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ON CONVEXITY FOR ENERGY DECAY RATES OF A VISCOELASTIC WAVE EQUATION WITH A DYNAMIC BOUNDARY AND NONLINEAR DELAY TERM *

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Abstract. In this paper, we are interested in the study of the initial boundary value problem for a system of viscoelastic wave equations in a bounded domain with dynamic boundary conditions related to the Kelvin Voigt damping and nonlinear delay term acting on the boundary. At first, the global existence of solutions is discussed using the potential well method and introducing suitable energy and Lyapunov functionals to establish general decay estimates for the energy.
Keywords: Boundary value problem; Lyapunov functionals; boundary conditions.

1. Introduction

In this paper, we consider the following wave equation with dynamic boundary conditions:

$$
(1.1)\qquad \begin{cases}
u_{tt} - \Delta u - \alpha \Delta u_t - \displaystyle\int_0^t g(t-s)\Delta u(s)ds = |u|^{p-1}u, \\
\qquad\qquad\qquad\qquad \text{in} \quad \Omega \times (0,+\infty), \\
u = 0, \qquad\qquad\qquad \textit{on} \qquad \Gamma_0 \times (0,+\infty), \\
u_{tt} = -a\left[\frac{\partial u}{\partial v}(x,t) + \delta \frac{\partial u_t}{\partial v}(x,t) - \displaystyle\int_0^t g(t-s)\Delta u(s)\frac{\partial u}{\partial v}(x,s)ds \right. \\
\qquad \left. +\mu_1 \psi(u_t(x,t)) + \mu_2 \psi(u_t(x,t-\tau))\right], \\
\qquad\qquad\qquad\qquad \text{on} \quad \Gamma_1 \times (0,+\infty) \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\
u_t(x,t-\tau) = f_0(x,t-\tau), \qquad \textit{on} \quad \Gamma_1 \times (0,+\infty),
\end{cases}
$$

where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$, Δ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of R^n, $(n \geq 1)$, $\partial\Omega = \Gamma_1 \cup \Gamma_0$,

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$\Gamma_1 \cap \Gamma_0 = \emptyset$ and $\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative, μ_1 and μ_2 are positive constants. Moreover, $\tau > 0$ represents the time delay and u_0, u_1, f_0 are given functions belonging to suitable spaces that will be explicated later. This type of problems arises (for example) in the modeling of longitudinal vibrations in a homogeneous bar on which there are viscous effects. The term Δu_t indicates that the stress is proportional not only to the strain, but also to the strain rate. See [5]. This type of problem without delay (i.e $\mu_i = 0$), has been considered by many authors during the past decades and many results have been obtained (see [2,4,6,7,14,37,38]).

The main difficulty of the problem considered is related to the non-ordinary boundary conditions defined on Γ_1. Very little attention has been paid to this type of boundary conditions. We mention a few particular results in the one dimensional without delay term for a linear damping (m=1)and $g = 0$ [15,17]. From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such types of boundary conditions are usually called dynamic boundary conditions. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, problem (1.1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represent the Newton's law for the attached mass (see [4, 1, 6] for more details), which arise when we consider the transverse motion of a flexible membrane whose boundary may be affected by the vibrations only in a region. Also, some of them, as in problem (1.1), appear when we assume that there is an exterior domain of R^3 in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [2] for more details). Among the early results dealing with the dynamic boundary conditions are those of Grobbelaar-Van Dalsen [7,8] in which the author has made contributions to this field and in [14] the authors have studied the following problem:

$$
(1.2) \qquad \begin{cases}
u_{tt} - \Delta u + \delta \Delta u_t = |u|^{p-1} u, \quad in \quad \Omega \times (0, +\infty), \\
u = 0, \qquad on \quad \Gamma_0 \times (0, +\infty), \\
u_{tt} = -a\left(\frac{\partial u}{\partial \nu}(x,t) + \delta \frac{\partial u_t}{\partial \nu}(x,t) + \alpha |u_t|^{m-1} u_t(x,t)\right), \\
\qquad on \quad \Gamma_1 \times (0, +\infty), \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\
u_t(x, t-\tau) = f_0(x, t-\tau), \qquad on \quad \Gamma_1 \times (0, +\infty),
\end{cases}
$$

and they have obtained several results concerning local existence which extended to the global existence by using stable sets. The authors have also obtained the energy decay and the blow up of the solutions for initial energy positive.

It is widely known that delay effects, which arise in many practical problems, result from some instabilities. The authors in [11-13] showed that a small delay in a boundary control turns a well-behaved hyperbolic system into a wild one which, in turn, becomes a source of instability. They also proved that the energy is exponentially stable under the condition

$$\mu_2 < \mu_1. \tag{1.3}$$

Recently, inspired by the works of Nicaise and C. Pignotti [10], Sthéphan Gherbi and Said el Houari [15] considered the following problem in a bounded domain:

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = 0, \quad in \quad \Omega \times (0, +\infty), \\ u = 0, \qquad\qquad on \quad \Gamma_0 \times (0, +\infty), \\ u_{tt} = -a\Big[\frac{\partial u}{\partial v}(x,t) + \alpha \frac{\partial u_t}{\partial v}(x,t) \\ \qquad + \mu_1 u_t(x,t)) + \mu_2(u_t(x,t-\tau))\Big] \text{ on } \quad \Gamma_1 \times (0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x, t-\tau) = f_0(x, t-\tau), \qquad on \quad \Gamma_1 \times (0, +\infty), \end{cases} \tag{1.4}$$

with a very particular restriction on $h(u_t)$, where they applied the Faedo-Galerkin method combined with the fixed point theorem and showed the existence and uniqueness of a local in time solution. Under some restrictions on the initial data, the solution continues to exist globally in time. On the other hand, they proved that, if the interior source dominates the boundary damping, the solution is unbounded and grows as an exponential function. In addition, they showed that, in the absence of the strong damping, the solution ceases to exist and blows up in finite time.

Motivated by the previous works, in the present paper we investigate problem (1.1) in which we generalize the results obtained in [16], supposing new conditions with which the global existence and stability are assured, by using the stable sets method to prove the existence result and introducing a suitable Lyapunov functional which helps us to prove energy decay rates.

The content of this paper is organized as follows. In Section 2, we provide assumptions that will be used later, state and prove the existence result in Theorem 3.1. In Section 4, we prove stability result that is given in Theorem 4.1.

