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THE CLASSICAL BERNOULLI-EULER ELASTIC CURVE IN A MANIFOLD

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Abstract. In this study, we describe the classical Bernoulli-Euler elastic curve in a manifold by the property that the velocity vector field of the curve is harmonic. Then, a condition is obtained for the elastic curve in a manifold. Finally, we give an example which provides the condition mentioned in this paper and illustrate it with a figure.
Keywords: Energy; energy of a unit vector field; elastic curve.

1. Introduction

The history of the elastica or the elastic curve is very old and many researchers have worked on this issue, for example [6, 11]. One can study a bent thin rod and consider the energy it stores. The classical Euler-Bernoulli model assigns a numerical value to this energy, which is proportional to $\int_0^s k^2(u)du$. The elastica is the critical point for this total squared curvature functional on regular curves with given boundary conditions [8].

In [1] the author calculated the energy of the Frenet vector fields in R^n, showing that the energy of the velocity vector field was $\mathcal{E}(V_1(s)) = \frac{1}{2}\int_a^s k_1^2(u)du$. By means of this result, we have seen that the speed vector field of the Bernoulli-Euler elastic curve is harmonic.

In this paper, using the above result, we give a condition for elastica on a manifold.

Definition 1.1. Let (M, g) be a Riemann manifold and $\alpha : I \to M$, be a unit speed curve.

If $\{E_i\}_{i=1}^r$ is an orthonormal frame along α and

$$E_1 = \frac{d\alpha}{ds},$$

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$$\nabla^{\alpha}_{\frac{\partial}{\partial s}} E_1 = k_1 E_2,$$
$$\nabla^{\alpha}_{\frac{\partial}{\partial s}} E_i = -k_{i-1} E_{i-1} + k_i E_{i+1}, \quad \forall i = 2, ..., r-1$$
$$\nabla^{\alpha}_{\frac{\partial}{\partial s}} E_r = -k_{r-1} E_{r-1},$$

where $k_1, ..., k_{r-1}$ are positive functions with a real value on I, then α is said to be an r-th order Frenet curve. These functions are called the curvature functions of the curve α.

Proposition 1.1. The connection map $K : T(T^1M) \to T^1M$ verifies the following conditions.

1) $\pi \circ K = \pi \circ d\pi$ and $\pi \circ K = \pi \circ \widetilde{\pi}$, where $\widetilde{\pi} : T(T^1M) \to T^1M$ is the tangent bundle projection.

2) For $\omega \in T_xM$ and a section $\xi : M \to T^1M$, we have

$$K(d\xi(\omega)) = \nabla_\omega \xi$$

where T^1M is the unit tangent bundle and ∇ is the Levi-Civita covariant derivative [3].

Definition 1.2. For $\eta_1, \eta_2 \in T_\xi(T^1M)$, we define

$$g_{\mathcal{S}}(\eta_1, \eta_2) =< d\pi(\eta_1), d\pi(\eta_2) > + < K(\eta_1), K(\eta_2) > . \tag{1.1}$$

This gives a Riemannian metric on tangent bundle TM. As mentioned, $g_{\mathcal{S}}$ is called *the Sasaki metric*. The metric g_s makes the projection $\pi : T^1M \to M$ a Riemannian submersion [3, 10].

Definition 1.3. Let $f : (M, <, >) \to (N, h)$ be a differentiable map between Riemannian manifolds. The energy of f is given by

$$\mathcal{E}(f) = \frac{1}{2} \int_M (\sum_{a=1}^{n} h(df(e_a), df(e_a)) \upsilon \tag{1.2}$$

where υ is the canonical volume form in M and $\{e_a\}$ is a local basis of the tangent space (see [12, 4], for example).

By a (smooth) variation of f we mean a smooth map $f : M \times (-\epsilon, \epsilon) \to N$, $(x, t) \to f_t(x)$ $(\epsilon > 0)$ such that $f_0 = f$. We can think of $\{f_t\}$ as a family of smooth mappings which depend 'smoothly' on a parameter $t \in (-\epsilon, \epsilon)$.

Definition 1.4. A smooth map $f : (M, g) \to (N, h)$ is said to be harmonic if

$$\frac{d}{dt} \mathcal{E}(f_t; D)|_{t=0} = o$$

where $\mathcal{E}(f; D) = \frac{1}{2} \int_D (\sum_{a=1}^{n} h(df(e_a), df(e_a)) \upsilon_g$, for all compact domains D and all smooth variations f_t of f supported in D, [2].

Definition 1.5. Let $\alpha : [a,b] \to R^n$ be a regular curve. Elastica is defined for the curve α over the each point on a fixed interval $[a,b]$ as a minimizer of the bending energy:

$$\mathcal{E}_B = \frac{1}{2}\int_a^b k_1^2(s)ds, \tag{1.3}$$

with some boundary conditions [5, 7].

