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STOCHASTIC EVOLUUTION EQUATIONS WITH MONOTONE NONLINEARITY IN Ł^p SPACES

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Abstract. In this paper, we study semilinear stochastic evolution equations with semimonotone nonlinearity and multiplicative noise in L^p spaces for $2 \leq p < \infty$. We do not impose any coercivity or Lipschitz condition on the nonlinear part of equations. We prove the existence, uniqueness and measurability of the mild solutions. The proofs of the existence and uniqueness are based on a version of the Itô type inequality which is stronger than analogous inequalities.
Keywords. Semilinear stochastic evolution equations; semimonotone nonlinearity; multiplicative noise; Lipschitz condition.

1. Introduction

Stochastic evolution equations (SEE's for short) describe the evolution in time of the stochastic phenomena and use to model dynamical systems with random effects such as problems arising in biology, chemistry, quantum mechanics, statistical physics, economics, etc. There are two approaches in the study of nonlinear SEE's. The first which is called the variational method, considers Hilbert space valued solutions in the framework of Gelfand triple under certain monotonicity and coercivity conditions on coefficients; see e.g., [20], [26] and [27]. The second approach, the one adopted in this paper, is the semigroup method in which we use the tools of semigroup theory to study mild solutions of semilinear SEE's. This approach gives a unified treatment of a wide class of parabolic, hyperbolic and functional stochastic partial differential equations. Furthermore, its advantage over the variational method is in that one does not require the coercivity condition. In semigroup method, one usually investigate existence, uniqueness and stability of mild solutions of semilinear SEE's under standard Lipshitz-type assumptions on coefficients. The Hilbert space theory of this case has been studied by many authors; see e.g., [8]

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and references therein. Brzeźniak extended a number of results of the same type to martingale type 2-spaces [3], [4]. van Neerven, Veraar and Weis studied stochastic equations in the setting of UMD Banach spaces [23].

On the other hand, some authors use semigroup method to study more general semilinear SEE's with (semi)monotone nonlinear drift instead of Lipschitz one. This approach has first followed by Browder [2] and Kato [18] for deterministic monotone-type semilinear evolution equations. Zangeneh [33, 35] applied this approach to prove the existence and uniqueness of mild solutions of monotone-type semilinear SEE's with multiplicative noise and studied [34] the measurability of mild solutions of these equations. Following this program, Jahanipur and Zangeneh [13] studied (sample-path and p-th mean) exponential asymptotic stability of solutions and Jahanipur [14] proved similar stability theorems for stochastic delay evolution equations. Hamedani and Zangeneh [10] considered a stopped version of monotone-type equations and obtained the existence, uniqueness and measurability of the solutions. Using the tools of random fixed point theory, Jahanipur [15, 16, 17] generalized this approach to study stochastic functional evolution equations. Moreover, Salavati and Zangeneh [28, 30] extended this method to investigate semilinear SEE's with Lévy (jump) noise.

In this paper, we consider monotone-type semilinear SEE's with multiplicative noise in $L^p(\mathbb{R}), 2 \leq p < \infty$, and we prove existence and uniqueness of mild solutions. Our results are remarkable from two points of view. First, we relax Lipschitz condition on nonlinearity drift to semimonotone one without imposing the coercivity hypothesis. Furthermore, while all the results for the semilinear SEE's obtained under our assumptions, have been restricted to the Hilbert space setting, we study the problem in the more general case $L^p(\mathbb{R}), 2 \leq p < \infty$, and therefore we extend some of the results mentioned above.

We make an iterative method to prove the existence and uniqueness of mild solutions in r-th moment for $r \geq 2$. This method is based on a version of Itô type inequality. This is a pathwise inequality for powers $r \geq 2$ of stochastic convolution integrals in $L^p(\mathbb{R}), 2 \leq p < \infty$, and generalizes corresponding inequalities (for example, Theorem 2 of [13]). We adopt the same approach as in [15] and we use a method based on random fixed point theory.