2. Preliminary Results

In this section, we present some material in the proof of our main result. For the relaxation function g we assume

(A_1) $g : R^+ \to R^+$ is a bounded C^1 function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l < 1, \; g'(t) \le -\xi(t)g(t),$$

(A_2) $\psi : R \to R$ is a nondecreasing function of the class $C^1(R)$ such that there exist ϵ_1(sufficiently small), $c_1, c_2, \alpha_1, \alpha_2 > 0$ and a convex and increasing function $H : R^+ \to R^+$ of the class $C^1(R_+) \cap C^2(]0,\infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon_1]$ or ($H^{'}(0) = 0$ and $H^{''} > 0$ on $[0, \epsilon_1]$), such that

$$c_1|s| \le |\psi(s)| \le c_2|s| \quad if \quad |s| \ge \epsilon_1, \tag{2.1}$$

$$\psi^2(s) \le H^{-1}(s\psi(s)) \quad if \quad |s| \le \epsilon_1, \tag{2.2}$$

$$\alpha_1 s\psi(s) \le G(s) \le \alpha_2 s\psi(s), \tag{2.3}$$

where

$$G(s) = \int_0^s \psi(r)dr,$$

(A_3)

$$\alpha_2\mu_2 < \alpha_1\mu_1.$$

We will also be using the embedding

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega),\ H_0^1(\Gamma_1) \hookrightarrow L^p(\Gamma_1),$$

and Poincaré's inequality. The same embedding constant c_s will be used.

As in [10] we choose ζ such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \zeta < \tau\frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}. \tag{2.4}$$

Lemma 2.1. *(Sobolev-Poincaré inequality). Let $2 \le p \le \frac{2n}{n-2}$. The inequality*

$$\|u\|_p \le c_s\|\nabla u\|_2 \quad for \quad u \in H_0^1(\Omega)$$

holds with some positive constant c_s.

Lemma 2.2. [28]. *For any $g \in C^1$ and $\phi \in H^1(0,T)$, we have*

$$\int_0^t \int_\Omega g(t-s)\varphi\varphi_t dxds = -\frac{1}{2}\frac{d}{dt}\left((go\varphi)(t) + \int_0^t g(s)ds\|\varphi\|_2^2\right) - g(t)\|\varphi\|_2^2 + (g'o\varphi)(t),$$

where

$$(go\varphi)(t) = \int_0^t g(t-s)\int_\Omega |\varphi(s) - \varphi(t)|^2 dxds.$$

Lemma 2.3. [28]. *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds \right)^2 dx \leq (1-l)c_s^2 (go\nabla u)(t), \tag{2.5}$$

where

$$(go\nabla u)(t) = \int_0^t g(t-s) \int_\Omega |u(s)-u(t)|^2 dxds,$$

and c_s^2 is the Poincaré constant and l is given in (A_1).

3. Global existence and energy decay results

In this section, we will prove the global existence and general decay results. For this reason, we introduce a new variable z as in [11],

$$z(x,k,t) = u_t(x, t-\tau k), \quad x \in \Gamma_1, \quad k \in (0,1),$$

which implies that

$$\tau z_t(x,k,t) + z_k(x,k,t) = 0, \quad in \quad \Gamma_1 \times (0,1) \times (0,\infty).$$

Therefore, problem (1.1) is equivalent to

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s)ds = |u|^{p-1}u, \\ \qquad\qquad\qquad\qquad in \quad \Omega \times (0,+\infty), \\ u = 0, \qquad\qquad\qquad on \quad \Gamma_0 \times (0,+\infty), \\ u_{tt} = -a\Big[\frac{\partial u}{\partial v}(x,t) + \delta \frac{\partial u_t}{\partial v}(x,t) - \displaystyle\int_0^t g(t-s)\Delta u(s)\frac{\partial u}{\partial v}(x,s)ds. \\ \qquad + \mu_1 \psi(u_t(x,t)) + \mu_2 \psi(z_k(x,1,t))\Big] \\ \qquad\qquad\qquad\qquad \text{on} \quad \Gamma_1 \times (0,+\infty) \\ \tau z_t(x,k,t) + z_k(x,k,t) = 0, \quad in \quad \Gamma_1 \times (0,1) \times (0,\infty), \\ z(x,k,0) = f_0(x,-\tau k), \quad x \in \Omega, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \Gamma_0, \quad t \geq 0. \end{cases} \tag{3.1}$$

Remark 3.1. For reasons of simplicity, we take $a = 1$ in (3.1).

Remark 3.2. By the mean value theorem for integrals and the monotonicity of ψ, we find

$$G(s) = \int_0^s \psi(r)dr \leq s\psi(s).$$

Then, $\alpha_1 \leq \alpha_2 \leq 1$.

Now inspired by [10, 28, 32], we define the energy functional related to problem (3.1) by

$$
\begin{aligned}
E(t) = {} & \tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{1}{2}\|u_t(t)\|_{2.\Gamma_1}^2 + \tfrac{1}{2}(go\nabla u)(t) + \tfrac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2. \\
& -\tfrac{1}{p+1}\|u(t)\|_{p+1}^{p+1} + \tfrac{\zeta}{2}\int_{\Gamma_1}\int_0^1 G(z(\gamma,k,s))dkd\gamma.
\end{aligned} \tag{3.2}
$$

Lemma 3.1. *Let* (u, z) *be the solution of* (3.1) *then, the energy equation satisfies*

$$
\begin{aligned}
E'(t) \leq {} & \tfrac{1}{2}(g'o\nabla u)(t) - \tfrac{1}{2}g(t)\|\nabla u(t)\|_2^2 - \left(\mu_1 - \tfrac{\zeta\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_{\Gamma_1} u_t(t)\psi(u_t(t))d\gamma \\
& -\left(\tfrac{\zeta}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_{\Gamma_1} z(x,1,t)\psi(z(x,1,t))d\gamma - \delta\|\nabla u_t(t)\|_2^2.
\end{aligned} \tag{3.3}
$$

Proof. By multiplying the first and second equation in (3.1) by $u_t(t)$, and integrating the first equation over Ω, and the second equation over Γ_1, using the Green's formula, we get

$$
\begin{aligned}
& \tfrac{d}{dt}\left[\tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{1}{2}\|u_t(t)\|_{2.\Gamma_1}^2 + \tfrac{1}{2}\|\nabla u(t)\|_2^2 + \delta\|\nabla u_t(t)\|_2^2 - \tfrac{1}{p+1}\|u(t)\|_{p+1}^{p+1}\right] \\
& +\mu_1\|u_t(t)\|_{2.\Gamma_1}^2 - \int_\Omega\int_0^t g(t-s)\nabla u(s)\nabla u_t(t)dsdx + \int_{\Gamma_1}\mu_2\psi(z(x,1,t))u_t(t)d\gamma.
\end{aligned} \tag{3.4}
$$