The right side of Equation (1.3) is the energy of the velocity vector field according to [1]. By combining this resultant with the definition 1.4 we can give the following definition

2. Elastica in a Manifold

Definition 2.1. A curve on a manifold is called a classical Bernoulli-Euler elastic curve if the velocity vector field of the curve is harmonic.

Theorem 2.1. Let M be a Riemann manifold, α be r-th order Frenet curve in M and $\alpha(a) = p,\ \ \alpha(b) = q$. If α is classical elastic curve, then the following equation is satisfied,

$$\int_a^b \lambda(s)k_1(s)k_1^{'}(s)ds = 0 \tag{2.1}$$

where k_1 is the 1^{th} curvature function and λ is the real-valued function on $[a,b]$.

Proof . Let $\alpha : I \to M$ be the r-th order Frenet curve C on $\varphi(U) \subset M$ and $\alpha = \varphi \circ \gamma,\ \gamma = (\gamma_1, ..., \gamma_m), \gamma : I \to U \subset R^m; \varphi : U \to M$. Let $(\{E_i\}_{i=1}^r)$ be the Frenet frame field on α.

We define the λ and v_i functions to create a curve family between two fixed points on the manifold. The functions are: $\lambda : [a,b] \subset I \to R,\ \lambda(s) = (s-a)(b-s)$, $\lambda(a) = 0,\ \lambda(b) = 0$ and $\lambda(s) \neq 0$ for all $s \in (a,b)$, of class C^2 and

$$\lambda(s)\ E_1(s) = (v_1(s), v_2(s), ..., v_n(s)).\ \ v_i : [a,b] \to R.$$

Since $\{\varphi_1(\gamma(s)), ..., \varphi_m(\gamma(s))\}$ is a local basis of the tangent space, where $\varphi_1, ..., \varphi_m$ are first-order partial derivatives, we have

$$\lambda(s)E_1(s) = \Sigma_{i=1}^m v_i(s)\varphi_i(\gamma(s));\ \ where\ \ v_i : [a,b] \to R. \tag{2.2}$$

Let the collection of the curve be

$$\alpha^t(s) = \varphi(\gamma_1(s) + tv_1(s), ..., \gamma_m(s) + tv_m(s)), \tag{2.3}$$

for $t = 0$, $\alpha^0(s) = \alpha(s)$ and

$$(\varphi^{-1} \circ \alpha^t)(s) = \gamma^t(s) = (\gamma_1(s) + tv_1(s), ..., \gamma_m(s) + tv_m(s)).$$

From (2.2) we get $\lambda(a)E_1(a) = \Sigma_{i=1}^m v_i(a)\varphi_i(\gamma(a))$. Since $\lambda(a) = 0$ we have $v_i(a) = 0$ and

$$\gamma^t(a) = (\gamma_1(a) + tv_1(a), ..., \gamma_m(a) + tv_m(a) = (\gamma_1(a), ..., \gamma_m(a)) = \gamma(a).$$

Similarly, we get $\gamma^t(b) = \gamma(b)$. Using these results in (2.3) we obtain

$\alpha^t(a) = (\varphi \circ \gamma^t)(a) = \alpha(a) = p$ and $\alpha^t(b) = (\varphi \circ \gamma^t)(b) = \alpha(b) = q$.

These results show that α^t is a curve segment from p to q on M. Take this collection $\alpha^t(s) = \alpha(s,t)$ for all curves. The expression for the energy of the velocity vector field E_{1t} of α^t from p to q on M becomes $\mathcal{E}(E_{1t})$.

Let TC_t be the tangent bundle. So we have $E_{1_t} : C_t \to TC_t$, where $TC_t = \cup_{j\in I} T_{\alpha^t(j)}C_t$, $C_t = \alpha^t(I)$ and $T_{\alpha^t(j)}C_t$ is the straight line through the point $\alpha^t(j)$ in the E_{1_t} direction. Let $\pi : TC_t \to C_t$ be the bundle projection. By using Equation (1.2) we calculate the energy of E_{1_t} as

$$\mathcal{E}(E_{1_t}) = \frac{1}{2}\int_a^b g_{\mathcal{S}}(dE_{1_t}(E_{1_t}(\alpha(s,t)), dE_{1_t}(E_{1_t}(\alpha(s,t)))ds \tag{2.4}$$

where ds is the element arc length. From (1.1) we have

$$\begin{aligned} g_{\mathcal{S}}(dE_{1_t}(E_{1_t}), dE_{1_t}(E_{1_t})) &= < d\pi(dE_{1_t}(E_{1_t})), d\pi(dE_{1_t}(E_{1_t})) > \\ &+ < K(dE_{1_t}(E_{1_t})), K(dE_{1_t}(E_{1_t})) > . \end{aligned}$$

