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RICCI SOLITONS AND GRADIENT RICCI SOLITONS ON NEARLY KENMOTSU MANIFOLDS

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Abstract. In this paper, we study nearly Kenmotsu manifolds with a Ricci soliton and we obtain certain conditions about curvature tensors.
Keywords: Contact manifold, Nearly Kenmotsu Manifold, Ricci Solitons.

1. Introduction and Preliminaries

Ricci solitons $\frac{\partial}{\partial t}g = -2S$ reflected on the modulo diffeomorphisms and scales from the space of the metrics are fixed points of the Ricci flow and mostly explosive limits for the Ricci flow in compact manifolds. Generally, physicists have studied Ricci solitons in relation with string theory. In particular, in differential geometry we use a Ricci soliton as a special type of the Riemannian metric. Such metrics builds from the Ricci flow only by symmetries of the flow so they can be viewed as generalizations of Einstein metrics. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of the Einstein metric such that [12]

$$\pounds V g + 2S + 2\lambda g = 0 \tag{1.1}$$

where S is a Ricci tensor and $\pounds V$ is the Lie derivative along the vector field V on M and λ is a real number.

Depending on whether λ is negative, zero or positive, a Ricci soliton is named shrinking, steady or expanding, respectively. In addition, if the vector field V is the gradient of a potential function $-f$, then the metric g is called a gradient Ricci soliton. We can regulate the (1.1) as

$$\nabla\nabla f = S + \lambda g. \tag{1.2}$$

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Ricci solitons firstly become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. [19] In particular, after Sharma had studied the Ricci solitons in contact geometry, Ricci flows in contact geometry gained a significant attention. They have been studied extensively ever since. The geometry of Ricci solitons in contact metric manifolds have been studied by authors such as Bagewadi, Bejan and Crasmareanu, Blaga, Hui et al., Chen, Deshmukh et al., Nagaraja and Premalatta, Tripathi and many others. In [11], Ricci solitons in K-contact manifolds were studied by Sharma. Ghosh, Sharma and Cho [11] studied gradient Ricci solitons in non-Sasakian (k, μ)-contact manifolds. In addition, in [21], Tripathi showed gradient Ricci solitons, compact Ricci solitons in $N(k)$-contact metric manifolds and (k, μ)-manifolds. Recently in [1], B. Barua and U. C. De focused on some properties of Ricci solitons in Riemannian manifolds.

Einstein solitons are open examples of Ricci solitons, where g is an Einstein metric and X is a Killing vector field. On a compact manifold, a Ricci soliton has a constant curvature, especially in dimension 2 and in dimension 3 [12, 13]. For details about these studies, we refer the reader to Chow and Knopf [8] and Derdzinski [10]. An important result by Perelman shows that on a compact manifold, the Ricci soliton is a gradient Ricci soliton.

Based on these studies, in this paper we review Ricci solitons $(R.S)$ and gradient Ricci solitons $(G.R.S)$ in a nearly Kenmotsu manifold. The paper progresses as follows. After some preliminary information and definitions in Section 2, we consider the case that in a nearly Kenmotsu manifold, if g admits a $(R.S)$ in the form of (g, V, λ) and V is point-wise collinear with ξ, then the manifold is an η-Einstein manifold. Furthermore, we show that if a nearly Kenmotsu manifold admits a compact $(R.S)$, then the manifold is Einstein. Finally, in the last section, we prove that when an η-Einstein nearly Kenmotsu manifold admits a $(G.R.S)$, the manifold transforms into an Einstein manifold under certain conditions.

Let M be an n-dimensional nearly Kenmotsu manifold with the (ϕ, ξ, η, g) structure that ϕ is a $(1,1)$ type tensor field, ξ is a contravariant vector field, η is a 1-form and g is a Riemannian metric. Then by definition, it satisfies the following relation [15]

$$\eta(\xi) = 1, \qquad \phi^2 = -I + \eta \otimes \xi, \tag{1.3}$$

$$\phi\xi = 0, \qquad \eta\phi = 0, \qquad \nabla_X \xi = X - \eta(X)\xi, \tag{1.4}$$

$$\eta(X) = g(\xi, X), \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.5}$$

$$(\nabla_X \eta)(Y) = \Omega(Y, X), \qquad \Omega(X, Y) = \Omega(Y, X) \qquad (\Omega(Y, X) = g(\phi Y, X)), \tag{1.6}$$

$$(\nabla_Z \Omega)(X, Y) = \{g(X, Z) + \eta(X)\eta(Z)\}\eta(Y) + \{g(Y, Z) + \eta(Y)\eta(Z)\}\eta(X) \tag{1.7}$$

for any vector fields X, Y and Z on M, where the $\otimes$ is the tensor product and I shows the identity map on T_pM.