We multiply the third equation in (3.1) by ζz and integrate over $\Gamma_1 \times (0,1)$ to obtain

$$
\begin{aligned}
\zeta\int_{\Gamma_1}\int_0^1 z_t\psi(z(\gamma,k,t))dkd\gamma &= -\tfrac{\zeta}{\tau}\int_{\Gamma_1}\int_0^1 \frac{\partial}{\partial k}G(z(\gamma,k,t))dkd\gamma \\
&= -\tfrac{\zeta}{\tau}\int_{\Gamma_1}[G(z(\gamma,1,t))d\gamma - G(z(\gamma,0,t))]d\gamma
\end{aligned} \tag{3.5}
$$

then

$$
\zeta\frac{d}{dt}\int_{\Gamma_1}\int_0^1 G(z(\gamma,k,t))dkd\gamma = -\frac{\zeta}{\tau}\int_{\Gamma_1} G(z(\gamma,1,t))d\gamma + \frac{\zeta}{\tau}\int_{\Gamma_1} G(u_t)d\gamma. \tag{3.6}
$$

From (3.4) , (3.6) and Young's inequality we get

$$
\begin{aligned}
& E(t) + \left(\mu_1 - \tfrac{\zeta\alpha_2}{\tau}\right)\int_0^t\int_{\Gamma_1} u_t\psi(u_t)d\gamma ds + \frac{\zeta}{\tau}\int_0^t\int_{\Gamma_1} G(z(\gamma,1,t))d\gamma ds \\
& +\mu_2\int_0^t\int_{\Gamma_1} u_t\psi(z(\gamma,1,t))d\gamma ds\frac{1}{2}\int_0^t (g'o\nabla u)(s)ds \\
& +\tfrac{1}{2}\int_0^t g(s)\|\nabla u(s)\|_2^2 ds + \delta\int_0^t \|\nabla u_t(s)\|_2^2 ds = E(0).
\end{aligned} \tag{3.7}
$$

Let us denote G^* to be the conjugate of the convex function G, ie., $G^*(s) = \sup_{t\in R_+}(st - G(t))$. Then G^* is the Legendre transform of G, which is given by (see Arnold [4], p.61 – 62)

$$
G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)], \forall s \geq 0, \tag{3.8}
$$

and satisfies the following inequality

$$st \leq G^*(s) + G(t), \quad \forall s, t \geq 0. \tag{3.9}$$

Then, from the definition of G, we get

$$G^*(s) = s\psi^{-1}(s) - G(\psi^{-1}(s)).$$

Hence

$$\begin{array}{rcl} G^*(\psi(z(\gamma,1,t))) & = & z(\gamma,1,t)\psi(z(\gamma,1,t)) - G(z(\gamma,1,t)) \\ & \leq & (1-\alpha_1)z(\gamma,1,t)\psi(z(\gamma,1,t)). \end{array} \tag{3.10}$$

Making use of (3.7), (3.9) and (3.10), we have

$$\begin{array}{l} E'(t) \leq -\left(\mu_1 - \frac{\zeta\alpha_2}{\tau}\right)\int_{\Gamma_1} u_t\psi(u_t)d\gamma - \dfrac{\zeta}{\tau}\int_{\Gamma_1} G(z(\gamma,1,t))d\gamma \\ -\frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - \delta\|\nabla u_t(t)\|_2^2 \\ +\mu_2\int_{\Gamma_1}(G(u_t) + G^*(z(\gamma,1,t)))d\gamma \\ \leq \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ \qquad\qquad -\left(\mu_1 - \frac{\zeta\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_{\Gamma_1} u_t(t)\psi(u_t(t))d\gamma \\ -\left(\frac{\zeta}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_{\Gamma_1} z(x,1,t)\psi(z(x,1,t))d\gamma - \delta\|\nabla u_t(t)\|_2^2. \end{array} \tag{3.11}$$

This ends the proof. $\square$

Now we are in a position to state the local existence result to problem (3.1), which can be established by combining arguments of [9, 10, 28, 29].

Theorem 3.1. *Let $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in L^2(\Gamma_1\times(0,1))$. Suppose that $(A_1)-(A_4)$ hold. Then the problem (3.1) admits a unique weak solution satisfying*

$$u \in L^\infty((0,T); H^1_{\Gamma_0}(\Omega)), \quad u_t \in L^\infty((0,T); H^1_{\Gamma_0}(\Omega)) \cap L^\infty((0,T); L^2(\Gamma_1)),$$

$$u_{tt} \in L^\infty((0,T); L^2(\Omega)) \cap L^\infty((0,T); L^2(\Gamma_1)).$$

Now we will prove that the solution obtained above is global and bounded in time. For this purpose, let us define

$$\begin{array}{l} I(t) = \left(1 - \displaystyle\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (go\nabla u)(t) - \|u\|_{p+1}^{p+1} \\ \qquad\qquad +\zeta\displaystyle\int_{\Gamma_1}\int_0^1 G(z(\gamma,k,s))dkd\gamma, \end{array} \tag{3.12}$$

$$J(t) = \tfrac{1}{2}(go\nabla u)(t) + \tfrac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \tfrac{1}{p+1}\|u\|_{p+1}^{p+1} + \tfrac{\zeta}{2}\int_{\Gamma_1}\int_0^1 G(z(\gamma,k,s))dkd\gamma, \tag{3.13}$$

where

$$E(t) = J(t) + \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_{2.\Gamma_1}^2. \tag{3.14}$$

Lemma 3.2. . *Suppose that* $(A_0) - (A_1)$ *and* (2.4) *hold. Let* (u, z) *be the solution of the problem* (3.1). *Assume further that* $I(0) > 0$ *and*

$$\alpha = \left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-2}{2}} < 1. \tag{3.15}$$

Then $I(t) > 0$ *for all* $t \geq 0$.

Proof. Since $I(0) > 0$, then there exists (by continuity of $u(t)$) $T^* < T$ such that

$$I(t) \geq 0, \tag{3.16}$$

for all $t \in [0, T^*]$. From (3.12) and (3.13) we have

$$\begin{aligned}
J(t) &\geq \tfrac{p-1}{2(p+1)}\left[\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (go\nabla u)(t)\right] \\
&\geq + \tfrac{(p-1)}{2(p+1)}\left[\zeta\int_0^1\int_{\Gamma_1} G(z(x,k,t))dkd\gamma\right] + \tfrac{1}{p+1}I(t) \\
&\geq \tfrac{p-1}{2(p+1)}\left[\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (go\nabla u)(t)\right] \\
&+ \tfrac{p-1}{2(p+1)}\left[\zeta\int_0^1\int_{\Gamma_1} G(z(\gamma,k,t))dkd\gamma\right] \\
&\geq \tfrac{p-1}{2(p+1)}l\|\nabla u\|_2^2.
\end{aligned} \tag{3.17}$$

