n \oplus \mathcal{K}_n^{\perp}$ and $\dim \mathcal{K}_n = \dim \mathcal{K}_n^{\perp} = \infty$. We have that $\mathcal{K}_n$ is isomorphic to $\mathcal{M}_n$, let us denote the isomorphism by J_n'. Without loss of generality we can assume that J_n' is unitary. This isomorphism is naturally extended to $J_n \in \mathcal{B}(\mathcal{M}_n)$ by

$$J_n x = \begin{cases} J_n' x, & x \in K_n \\ 0, & x \in \mathcal{K}_n^{\perp} \end{cases}.$$

We have that $\mathcal{N}(J_n) = \mathcal{K}_n^{\perp}$, and $\mathcal{R}(J_n) = \mathcal{M}_n$. Define T_n by

$$T_n = r_n J_n + \lambda_n.$$

We have that $T_n - \lambda_n = r_n J_n$, so $n\,(T_n - \lambda_n) = n\,(J_n) = \infty$ and $\mathcal{R}\,(T_n - \lambda_n) = \mathcal{R}(J_n) = \mathcal{M}_n$, so $\lambda_n \in \sigma_{FC}\,(T_n)$. Since $|\lambda_n|, |\lambda_n - \lambda_m| > |r_n| = \|r_n J_n\|$ for $m \neq n$ we have that T_n and $T_n - \lambda_m,\ m \neq n$ are invertible, and $\|T_n\| \leq r_n + \lambda_n \leq 1 + M$ for $n \in \mathbb{N}$ where M is any upper bound for the convergent sequence $(\lambda_n)_n$ which proves that the family $(T_n)_n$ is uniformly bounded.

Since $\sigma_{FC}(T) = \emptyset$ for all $T \in \mathcal{B}(\mathcal{H})$ when $\mathcal{H}$ is finite dimensional the set of Fredholm consistent operators will coincide with $\mathcal{B}(\mathcal{H})$ and will thus be a regularity. We have that the following Theorem analogous to Theorem 2.4 holds:

Theorem 3.3. *The set of all Fredholm consistent operators in $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a regularity of and only if $\mathcal{H}$ is finite dimensional.*

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