In an n-dimensional nearly Kenmotsu manifold with (ϕ, ξ, η, g) structure, the following relations hold.

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \qquad S(X, \xi) = -(n-1)\eta(X), \tag{1.8}$$

$$R(\xi,Y)X=-g(Y,X)\xi+\eta(X)Y,\quad R(Y,X)\xi=\eta(Y)X-\eta(X)Y, \tag{1.9}$$

$$\begin{aligned}\phi(R(X,\phi Y)Z)&=R(X,Y)Z+2\{\eta(Y)X-\eta(X)Y\}\eta(Z)+2\{g(X,Z)\eta(Y)\\&-g(Y,Z)\eta(X)\}+\Omega(X,Z)\phi(Y)-\Omega(Y,Z)\phi X+g(Y,Z)X-g(X,Z)Y,\end{aligned} \tag{1.10}$$

where R is the curvature tensor and S is the Ricci tensor with respect to g. If the Ricci tensor S satisfies the condition

$$S=ag+b\eta\otimes\eta, \tag{1.11}$$

then an n-dimensional nearly Kenmotsu manifold is said to be η-Einstein. In the Ricci tensor equation, a and b are smooth functions on M.

In an η-Einstein nearly Kenmotsu manifold, the Ricci tensor S and Ricci operator Q are shown in the form below.

$$S(X,Y)=\left[\frac{r}{n-1}-1\right]g(X,Y)+\left[\frac{r}{n-1}-n\right]\eta(X)\eta(Y) \tag{1.12}$$

$$QX=\left[\frac{r}{n-1}-1\right]X+\left[\frac{r}{n-1}-n\right]\eta(X)\xi. \tag{1.13}$$

2. $(R.S)$ on Nearly Kenmotsu Manifolds

Suppose that a nearly Kenmotsu manifold admits a $(R.S)$. Considering the properties of nearly Kenmotsu manifolds with $(R.S)$, we know that $\nabla g=0$. Since λ in the $(R.S)$ equation is a constant, we can specify that $\nabla\lambda g=0$. Because of this, it is easy to say that $\pounds_V g+2S$ is parallel.

It was proved in [16] that if a nearly Kenmotsu manifold with a symmetric parallel $(0,2)$ type tensor, then the tensor is a constant multiple of the metric tensor. As a result of this theorem, we can say that $\pounds_V g+2S$ is a constant multiple of metric tensors g, i.e., and $\pounds_V g+2S=ag$, such that a is constant.

From the above equations, we can write $\pounds_V g+2S+2\lambda g$ as $(a+2\lambda)g$. Then using $(R.S)$, we get $\lambda=-a/2$.

Based on these results we can write the following proposition.

Proposition 2.1. *In a nearly Kenmotsu manifold, depending on whwther a is positive or negative, $(R.S)$ with the form of (g,λ,V) is shrinking or expanding. Particularly, let V be point-wise collinear with ξ i.e. $V=b\xi$, where b is a function on a nearly Kenmotsu manifold. Then*

$$(\pounds_V g+2S+2\lambda g)(X,Y)=0, \tag{2.1}$$

which adds up to

$$g(\nabla_X b\xi,Y)+g(\nabla_Y b\xi,X)+2S(X,Y)+2\lambda g(X,Y)=0,$$

or,

$$bg((\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0.$$

Using (1.4), we get

$$2bg(X,Y) - 2b\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{2.2}$$

Then putting $Y = \xi$ in (2.2) we obtain

$$(Xb) + \eta(X)\xi b + 2(1-n)\eta(X) + 2\lambda\eta(X)$$

or,

$$(Xb) = (1 - n - \lambda)\eta(X). \tag{2.3}$$

We know that in a nearly Kenmotsu manifold $d\eta = 0$ and from (2.3) we get

$$Xb = 0$$

if

$$\lambda = 1 - n.$$

Theorem 2.1. *If in a nearly Kenmotsu manifold, the metric g is a $(R.S)$ and V is point-wise collinear with ξ, then V is a constant multiple of ξ on condition that $\lambda = 1 - n$. Especially, if we take $V = \xi$. Then*

$$(\pounds_V g + 2S + 2\lambda g)(X,Y) = 0,$$

implies that

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{2.4}$$

Substituting $X = \xi$, we get $\lambda = -(n-1) < 0$. Because it is negative, we can say that the $(R.S)$ is shrinking.