Thus by (3.7),(3.17), we deduce

$$\begin{aligned}
l\|\nabla u\|_2^2 &\leq \left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\
&\leq \tfrac{2(p+1)}{(p-1)}E(t) \\
&\leq \tfrac{2(p+1)}{(p-1)}E(0), \ \forall t \in [0, T^*].
\end{aligned} \tag{3.18}$$

Exploiting Lemma 2.1,(3.15), we obtain

$$
\begin{aligned}
\|u\|_{p+1}^{p+1} &\leq c_s^{p+1}\|\nabla u\|_2^{p+1} \\
&\leq \frac{c_s^{p+1}}{l}\|\nabla u\|_2^{p-1}\|\nabla u\|_2^2 \\
&\leq \frac{c_s^{p+1}}{l}\left(\frac{2(p+1)}{(p-1)l}E(0)\right)^{\frac{p-1}{2}} l\|\nabla u\|_2^2 \\
&\leq \leq \alpha l\|\nabla u\|_2^2 \\
&< \left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2,\ \forall t \in [0, T^*].
\end{aligned} \tag{3.19}
$$

Hence

$$
I(t) = \left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (go\nabla u)(t) - \|u\|_{p+1}^{p+1} + \zeta\int_0^1\int_{\Gamma_1} G(z(\gamma,k,t))dkd\gamma > 0,
$$

$\forall t \in [0, T^*]$. Repeating this procedure and using the fact that

$$
\lim_{t\to T^*}\frac{c_s^{p+1}}{l}\left(\frac{2(p+1)}{2l(p-1)}E(u(t))\right)^{\frac{p-1}{2}} \leq \alpha < 1.
$$

We can take $T^* = T$. This completes the proof. □

4. Asymptotic behavior

In this section, we prove the energy decay result by constructing a suitable Lyapunov functional.

Now we define the following functional

$$
L(t) = ME(t) + \epsilon\phi(t) + \epsilon\varphi(t) + \epsilon I(t), \tag{4.1}
$$

where

$$
\phi(t) = \int_\Omega uu_t dx + \int_{\Gamma_1} uu_t d\gamma, \tag{4.2}
$$

$$
\varphi(t) = -\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))dsdx, \tag{4.3}
$$

and

$$
I(t) = \int_{\Gamma_1}\int_0^1 e^{-2k\tau}G(z(x,k,t))dkd\gamma. \tag{4.4}
$$

We also need the following lemma

Lemma 4.1. *Let (u,z) be a solution of problem (3.1). Then there exists two positive constants λ_1,λ_2 such that*

$$\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t), \quad t \geq 0, \tag{4.5}$$

for M sufficiently large.

Proof. Thanks to the Holder and Young's inequalities, lemma 2.1 and using the fact that $\|u\|_{2.\Gamma_1} \leq B\|\nabla u\|_2$, we have

$$|\phi(t)| \leq \frac{1}{\omega}\|u_t\|_2^2 + \frac{1}{4\omega}\|u_t\|_{2.\Gamma}^2 + \omega c_s^2\|\nabla u\|_2^2 + \omega B^2\|\nabla u\|_2^2, \tag{4.6}$$

$$\begin{aligned} \varphi(t) &= \left|-\int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))dsdx\right| \\ &\leq \tfrac{1}{2}\|u_t\|_2^2 + \tfrac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2 dx \\ &\leq \tfrac{1}{2}\left(\|u_t\|_2^2 + (1-l)c_s^2 \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds\right) \\ &\leq \tfrac{1}{2}\left(\|u_t\|_2^2 + (1-l)c_s^2 (go\nabla u)(t)\right), \end{aligned} \tag{4.7}$$

and it follows from (4.4) that $\forall c > 0$

$$\begin{aligned} |I(t)| &= \left|\int_\Omega \int_0^1 e^{-2k\tau} G(z(x,k,s))dkd\gamma\right| \\ &\leq c\int_{\Gamma_1}\int_0^1 G(z(x,k,s))dkd\gamma. \end{aligned} \tag{4.8}$$

Hence, combining (4.6)-(4.8). This yields

$$\begin{aligned} |L(t) - ME(t)| &= \epsilon\phi(t) + \varphi(t) + \epsilon I(t) \\ &\leq \left(\tfrac{\epsilon}{\omega} + \tfrac{\epsilon}{2}\right)\|u_t\|_2^2 + \left(\epsilon\omega + \epsilon B^2\right)\|\nabla u\|_2^2 \\ &+ \quad + \tfrac{\epsilon}{4\omega}\|u_t\|_{2.\Gamma_1}^2 + c\int_{\Gamma_1}\int_0^1 G(z(x,k,t))dkd\gamma \\ &+ \quad \tfrac{(1-l)c_s^2}{2}(go\nabla u)(t). \end{aligned} \tag{4.9}$$

Where

$$c_1 = \left(\frac{\epsilon}{\omega} + \frac{\epsilon}{2}\right), \quad c_2 = \left(\epsilon\omega + \epsilon B^2\right), \quad c_3 = \frac{\epsilon}{4\omega}, \quad c_4 = \frac{(1-l)c_s^2}{2},$$

$$|L(t) - ME(t)| \leq c_5 E(t), \tag{4.10}$$

where $c_5 = max(c_1, c_2, c_3, c_4)$. Thus, from the definition of E(t) and selecting M sufficiently large,

$$\beta_2 E(t) \leq L(t) \leq \beta_1 E(t). \tag{4.11}$$

Where $\beta_1 = (M - \epsilon c_5)$, $\beta_2 = (M + \epsilon c_5)$. This completes the proof. □

Lemma 4.2. *Let (u,z) be the solution of (3.1). Then it holds*

$$\begin{aligned}
&\tfrac{d}{dt}\phi(t) \leq \epsilon\left(\tfrac{(1+l)(1-l)^2+(\mu_1+\mu_2)\alpha c_s^2 B^2}{2}-1\right)\|\nabla u\|_2^2\\
&+\epsilon\tfrac{(1-l)}{2}(go\nabla u)(t)+\epsilon\tfrac{\mu_2}{4\alpha}\|\psi(z(x,1,t))\|_2^2+\epsilon\tfrac{1}{p+1}\|u\|_{p+1}^{p+1}+\|u_t\|_2^2\\
&+\epsilon\left(\tfrac{\mu_1}{4\alpha}+1\right)\|u_t\|_{2.\Gamma_1}^2\|u_t\|_2^2+\epsilon\tfrac{\mu_1}{4\alpha}\|\psi(u_t)\|_{2.\Gamma_1}^2.
\end{aligned}\tag{4.12}$$