The organization of the paper is as follows. We begin by recalling some preliminary materials in the Section 2. Section 3 is devoted to prove an Itô type inequality inequality. In Section 4, we study the measurability of the solutions of the random integral equation. In Section 5, we introduce the semilinear SEE of monotone-type and prove the existence and uniqueness of it's mild solution.

2. Preliminaries

Throughout the paper, $(\Omega, \mathcal{F}, \mathcal{P})$ denotes a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, $p \geq 2$, $T > 0$ and $r \geq 2$ are given constants. The conjugate exponent of p will be denoted by q and we will simply write L^p for $L^p(\mathbb{R})$. Moreover, H is a real separable Hilbert space with inner product $\langle .\,,\,.\rangle_H$, E is a real Banach space with dual E^*, and $\mathcal{L}(H,E)$ stands for the space of all bounded linear operators from H to E. We recall that the duality mapping $J : E \longrightarrow E^*$ is defined for every $x \in E$ by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x^*\|^2 = \|x\|^2\ \},$$

in which $\langle x\,,\,x^*\rangle$ is the duality pairing between E and E^*. It is well-known that if E is uniformly convex, then J is single valued and continuous. Also, $W_H := (W_H(t))_{t\in[0,T]}$ denotes an H-cylindrical Brownian motion, i.e., $W_H(t)$ is a bounded operator from H to $L^2(\Omega)$, for each $h \in H$ the process $W_H h := (W_H(t)h)_{t\in[0,T]}$ is a real Brownian motion, and for all $h_1, h_2 \in H$ and $t_1, t_2 \in [0,T]$ we have

$$\mathbb{E}(W_H(t_1)h_1 \cdot W_H(t_2)h_2) = (t_1 \wedge t_2)\langle h_1, h_2\rangle_H\cdot$$

Furthermore, we assume that $A : D(A) \subseteq L^p \longrightarrow L^p$ is the generator of a C_0-semigroup $\big(S(t)\big)_{t\geq 0}$ of bounded linear operators satisfying an exponential growth condition with parameter $\lambda > 0$; that is,

$$\|S(t)\| \leq e^{\lambda t} \qquad \forall t \geq 0\cdot$$

If $\|S(t)\| \leq 1$ for all $t \geq 0$, then $S(t)$ is called a contraction semigroup.

2.1. Derivative of L^p-norm

Here we calculate the first and second Fréchet derivatives of L^p-norm function. These results are used in the next sections. Let

$$h(x) = \|x\|^r_{L^p} \qquad \forall x \in L^p(\mathbb{R})\cdot$$

The first and second Fréchet derivatives of h at the point $x \in L^p(\mathbb{R})$ are defined as mappings $Dh(x) : L^p \longrightarrow \mathbb{R}$ and $\big(D^2h(x)\big)(y) : L^p \longrightarrow \mathbb{R}$ such that for any $y, z \in L^p$,

$$\begin{aligned}
\langle y, Dh(x)\rangle &= \lim_{t\downarrow 0} \frac{1}{t}\big(\|x+ty\|^r_{L^p} - \|x\|^r_{L^p}\big) \\
&= \lim_{t\downarrow 0} \frac{r}{p}\Big(\int |x+ty|^p\Big)^{\frac{r}{p}-1} \cdot p \int |x+ty|^{p-2}(x+ty)y \\
&= r\|x\|^{r-p}_{L^p} \int |x|^{p-2}xy = r\|x\|^{r-2}_{L^p} \int \|x\|^{2-p}_{L^p}|x|^{p-2}xy \\
&= r\|x\|^{r-2}_{L^p}\langle y, J(x)\rangle,
\end{aligned}$$

where $J(x)$ is the value at x of the duality mapping of J, and similarly

$$\big\langle z, \big(D^2h(x)\big)(y)\big\rangle \;=\; \big\langle z, D(r\|x\|^{r-2}_{L^p}\langle y, J(x)\rangle)\big\rangle$$