Since E_{1_t} is a section, we have $d(\pi)\circ d(E_{1_t}) = d(\pi\circ E_{1_t}) = d(id_{C_t}) = id_{TC_t}$. By Proposition 1.1, we also have that

$$K(dE_{1_t}(E_{1_t})) = \nabla^{\alpha}_{E_{1_t}} E_{1_t} = E'_{1_t} = \frac{\partial E_{1_t}}{\partial s},$$

giving

$$g_{\mathcal{S}}(dE_{1_t}(E_{1_t}), dE_{1_t}(E_{1_t})) =< E_{1_t}, E_{1_t} > + < E'_{1_t}, E'_{1_t} > .$$

Using these results in (2.4) we get

$$\mathcal{E}(E_{1_t}) = \frac{1}{2}\int_a^b (< E_{1_t}, E_{1_t} > + < E'_{1_t}, E'_{1_t} >)ds \tag{2.5}$$

By Definition 1.4, if E_{1_t} is a harmonic, then $t = 0$ should be the critical point of $\mathcal{E}(E_{1_t})$. Supposing that $\frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} = 0$, from (2.5) we obtain:

$$\begin{aligned} \frac{\partial \mathcal{E}(E_{1_t})}{\partial t} &= \frac{\partial}{\partial t}[\frac{1}{2}\int_a^b (< E_{1_t}, E_{1_t} > + < E'_{1_t}, E'_{1_t} >)ds] \\ &= \frac{1}{2}[\int_a^b \frac{\partial}{\partial t}[(< E_{1_t}, E_{1_t} > + < \frac{\partial E_{1_t}}{\partial s}, \frac{\partial E_{1_t}}{\partial s} >]ds. \end{aligned}$$

Since $< E_{1_t}, E_{1_t} >= 1$ we have $\frac{\partial}{\partial t} < E_{1_t}, E_{1_t} >= 0$ and we get

$$\text{(2.6)}\quad \frac{\partial \mathcal{E}(E_{1_t})}{\partial t} = \frac{1}{2}\int_a^b \frac{\partial}{\partial t} < \frac{\partial E_{1_t}}{\partial s}, \frac{\partial E_{1_t}}{\partial s} > ds = \int_a^b < \frac{\partial^2 E_{1_t}}{\partial s \partial t}, \frac{\partial E_{1_t}}{\partial s} > ds.$$

We can write

$$\frac{\partial}{\partial s} < \frac{\partial E_{1_t}}{\partial t}, \frac{\partial E_{1_t}}{\partial s} >=< \frac{\partial^2 E_{1_t}}{\partial s \partial t}, \frac{\partial E_{1_t}}{\partial s} > + < \frac{\partial E_{1_t}}{\partial t}, \frac{\partial^2 E_{1_t}}{\partial s^2} > .$$

Thus, we can deduce,

$$\text{(2.7)}\qquad < \frac{\partial^2 E_{1_t}}{\partial s \partial t}, \frac{\partial E_{1_t}}{\partial s} >= \frac{\partial}{\partial s} < \frac{\partial E_{1_t}}{\partial t}, \frac{\partial E_{1_t}}{\partial s} > - < \frac{\partial E_{1_t}}{\partial t}, \frac{\partial^2 E_{1_t}}{\partial s^2} >$$

Substituting (2.7) in (2.6), for, $t = 0$, we have

$$\frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} = \int_a^b [\frac{\partial}{\partial s} < \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial E_{1_k}}{\partial s}(s,0) > - < \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial^2 E_{1_t}}{\partial s^2}(s,0) >]ds$$

and

$$\text{(2.8)}\qquad \begin{aligned} \frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} &= < \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial E_{1_t}}{\partial s}(s,0) >|_a^b \\ &\quad - \int_a^b < \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial^2 E_{1_t}}{\partial s^2}(s,0) > ds. \end{aligned}$$

From (2.2) and (2.3), we obtain,

$$\text{(2.9)}\qquad \frac{\partial \alpha}{\partial t}(s,t) = \lambda(s) E_{1_t}(s).$$

and

$$\text{(2.10)}\qquad \frac{\partial \alpha}{\partial s}(s,t)_{|t=0} = \alpha^{'}(s) = E_1(s).$$

Now we calculate the partial derivatives of (2.10) with respect to s and t; using Frenet formulas, we get

$$\text{(2.11)}\qquad \frac{\partial E_{1_t}}{\partial s}(s) = \frac{\partial^2 \alpha}{\partial s^2}(s,t)_{|t=0} = \alpha^{''}(s) = E_1^{'}(s) = k_1(s) E_2(s)$$

and

$$\frac{\partial E_{1_t}}{\partial t}(s,t) = \frac{\partial^2 \alpha}{\partial s \partial t}(s,t) = \frac{\partial^2 \alpha}{\partial t \partial s}(s,t).$$