Proof. We take the derivative of $\phi(t)$. It follows from (4.2) that

$$\tfrac{d}{dt}\phi(t)=\int_\Omega u_{tt}u dx+\int_\Omega u_{tt}u d\gamma+\|u_t\|_2^2+\|u_t\|_{2.\Gamma_1}^2,\tag{4.13}$$

using the problem (3.1), then we have

$$\begin{aligned}
&\tfrac{d}{dt}\phi(t)=\|u_t\|_2^2+\|u_t\|_{2.\Gamma_1}^2-\|\nabla u\|_2^2+\int_\Omega\int_0^t g(t-s)\nabla u(s)\nabla u(t)dsdx\\
&-\mu_2\int_{\Gamma_1}\psi(z(x,1,t))u(t)d\gamma-\mu_1\int_{\Gamma_1}\psi(u_t)u(t)d\gamma+\frac{1}{p+1}\|u_t\|_{p+1}^{p+1},
\end{aligned}\tag{4.14}$$

we estimate the fourth term in the right hand side of (4.14) as follows

$$\begin{aligned}
&\left|\int_\Omega\nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx\right|\\
&\leq\tfrac{1}{2}\|\nabla u\|_2^2+\tfrac{1}{2}\int_\Omega\left(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|)ds\right)^2dx\\
&\leq\tfrac{1+(1+\lambda)(1-l)^2}{2}\|\nabla u\|_2^2+\tfrac{(1+\frac{1}{\lambda})(1-l)}{2}(go\nabla u)(t),
\end{aligned}\tag{4.15}$$

for the fifth and sixth term in (4.14), Holder and Young's and trace operator inequalities to get

$$\left|\int_{\Gamma_1}\psi(u_t)ud\gamma\right|\leq\alpha c_s^2B^2\|\nabla u\|_2^2+\frac{1}{4\alpha}\|\psi(u_t)\|_{2.\Gamma_1}^2,\tag{4.16}$$

and

$$\left|\int_{\Gamma_1}\psi(z(x,1,t))ud\Gamma_1\right|\leq\alpha c_s^2B^2\|\nabla u\|_2^2+\frac{1}{4\alpha}\|\psi(z(x,1,t))\|_2^2.\tag{4.17}$$

Let $\lambda=\frac{l}{1-l}$ in (4.14) and using (4.16), (4.17), then (4.14) becomes

$$\begin{aligned}
&\tfrac{d}{dt}\phi(t) \leq \epsilon\left(\tfrac{(1+l)(1-l)^2+(\mu_1+\mu_2)\alpha c_s^2 B^2}{2}-1\right)\|\nabla u\|_2^2\\
&+\epsilon\tfrac{(1-l)}{2}(go\nabla u)(t)+\epsilon\tfrac{\mu_2}{4\alpha}\|\psi(z(x,1,t))\|_2^2+\epsilon\tfrac{1}{p+1}\|u\|_{p+1}^{p+1}+\|u_t\|_2^2\\
&+\epsilon\left(\tfrac{\mu_1}{4\alpha}+1\right)\|u_t\|_{2.\Gamma_1}^2\|u_t\|_2^2+\epsilon\tfrac{\mu_1}{4\alpha}\|\psi(u_t)\|_{2.\Gamma_1}^2.
\end{aligned}\tag{4.18}$$

This completes the proof. □

Lemma 4.3. *Let (u, z) be the solution of (3.1). Then $\varphi(t)$ satisfies*

$$
\begin{aligned}
&\tfrac{d}{dt}\varphi(t) \le \alpha\left(1+2(1-l)^2+c_s^{2p}\left(\tfrac{\beta E(0)}{l}\right)^{2p}\right)\|\nabla u\|_2^2\\
&+\alpha\|u_t\|_2^2+\mu_1\|\psi(u_t)\|_{2.\Gamma_1}^2+\tfrac{g(0)c_s^2}{4\alpha}(-g'o\nabla u)(t)\\
&+\tfrac{\mu_2}{4\alpha}(1-l)\left(2(\alpha+1)+c_s^2\right)(go\nabla u)(t)\\
&+\tfrac{1}{4\alpha}c_s^2(1-l)^2\mu_2\int_{\Gamma_1}\psi^2(z(x,1,t))d\gamma.
\end{aligned}
\tag{4.19}
$$

Proof. Now taking the derivatives of $\varphi(t)$, and using the problem (3.1), we obtain

$$
\begin{aligned}
&\varphi'(t) = -\int_\Omega u_{tt}\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&-\int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx\\
&-\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2dx\\
&=\int_\Omega \nabla u(t)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\\
&-\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\\
&-\int_\Omega |u|^{p-1}u\left(\int_0^t g(t-s)u(t)-u(s)ds\right)dx\\
&+\int_\Omega \mu_1\psi(u(t))\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&+\int_\Omega \mu_2\psi(z(x,1,t))\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&-\int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx-\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2dx.
\end{aligned}
\tag{4.20}
$$

Next we will estimate the right hand side of (4.20), using Holder, Young's inequalities and (A_1) to have

$$
\begin{aligned}
&\int_\Omega \nabla u\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\\
&\le \alpha\|\nabla u\|_2^2+\tfrac{(1-l)}{4\alpha}(go\nabla u)(t),
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
& -\int_\Omega\left(\int_0^t g(t-s)\nabla u_s ds\right)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx \\
& \leq \alpha\int_\Omega\left(\int_0^t g(t-s)(|\nabla u(t)|_2^2 ds\right)^2 dx \\
& +\tfrac{1}{\alpha}\int_\Omega\left|\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right|^2 dx \\
& \leq \alpha\int_\Omega\left(\int_0^t g(t-s)(|\nabla u(t)-\nabla u(s)|+|\nabla u(t)|)ds\right)^2 dx \\
& +\tfrac{1}{\alpha}\left(\int_0^t g(t-s)ds\right)\int_\Omega\int_0^t g(t-s)(|\nabla u(t)|-\nabla u(s)|^2 dsdx \\
& \leq \alpha\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2 dx \\
& +2\alpha(1-l)^2|\nabla u(t)|_2^2+\tfrac{1}{4\alpha}(1-l)(go\nabla u)(t) \\
& \leq 2\alpha(1-l)^2\|\nabla u(t)\|_2^2+\left(2\alpha+\tfrac{1}{4\alpha}\right)(1-l)(go\nabla u)(t).
\end{aligned}
\tag{4.22}
$$