$$
\begin{aligned}
&= \lim_{t\downarrow 0}\frac{1}{t}\Bigg[\Big(r\|x+tz\|_{L^p}^{r-p}\int|x+tz|^{p-2}(x+tz)y\Big)\\
&\qquad -\Big(r\|x\|_{L^p}^{r-p}\int|x|^{p-2}xy\Big)\Bigg]\\
&= \lim_{t\downarrow 0} r\Bigg[\Big(\frac{d}{dt}\|x+tz\|_{L^p}^{r-p}\Big)\int|x+tz|^{p-2}(x+tz)y\\
&\qquad +\|x+tz\|_{L^p}^{r-p}\Big(\frac{d}{dt}\int|x+tz|^{p-2}(x+tz)y\Big)\Bigg]\\
&= r(r-p)\|x\|_{L^p}^{r-2p}\int|x|^{p-2}xz\int|x|^{p-2}xy\\
&\qquad +r(p-1)\|x\|_{L^p}^{r-p}\int|x|^{p-2}zy\\
&= r(r-p)\|x\|_{L^p}^{r-4}\langle z,J(x)\rangle\langle y,J(x)\rangle\\
&\qquad +r(p-1)\|x\|_{L^p}^{r-p}\int|x|^{p-2}zy.
\end{aligned}
$$

So, by Hölder's inequality

$$\Big|\Big\langle z,\big(D^2h(x)\big)(y)\Big\rangle\Big| \le r(r-p)\|x\|_{L^p}^{r-2}\|z\|_{L^p}\|y\|_{L^P}+r(p-1)\|x\|_{L^p}^{r-2}\|z\|_{L^p}\|y\|_{L^p}$$

and therefore,

$$\|D^2(h(x))\| \le r(r-1)\|x\|_{L^p}^{r-2}. \tag{2.1}$$

2.2. γ-radonifying operators

Suppose $(\gamma_n)_{n\geq 1}$ is a Gaussian sequence; i.e., a sequence of independent real-valued standard Gaussian random variables. A linear operator $R: H \longrightarrow E$ is called γ-radonifying if for some (and consequently for every) orthonormal basis $(h_n)_{n\geq 1}$ of H, the series $\sum_{n=1}^{\infty}\gamma_n R h_n$ converges in $L^2(\Omega, E)$. We denote by $\gamma(H,E)$ the set of all γ-radonifying operators from H to E. For any $R\in\gamma(H,E)$ the norm of R is defined by

$$\|R\|_{\gamma(H,E)} := \Big(\mathbb{E}\Big\|\sum_{n=1}^{\infty}\gamma_n R h_n\Big\|^2\Big)^{\frac{1}{2}}.$$

Note that $\|\cdot\|_{\gamma(H,E)}$ is independent of the orthonormal basis $(h_n)_{n\geq 1}$ for H. Endowed with this norm, $\gamma(H,E)$ is a Banach space. If $R\in\gamma(H,E)$, then R is bounded and $\|R\| \le \|R\|_{\gamma(H,E)}$. If E is also a Hilbert space, then $\gamma(H,E)$ is isometrically isomorphic to $\mathcal{L}_2(H,E)$, where $\mathcal{L}_2(H,E)$ denotes the space of all Hilbert-Schmidt operators from H to E. Specially if E is finite dimensional, then γ-radonifying norm is the same as operator norm. For more information about γ-radonifying operators and their properties, see [24].

2.3. Itô formula in UMD Banach spaces

A Banach space E is said to have the unconditional martingale difference property, or briefly, E is a UMD space, if for some (equivalently, for all) $p \in (1, \infty)$ there exists a real positive constant C such that

$$\mathbb{E}\Big\|\sum_{n=1}^{N} \varepsilon_n d_n\Big\|^p \leq C\Big\|\sum_{n=1}^{N} d_n\Big\|^p \qquad \forall N \geq 1$$

for all $(\varepsilon_n)_{n=1}^N \in \{-1, 1\}^N$ and every L^p-integrable E-valued martingale difference sequence $(d_n)_{n\geq 1}$. For example every Hilbert space is a UMD space. Also, the spaces $L^p(S)$ for $1 < p < \infty$ and σ-finite measure space $(S, \mathcal{A}, \mu)$ are UMD spaces. If E is a UMD Banach space, then it is well-known that for a suitable class of functions $\Phi : [0, T] \times \Omega \longrightarrow \gamma(H, E)$ the stochastic integral with respect to W_H is well-defined (see, e.g., [22]).