Particularly, if the manifold is a nearly Kenmotsu manifold, then we have

$$(\nabla_X \eta)(Y) = g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.5}$$

Hence using (2.3), (2.5) Equation (2.2) becomes

$$S(X,Y) = (n-2)g(X,Y) + \eta(X)\eta(Y), \tag{2.6}$$

that is, it is an $\eta-$Einstein manifold. In addition, we have the following theorem.

Theorem 2.2. *If in a nearly Kenmotsu manifold the metric g is a $(R.S)$ and V is point-wise collinear with ξ, then the manifold is an η-Einstein manifold.*

Conversely, if we have a nearly Kenmotsu η-Einstein manifold M with the following form in which γ and δ constants

$$S(X,Y) = \delta g(X,Y) + \gamma\eta(X)\eta(Y), \tag{2.7}$$

then taking $V = \xi$ in (2.1) and using the above equation, we obtain

$$\begin{aligned} &(\pounds\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) \\ = \; & 2(1+\lambda+\delta)g(X,Y) + 2(\gamma-1)\eta(X)\eta(Y). \end{aligned} \tag{2.8}$$

From Equation (2.8) it follows that M with a $(R.S)$ with the form of (g,ξ,λ) such that $\lambda = \gamma - \delta$.

So we have the following theorem.

Theorem 2.3. *If a nearly Kenmotsu Manifold is η-Einstein, then the manifold admits a $(R.S)$ of type $(g,\xi,(\gamma-\delta))$.*

Again, as a result of some adjustments, we get from (2.6)

$$r = (n-1)^2 = constant.$$

By the last equation, the scalar curvature is constant.

In [11] Sharma proved that a compact Ricci soliton with a constant scalar curvature is Einstein. Therefore, from this theorem, we give the following result.

Corollary 2.1. *Let M be a nearly Kenmotsu manifold with a compact $(R.S)$, then the manifold is Einstein.*

3. $(G.R.S)$ on Nearly Kenmotsu Manifolds

If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci Soliton and we can regulate (1.1) as

$$\nabla\nabla f = S + \lambda g. \tag{3.1}$$

This can be written as

$$\nabla_Y Df = QY + \lambda Y, \tag{3.2}$$

where D shows the gradient operator of g. From (3.2) it is clear that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{3.3}$$

This implies that

$$g(R(\xi,Y)Df,\xi) = g((\nabla\xi Q)Y,\xi) - g((\nabla_Y Q)\xi,\xi). \tag{3.4}$$

Now using (1.13) and (1.4) we have

$$(\nabla_Y Q)(X) = \left[\frac{r}{1-n} - n\right](-2\eta(X)\eta(Y)\xi + g(X,Y)\xi + \eta(X)Y). \tag{3.5}$$

Then clearly

$$g((\nabla_X Q)\xi - (\nabla\xi Q)X, \xi) = 0. \tag{3.6}$$

Then we have from (3.4)

$$g(R(\xi, X)Df, \xi) = 0. \tag{3.7}$$

From (1.9) and (3.7) we get

$$g(R(\xi, Y)Df, \xi) = -g(Y, Df) + \eta(Df)\eta(Y) = 0.$$

Hence

$$Df = \eta(Df)\xi = g(Df, \xi)\xi = (\xi f)\xi. \tag{3.8}$$

Using (3.8) in (3.2) we get

$$S(X,Y) + \lambda g(X,Y) = Y(\xi f)\eta(X) + \xi f g(\phi X, \phi Y). \tag{3.9}$$

Putting $X = \xi$ in (3.9) and using (2.3) we get

$$Y(\xi f) = (1 - n + \lambda)\eta(Y). \tag{3.10}$$

With this equation, it is clear that if $\lambda = n - 1$.

So from here, $\xi f = constant$. Then using (3.8) we have

$$Df = (\xi f)\xi = c\xi.$$

Particularly, taking a frame field $\xi f = 0$, we get from (3.8), $f = constant$. Therefore, Equation (3.1) can be shown as

$$S(X,Y) = (1-n)g(X,Y),$$

that is M is an Einstein manifold.

Theorem 3.1. *If an η-Einstein nearly Kenmotsu manifold admits a $(G.R.S)$ then the manifold transforms to an Einstein manifold provided $\lambda = 1 - n$ and with the frame field $\xi f = 0$.*

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