Employing Young's inequality, lemma 2.1, we have

$$
\begin{aligned}
& \int_\Omega |u|^{p-1}u\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)dsdx \\
& \leq \alpha\int_\Omega |u|^{2p}dx+\frac{1}{4\alpha}\int_\Omega\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2 dx,
\end{aligned}
\tag{4.23}
$$

using lemma 2.1 to obtain

$$
\int_\Omega |u|^{2p}dx \leq c_s^{2p}\|\nabla u\|_2^{2p} \leq c_s^{2p}\left(\frac{\beta E(0)}{l}\right)^{2p}\|\nabla u\|_2^2. \tag{4.24}
$$

By inserting (4.24) in (4.23), we get

$$
\begin{aligned}
& \int_\Omega |u|^{p-1}u\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)dsdx \\
& \leq \alpha c_s^{2p}\left(\tfrac{\beta E(0)}{l}\right)^{2p}\|\nabla u\|_2^2+c_s^{2p}\tfrac{(1-l)}{4\alpha}(go\nabla u)(t),
\end{aligned}
\tag{4.25}
$$

where g is positive, continuous and $g(0)>0$, for any t_0, we have

$$
\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0, \quad \forall t \geq t_0, \tag{4.26}
$$

then we use (4.26) to get

$$
\begin{aligned}
& \int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx-\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2 dx \\
& \leq \alpha\|u_t\|_2^2+\tfrac{g(0)c_s^2}{4\alpha}(-g'o\nabla u)(t)-g_0\|u_t\|_2^2,
\end{aligned}
\tag{4.27}
$$

and

$$\left|-\int_{\Gamma_1}\mu_1\psi(u_t)\int_0^t g(t-s)(u(t)-u(s))dsdx\right| \le \mu_1\|\psi(u_t)\|^2_{2.\Gamma_1}+\tfrac{\mu_1(1-l)c_s^2}{4\alpha}(go\nabla u)(t), \tag{4.28}$$

and

$$\left|-\int_{\Gamma_1}\mu_2\psi(z(x,1,t))\int_0^t g(t-s)(u(t)-u(s))dsd\gamma\right| \le \mu_2\int_{\Gamma_1}\psi^2(z(x,1,t))d\gamma+\frac{\mu_2(1-l)c_s^2}{4\alpha}(go\nabla u)(t). \tag{4.29}$$

A substitution of (4.21)-(4.29) into (4.20) yields

$$\begin{aligned}\varphi'(t) &\le \alpha\left(1+2(1-l)^2+c_s^{2p}(\tfrac{\beta E(0)}{l})^{2p}\right)\|\nabla u\|_2^2\\ &-(g_0-\alpha)\|u_t\|_2^2+\mu_1\|\psi(u_t)\|^2_{2.\Gamma_1}\\ &+\tfrac{\mu_2}{4\alpha}(1-l)(2(\alpha+1)+c_s^2)(go\nabla u)(t)\\ &+\tfrac{1}{4\alpha}c_s^2(1-l)^2\mu_2\int_{\Gamma_1}\psi^2(z(x,1,t))d\gamma.\\ &+\tfrac{g(0)c_s^2}{4\alpha}(-g'o\nabla u)(t)\end{aligned} \tag{4.30}$$

□

As in [10], the derivative of (4.4) can be estimated as follows

$$\frac{d}{dt}I(t)\le -2I(t)-\frac{e^{-2\tau}}{\tau}\int_{\Gamma_1}G(z(\gamma,1,t))d\gamma+\frac{1}{\tau}\int_{\Gamma_1}G(z(\gamma,0,t))d\gamma. \tag{4.31}$$

Theorem 4.1. *Let $(A_1)-(A_4)$ holds and let $u_0\in H^1_{\Gamma_0}(\Omega)$, $u_1\in L^2(\Omega)$, $f_0\in L^2(\Gamma_1\times(0,1))$ be given. Suppose that* (2.4) *holds. Then the solution of the problem (3.1) is global and bounded in time. Furthermore, we have the following decay estimates:*

$$E(t)\le \omega_1 H_1^{-1}(\omega_2 t+\omega_3),\quad \forall t>0,$$

where

$$H_1(t)=\int_t^1\frac{1}{H_2(s)}ds. \tag{4.32}$$

Proof. . First, we prove $T=\infty$. It is sufficient to show that $l\|\nabla u\|_2^2$ is bounded independently of t. We have from (3.13) and (3.14)

$$\begin{aligned}E(0)\ge E(t) &= \tfrac12\|u_t\|_2^2+\|u_t\|^2_{2.\Gamma_1}+J(t)\\ &\ge \tfrac12\|u_t\|_2^2+\|u_t\|^2_{2.\Gamma_1}+\left(\tfrac{p-1}{2(p+1)}\right)(l\|\nabla u\|_2^2)\\ &\ge l\|\nabla u\|_2^2.\end{aligned}$$

Therefore

$$l\|\nabla u\|_2^2\le \beta E(0). \tag{4.33}$$

where β is a positive constant which depends only on p, so we obtain the global existence result.
Hence we conclude from lemma 4.2, lemma 4.3 and 4.31 that

$$\begin{aligned}\frac{dL(t)}{dt} &\leq \left(\epsilon\left(\frac{\mu_2(1-l)}{4\alpha}\left(2(\alpha+1)+c_s^2+\frac{1-l}{2}\right)\right)\right)(go\nabla u)(t)\\
&-\left(\epsilon\left((1-\alpha-(1-l)^2(2+(1+l)))\right)\right)\|\nabla u\|_2^2\\
&-\left(\epsilon(c_s^{2p}\left(\frac{\beta E(0)}{l}\right)-(\mu_1+\mu_2)\alpha c_s^2 B^2\right)\|\nabla u\|_2^2\\
&+\frac{\epsilon}{p+1}\|u\|_{p+1}^{p+1}-\epsilon(g_0-\alpha-1)\|u_t\|_2^2-M\delta\|\nabla u_t\|_2^2\\
&-Mc_1\int_{\Gamma_1}u_t\psi(u_t)d\gamma-Mc_2\int_{\Gamma_1}z(\gamma,1,t)\psi(z(\gamma,1,t))d\gamma\\
&+\frac{\epsilon\mu_1}{4\alpha}\|\psi(u_t)\|_{2.\Gamma_1}^2+\epsilon\left(\frac{\mu_2 c_s^2(1-l)^2}{4\alpha}+\frac{\mu_2}{4\alpha}\right)\|\psi(z(\gamma,1,t))\|_{2.\Gamma_1}^2\\
&-2\epsilon\int_{\Gamma_1}\int_0^1 e^{-2k\tau}G(z(\gamma,k,t))dkd\gamma-\epsilon\frac{e^{-2\tau}}{\tau}\int_{\Gamma_1}G(z(\gamma,1,t))d\gamma\\
&+\frac{\epsilon}{\tau}\int_{\Gamma_1}G(z(\gamma,0,t))d\gamma+\left(\frac{M}{2}-\frac{\epsilon g(0)c_s^2}{4\alpha}\right)(g'o\nabla u)(t)\\
&+\epsilon\left(\frac{c\mu_1}{4\alpha}+1\right)\|u_t\|_{2.\Gamma_1}^2\end{aligned}\tag{4.34}$$