Let E and F be two normed linear spaces and $h : [0, T] \times E \longrightarrow F$ be a function. We say that h is of class $C^{1,2}$ if h is Fréchet differentiable with respect to the first variable and twice Fréchet differentiable with respect to the second variable and h, $D_1 h$, $D_2 h$ and $D_2^2 h$ are continuous functions on $[0, T] \times \Omega$. Now, we recall the main result of [6].

Theorem 2.1. *(Itô formula) Let E and F be UMD spaces. Assume that $h : [0, T] \times E \longrightarrow F$ is of class $C^{1,2}$. Let $\Phi : [0, T] \times \Omega \longrightarrow \mathcal{L}(H, E)$ be an H-strongly measurable and adapted process which is stochastically integrable with respect to W_H and assume that the paths of Φ belong to $L^2(0, T; \gamma(H, E))$ almost surely. Let $\psi : [0, T] \times \Omega \longrightarrow E$ be strongly measurable and adapted with paths in $L^1(0, T; E)$ almost surely. Let $\xi : \Omega \longrightarrow E$ be strongly $\mathcal{F}_0$-measurable. Define $\zeta : [0, T] \times \Omega \longrightarrow E$ by*

$$\zeta = \xi + \int_0^{\cdot} \psi(s)ds + \int_0^{\cdot} \Phi(s)dW_H(s)\cdot$$

Then $s \longmapsto D_2 h(s, \zeta(s))\Phi(s)$ is stochastically integrable and almost surely we have, for all $t \in [0, T]$,

$$\begin{aligned} h(t, \zeta(t)) - h(0, \zeta(0)) &= \int_0^t D_1 h(s, \zeta(s))ds + \int_0^t D_2 h(s, \zeta(s))\psi(s)ds \\ &+ \int_0^t D_2 h(s, \zeta(s))\Phi(s)dW_H(s) + \frac{1}{2}\int_0^t Tr_{\Phi(s)}\big(D_2^2 h(s, \zeta(s))\big)ds\cdot \end{aligned}$$

Moreover,

$$\int_0^t \Big\|Tr_{\Phi(s)}\Big(D_2^2 h(s, \xi(s))\Big)\Big\|ds \leq \int_0^t \Big\|D_2^2 h(s, \xi(s))\Big\|\Big\|\Phi(s)\Big\|^2_{\gamma(H,E)} ds\cdot \tag{2.2}$$

The following theorem is a maximal inequality for stochastic convolution integrals to which we refer several times in the next sections. we recall it from [25].

Theorem 2.2. *Let E be a 2-smooth Banach space and let Φ be a progressively measurable process in $\gamma(H,E)$.If*

$$\int_0^T \left\|\Phi(t)\right\|^2_{\gamma(H,E)} dt < \infty \qquad a.s.,$$

then the stochastic convolution process $X(t) = \int_0^t S(t-s)\Phi(s)dW_H(s)$ is well-defined and has a continuous version. Moreover, for all real positive b there exists a constant D, depending only on b and E, such that

$$\mathbb{E} \sup_{0\le t\le T} \left\|X(t)\right\|^b \le D^b \mathbb{E}\bigg(\int_0^T \left\|\Phi(t)\right\|^2_{\gamma(H,E)} dt\bigg)^{\frac{b}{2}}. \tag{2.3}$$

We conclude this section by recalling the well-known Burkholder-Davis-Gundy inequality for stochastic integrals in UMD Banach spaces from [32].