From (2.9), we have

$$\text{(2.12)}\qquad \frac{\partial E_{1_t}}{\partial t}(s,t)_{|t=0} = \frac{\partial E_{1_t}}{\partial t}(s,0) = \lambda^{'}(s) E_1(s) + \lambda(s) k_1(s) E_2(s).$$

It follows from (2.11) and (2.12) that

$$< \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial E_{1_t}}{\partial s}(s,0) >= \lambda(s)k_1^2(s).$$

Considering the candidate function $\lambda(a) = \lambda(b) = 0$, we get:

$$< \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial E_{1_t}}{\partial s}(s,0) >|_a^b = \lambda(b)k_1^2(b) - \lambda(a)k_1^2(a) = 0. \tag{2.13}$$

From (2.11), we get

$$\frac{\partial^2 E_{1_t}}{\partial s^2}(s,0) = -k_1^2(s)E_1(s) + k_1^{'}(s)E_2(s) + k_1(s)k_2(s)E_3(s) \tag{2.14}$$

Therefore, (2.12) and (2.14) gives

$$< \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial^2 E_{1_t}}{\partial s^2}(s,0) >= [-\lambda(s)k_1^2(s)]^{'} + 3\lambda(s)k_1(s)k_1^{'}(s) \tag{2.15}$$

Substituting (2.13) and (2.15) in (2.8) yields

$$\frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} = -\int_a^b ([-\lambda(s)k_1^2(s)]^{'} + 3\lambda(s)k_1(s)k_1^{'}(s))ds = 0$$

and

$$\frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} = [\lambda(s)k_1^2(s)]\ |_a^b\ -3\int_a^b \lambda(s)k_1(s)k_1^{'}(s)ds = 0$$

We are looking the candidate function $\lambda(a) = \lambda(b) = 0$,
which given $[\lambda(s)k_1^2(s)]\ |_a^b = 0$ and

$$\frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} = -3\int_a^b \lambda(s)k_1(s)k_1^{'}(s)ds = 0$$

This completes the proof of the theorem. ■

Example 1. Let $\varphi : R^2 \to R^3, \varphi = (x, y, \frac{1}{3}xy), \varphi(R^2) = M$ and $\alpha(s) = (3s, s^2, s^3)$. If we can choose $\lambda : [-10, 10] \to R, \lambda(s) = 10^2 - s^2$ then $\lambda(-10) = 0\lambda(10) = 0$ and $\lambda(s) \neq 0$ for all $s \in (-10, 10)$. We calculate

$$k_1(s) = \frac{6\sqrt{s^4 + 9s^2 + 1}}{(\sqrt{9s^4 + 4s^2 + 9})^3},$$

$$k_1^{'}(s) = 6\frac{\frac{2s^3+9s}{\sqrt{s^4+9s^2+1}}(\sqrt{9s^4 + 4s^2 + 9})^3 - 3\sqrt{s^4 + 9s^2 + 1}(\sqrt{9s^4 + 4s^2 + 9})^2(35s^3 + 8s)}{(9s^4 + 4s^2 + 9)^3},$$

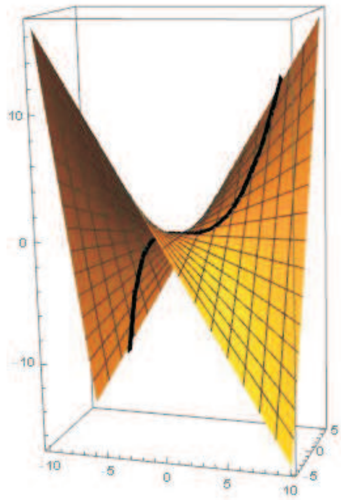

FIG. 2.1:

and

$$\frac{\partial \mathcal{E}(T_k)}{\partial k}_{|k=0} = -\int_{-10}^{10} (10^2 - s^2) k_1(s) k_1^{'}(s) ds = 0.$$

Thus α is an elastica on M, Figure 2.1.

Conclusion. In this paper, we have determined the classical Bernoulli-Euler elastic curve that is the harmonic of the velocity vector field of the curve on a manifold. We have obtained the collection of curves passing through p and q points using λ and v_i functions on the manifold. We have also proposed a novel condition to be the classical Bernoulli-Euler elastic curve in the collection of curves. In the end, we have given an example of the elastic curve satisfying the novel condition on a two-dimensional manifold and shown the graphs of both the manifold and the elastic curve.

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