Using the fact that

$$\|u_t\|_{2.\Gamma_1}^2\leq c\|\nabla u_t\|_2^2,$$

and adding and subtracting the following term $\frac{M\epsilon}{2}\|u_t\|_{2.\Gamma_1}^2$ in the right hand side of (4.34), we get

$$\begin{aligned}\frac{dL(t)}{dt} &\leq \left(\epsilon\left(\frac{\mu_2(1-l)}{4\alpha}\left(2(\alpha+1)+c_s^2+\frac{1-l}{2}\right)\right)\right)(go\nabla u)(t)\\
&-\left(\epsilon\left((1-\alpha-(1-l)^2(2+(1+l)))\right)\right)\|\nabla u\|_2^2\\
&-\left(\epsilon(c_s^{2p}\left(\frac{\beta E(0)}{l}\right)-(\mu_1+\mu_2)\alpha c_s^2 B^2\right)\|\nabla u\|_2^2\\
&+\frac{\epsilon}{p+1}\|u\|_{p+1}^{p+1}-\epsilon(g_0-\alpha-1)\|u_t\|_2^2-\frac{M\epsilon}{2}\|u_t\|_{2.\Gamma_1}^2\\
&+\epsilon\left(M(c_1\alpha+\frac{\mu_1}{4\alpha})\right)\|\psi(u_t)\|_{2.\Gamma_1}^2-\epsilon\left(M\left(\delta-\frac{c}{2}\right)+\frac{c\mu_1}{4\alpha\tau}\right)\|\nabla u_t\|_2^2\\
&+\epsilon\left(\frac{\mu_2}{4\alpha}(1+c_s^2(1-l)^2)\right)\|\psi(z(\gamma,1,t))\|_{2.\Gamma_1}^2\\
&+\epsilon\left(\frac{\alpha_1 e^{-2\tau}}{\tau}+Mc_2\right)\int_{\Gamma_1}z(\gamma,1,t)\psi(z(\gamma,1,t))d\gamma\\
&-2\epsilon\int_{\Gamma_1}\int_0^1 e^{-2k\tau}G(z(\gamma,k,t))dkd\gamma\\
&+\left(\frac{M}{2}-\frac{\epsilon g(0)c_s^2}{4\alpha}\right)(g'o\nabla u)(t)\end{aligned}\tag{4.35}$$

. Choosing carefully ϵ to be sufficiently small and M sufficiently large and putting

$$\left(\frac{M}{2}-\frac{\epsilon g(0)c_s^2}{4\alpha}\right)=\eta_0>0,$$

$$\left(\frac{\mu_2(1-l)}{2\alpha}\left(2(\alpha+1)+c_s^2+\frac{1-l}{2}\right)\right)=\eta_1>0,$$

$$\left((1-\alpha-(1-l)^2(2+(1+l)))-c_s^{2p}\left(\frac{\beta E(0)}{l}\right)-(\mu_1+\mu_2)\alpha c_s^2 B^2\right)=\eta_2>0,$$

$$(g_0-\alpha-1)=\eta_3>0,$$
$$\left(M\left(\delta-\frac{c}{2}\right)+\frac{c\mu_1}{4\alpha\tau}\right)=\eta_4>0,$$

(4.35) takes the form

$$\frac{dL(t)}{dt}\leq -\theta\epsilon E(t)+\epsilon\frac{\eta_1}{2}(go\nabla u)(t)+\epsilon c||\psi(u_t)||^2_{2.\Gamma_1}, \tag{4.36}$$

where θ is a positive constant. Setting

$$\lambda_1=\frac{\theta\epsilon}{\beta_2},\ \lambda_2=\frac{\eta_1\epsilon}{2},\ \lambda_3=\epsilon c,$$

the last inequality becomes

$$\frac{dL(t)}{dt}\leq -\lambda_1 E(t)+\lambda_2(go\nabla u)(t)+\lambda_3||\psi(u_t)||^2_{2.\Gamma_1}, \tag{4.37}$$

. Multiplying the last inequality by $\xi(t)$, we get

$$\begin{aligned}\xi(t)\tfrac{dL(t)}{dt} &\leq -\lambda_1\xi(t)E(t)+\lambda_2\xi(t)(go\nabla u)(t)+\lambda_3\xi(t)||\psi(u_t)||^2_{2.\Gamma_1}\\ &\leq -\lambda_1\xi(t)E(t)-\lambda_2\xi(t)(g^{'}o\nabla u)(t)+\lambda_3\xi(t)||\psi(u_t)||^2_{2.\Gamma_1}\\ &\leq -\lambda_1\xi(t)E(t)-cE^{'}(t)+\lambda_3\xi(t)||\psi(u_t)||^2_{2.\Gamma_1},\end{aligned} \tag{4.38}$$

. We consider the following partition on Γ_1

$$\Gamma_{11}=\{\gamma\in\Gamma_1;|u_t|\geq\epsilon^{'}\},\quad \Gamma_{12}=\{\gamma\in\Gamma_1;|u_t|\leq\epsilon^{'}\},$$

. It is clear that $F=L(t)+c\xi(t)E(t)$ is equivalent to E(t). Then

$$F^{'}(t)\leq -\lambda_1\xi(t)E(t)+\lambda_3\xi(t)||\psi(u_t)||^2_{2.\Gamma_1}\quad \forall t\geq t_0 \tag{4.39}$$

. From (2.1) and (2.2), it follows that

$$\int_{\Gamma_{12}}|\psi(u_t)|^2d\gamma\leq\mu_1\int_{\Gamma_{12}}u_t||\psi(u_t)||^2_2 d\gamma\leq -\mu_1 E^{'}(t). \tag{4.40}$$

1. Case 1

   H is linear then, according to (A_1),

$$c_1^{'}|s|\leq|\psi(s)|\leq c_2^{'}|s|,\quad \forall s,$$

   and so

$$\psi^2(s)\leq c_2^{'}s\psi(s),\quad \forall s.$$

H is linear on $[0,\epsilon^{'}]$. In this case one can easily check that there exists $\mu_1^{'} > 0$, such that $|\psi(s)| \leq \mu_1^{'}|s|$ for all $|s| \leq \epsilon^{'}$, and thus

$$\int_{\Gamma_{11}} \|\psi(u_t)\|^2 d\gamma \leq \mu_1^{'} \int_{\Gamma_{11}} u_t\psi(u_t)d\gamma \leq -\mu_1^{'}E^{'}(t), \tag{4.41}$$

. Using (4.40), (4.41) and the fact that $\xi^{'} \leq 0$,, it is clear that $\vartheta = L(t)\xi(t) + c(\mu_1 + \mu_1^{'})E$ is equivalent to E(t). Then from (4.39) it produces

$$E(t) \leq ce^{-c\int_0^t \xi(s)ds} = H_1^{-1}\left(\int_0^t \xi(s)ds\right). \tag{4.42}$$

.