Theorem 2.3. *(B.D.G inequality) Let E be a UMD Banach space and $\Phi : [0,T]\times \Omega \longrightarrow \gamma(H,E)$ be an H-strongly measurable and $\mathcal{F}_t$-adapted process which is scalarly in $L^0\big(\Omega, L^2(0,T;H)\big)$. If Φ is stochastically integrable with respect to W_H, then for $0 < b < \infty$ we have*

$$\mathbb{E} \sup_{0\le t\le T} \left\| \int_0^t \Phi(s)dW_H(s)\right\|^b \le C_{p,E}\mathbb{E}\left\|\Phi\right\|^b_{\gamma(L^2(0,T;H),E)}, \tag{2.4}$$

where $C_{p,E}$ is a constant depending only on E and p.

3. Itô type inequality

In this section, we prove an Itô type inequality. This is a pathwise inequality for the norm of the stochastic convolution integral. We use this result to prove the existence and uniqueness of the mild solutions of stochastic evolution equations. One of the first attempts to obtain inequalities for the stochastic convolution integrals was the one made by Kotelenez [19], where he considered Hilbert space valued processes and power $r = 2$ for stochastic convolution integral.Tubaro [31] extended this result to exponents $r \ge 2$ and Ichikawa [11] proved it for the case $0 < r < 2$. van Neerven [25] and Brzeźniak [5] considered such inequalities for processes with values in some Banach spaces of special kind (Theorem 2.2).

While all of these inequalities are for moments and involve expectations, we need a pathwise inequality for studying monotone-type semilinear SEE's. There exist several results of this type for Hilbert space valued processes. In particular, Zangeneh [33] proved a pathwise inequality for the square of the norm of stochastic convolution integral in a Hilbert space. Jahanipur and Zangeneh [13] extended this inequality to the powers $r \geq 2$ in a special case that the stochastic convolution integral is an Itô integral with respect to the Wiener process. Salavati and Zangeneh [29] proved more general case where integrator is a general martingale.

We adopt the same approach as in [13] to prove a pathwise inequality for powers $r \geq 2$ of the norm of stochastic convolution integral in L^p, $p \geq 2$. First, we recall our main assumptions.

Hypothesis 3.1. *(a) X_0 is an $\mathcal{F}_0$-measurable random variable.*

(b) $f : [0,T] \times \Omega \longrightarrow L^p$ is strongly measurable and adapted process with paths in $L^1(0,T;L^p)$ almost surly and $\int_0^T \mathbb{E}\|f(t)\|^r dt < \infty$.

(c) $g : [0,T] \times \Omega \longrightarrow \mathcal{L}(H, L^p)$ is an H-strongly measurable and adapted process which is stochastically integrable with respect to W_H, almost every path of g belong to $L^2(0,T;\gamma(H,L^p))$ and $\int_0^T \mathbb{E}\|g(t)\|^r_{\gamma(H,L^p)} < \infty$.

Theorem 3.2. *(Itô type inequality) Let hypotheses 3.1 hold and*

$$X(t) := S(t)X_0 + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dW_H(s), \quad 0 \leq t \leq T.$$

Then for all $t \in [0,T]$ we have

$$\begin{aligned}
\big\|X(t)\big\|^r_{L^p} \quad &\leq e^{r\lambda t}\big\|X_0\big\|^r_{L^p} + r\int_0^t e^{r\lambda(t-s)}\big\|X(s)\big\|^{r-2}_{L^p}\langle f(s), J(X(s))\rangle ds \\
&\quad + r\int_0^t e^{r\lambda(t-s)}\big\|X(s)\big\|^{r-2}_{L^p}\langle g(s), J(X(s))\rangle dW_H(s) \\
&\quad + \tfrac{1}{2}r(r-1)\int_0^t e^{r\lambda(t-s)}\big\|X(s)\big\|^{r-2}_{L^p}\big\|g(s)\big\|^2_{\gamma(H,L^p)}ds,
\end{aligned} \tag{3.1}$$

where $J(X(s))$ denotes the value of the duality mapping J at $X(s)$.