2. Case 2

$H^{'}(0) = 0$ and $H^{''} > 0$ on $[0,\epsilon^{'}]$ since H is convex and increasing, and H^{-1} is concave and increasing. By Jensen's inequality

$$\begin{aligned} \int_{\Gamma_{12}} |\psi(u_t)|^2 d\gamma &\leq \int_{\Gamma_{12}} H^{-1}\left(u_t\psi(u_t)d\gamma\right) \\ &\leq |\Gamma_{12}|H^{-1}\left(\tfrac{1}{|\Gamma_{12}|}u_t\psi(u_t)d\gamma\right) \\ &\leq cH^{-1}(-c^{'}E^{'}(t)), \end{aligned} \tag{4.43}$$

. Then, using (2.1), (3.3), we get

$$\begin{aligned} \int_{\Gamma_1} |\psi(u_t)|^2 d\gamma &= \int_{\Gamma_{11}} |\psi(u_t)|^2 d\gamma + \int_{\Gamma_{12}} |\psi(u_t)|^2 d\gamma \\ &\leq \int_{\Gamma_{12}} H^{-1}u_t\psi(u_t)d\gamma + \int_{\Gamma_{12}} u_t\psi(u_t)d\gamma \\ &\leq |\Gamma_{12}|H^{-1}\left(\tfrac{1}{|\Gamma_{12}|}u_t\psi(u_t)d\gamma\right) + \int_{\Gamma_{12}} u_t\psi(u_t)d\gamma \\ &\leq cH^{-1}(-c^{'}E^{'}(t)) - c\xi(t)\mu_1^{'}E^{'}(t), \end{aligned} \tag{4.44}$$

. It is clear that $F = L(t) + c\mu_1 E(t)$ is equivalent to E(t). Therefore, (4.39) becomes

$$F^{'}(t) \leq \lambda_1\xi(t)E(t) + cH^{-1}(-c^{'}E^{'}(t)), \quad \forall t \geq t_0. \tag{4.45}$$

. Let us denote by H^* the conjugate function of the convex function H, i.e,

$$H^* = \sup_{t\in R_+}(st - H(t)). \tag{4.46}$$

Then H^* is the Legendre transform of H which satisfies the following inequality

$$st \leq H^* + H(t), \quad \forall s,t \geq 0, \tag{4.47}$$

and

$$H^* = s(H^{'})^{-1}(s) - H[(H^{'})^{-1}(s)], \forall s \geq 0, \tag{4.48}$$

. The relation (4.48) and the fact that $H^{'}(0) = 0$ and $(H^{'})^{-1}$, H are increasing functions yield

$$H^{*}(s) \leq s(H^{'})^{-1}(s), \quad \forall s \geq 0, \tag{4.49}$$

. Using the fact that $E^{'} \leq 0$, $H^{'} \geq 0$, $H^{''} \geq 0$, we derive $\epsilon_0 > 0$ small enough and we find that the functional F_1 defined by

$$F_1(t) = H^{'}(\epsilon_0 E(t))F(t) + c_3 E(t), \tag{4.50}$$

satisfies for some $\nu_1, \nu_2 > 0$

$$\nu_1 F_1(t) \leq E(t) \leq \nu_2 F_1(t), \tag{4.51}$$

. Taking the derivative of (4.50)

$$\begin{aligned} F_1^{'}(t) &= \epsilon_0 E^{'}(t)H^{''}(\epsilon_0 E(t))(H^{'}(\epsilon_0 E(t))F(t) + c_3 E(t)) \\ &+ H^{'}(\epsilon_0 E(t))(L^{'}(t) + c\mu_1 E^{'}(t)) + c_3 E^{'}(t) \\ &\leq -\lambda_1 \xi(t)E(t)H^{'}(\epsilon_0 E(t)) \\ &+ \hat{c}_3 H^{'}(\epsilon_0 E(t))H^{-1}(-c^{'}E^{'}(t)) + \hat{c}_3 c^{'}E^{'}(t) \\ &\leq -\lambda_1 \xi(t)E(t)H^{'}(\epsilon_0 E(t)) + \hat{c}_3 H^{*}(H^{'}(\epsilon_0 E(t))) \\ &+ -\hat{c}_3 \xi(t)E^{'}(t) + c_3 E^{'}(t) \\ &\leq -\lambda_1 \xi(t)E(t)H^{'}(\epsilon_0 E(t)) + \epsilon_0 \hat{c}_3 \xi(t)E(t)(H^{'}(\epsilon_0 E(t))) \\ &- \hat{c}_3 \xi(t)E^{'}(t) + c_3 E^{'}(t) \\ &\leq -c\xi(t)H_2 E(t), \end{aligned} \tag{4.52}$$

where $H_2(t) = tH^{'}(\epsilon_0 t)$. We can easily observe from lemma 4.1 and 2.1 that L(t)is equivalent to E(t), so $F_1(t)$ is also equivalent to E(t). By the fact that H_2 is increasing we obtain

$$F_1^{'}(t) \leq -\hat{c}\xi(t)H_2 F_1(t), \forall t \geq 0. \tag{4.53}$$

Noting that $H_1^{'} = \frac{-1}{H_2}$, we infer from (4.53)

$$[F_1(t)H_1(F_1(t))]^{'} \geq \hat{c}\xi(t), \forall t \geq 0. \tag{4.54}$$

A simple integration over (0,t) yields

$$H_1(F_1(t)) \geq \hat{c} \int_0^t \xi(s)ds + H_1(F_1(0)), \tag{4.55}$$

. Exploiting the fact that H_1^{-1} is decreasing, we infer

$$F_1(t) \leq H_1^{-1}\left(\hat{c}\int_0^t \xi(s)ds + H_1(F_1(0))\right), \tag{4.56}$$

. The equivalence of L, F_1 and E yields the estimate

$$E(t)(t) \leq H_1^{-1}\left(\hat{c}\int_0^t \xi(s)ds + H_1(F_1(0))\right). \tag{4.57}$$

This completes the proof. □

Remark 4.1. Our work is based on some manipulations in [10,32] and our Lyapunov functional is different from the one used in [32]. This work can be viewed as a continuation of the works of S. Messaoudi and M. Mustafa [32].

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