It is easy to see that by an appropriate transformation, we may assume that $\lambda = 0$ (see, e.g., Lemma 1 of [13]). Then according to the Lumer-Phillips theorem, we have $\langle Ax, J(x)\rangle \leq 0$ for each $x \in D(A)$.

The main idea of the proof is to approximate $X(t)$ using the Yosida method. For each $n \in \mathbb{N}$ we define the mapping $R_n : L^p \longrightarrow D(A)$ by $R_n = nR(n,A)$ where $R(n,A) = (nI - A)^{-1}$; hence $\|R_n\| \leq 1$. Let

$$X_0^n = R_n X_0, \qquad f_n = R_n f, \qquad g_n = R_n g$$

and define

$$X_n(t) = S(t)X_0^n + \int_0^t S(t-s)f_n(s)ds + \int_0^t S(t-s)g_n(s)dW_H(s)\cdot$$

Now, we state and prove some lemmas.

Lemma 3.1. *Under the above conditions,*

$$\big\|X_n(t)-X(t)\big\|_{\infty} := \sup_{0\le t\le T}\big\|X_n(t)-X(t)\big\|_{L^p} \longrightarrow 0$$

in L^r as $n\to\infty$. Moreover, there exists a subsequence, again denoted by $\{X_n\}$, such that

$$\mathbb{E}\int_0^T \Big|\big\|X_n(t)\big\|_{L^p}^r - \big\|X(t)\big\|_{L^p}^r\Big| dt \longrightarrow 0\cdot$$

Proof. By Theorem 2.2, we have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t S(t-s)(g_n(s)-g(s))dW_H(s)\Big\|_{L^p}^r\Big]\\ &\le D\mathbb{E}\Big[\int_0^T \big\|g_n(s)-g(s)\big\|_{\gamma(H,L^p)}^2\Big]^{\frac{r}{2}}.\end{aligned}$$

Since

$$\big\|g_n(s)-g(s)\big\|_{\gamma(H,L^p)} \le \big\|R_n-I\big\|\big\|g(s)\big\|_{\gamma(H,L^p)} \le 2\big\|g(s)\big\|_{\gamma(H,L^p)} \quad a.s.$$

by Hypothesis 3.1(c) and the fact that $R_n\longrightarrow I$ strongly, the dominated convergence theorem implies

$$\mathbb{E}\Big[\int_0^T \big\|g_n(s)-g(s)\big\|_{\gamma(H,L^p)}^2\Big]^{r/2} \longrightarrow 0.$$

So,

$$\Big\|\int_0^{\cdot} S(t-s)(g_n(s)-g(s))dW_H(s)\Big\|_{\infty} \longrightarrow 0 \quad in \;\; L^r\cdot$$

On the other hand, by Hölder's inequality,

$$\mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t S(t-s)(f_n(s)-f(s))ds\Big\|_{L^p}^r\Big] \le T^{r-1}\mathbb{E}\int_0^T \|f_n(s)-f(s)\|_{L^p}^r ds$$

and the right hand side of the above inequality tends to zero by the dominated convergence theorem. Hence,

$$\Big\|\int_0^{\cdot} S(t-s)(f_n(s)-f(s))ds\Big\|_{\infty} \longrightarrow 0, \qquad in \;\; L^r\cdot$$

Moreover,

$$\big\|S(t)(X_0^n-X_0)\big\|_{L^p}^r \le \|X_0^n-X_0\|_{L^p}^r \longrightarrow 0 \qquad boundedly.$$

From (1) we imply that there exists a subsequence, again denoted by $\{X_n\}$, such that for each $t\in[0,T]$,

$$\|X_n(t)\|_{L^p}^r \longrightarrow \|X(t)\|_{L^p}^r \quad a.s.$$

and

$$\|X_n(\omega) - X(\omega)\|_\infty^r \longrightarrow 0 \quad a.s.$$

Furthermore, we have

$$\begin{aligned}\Big|\big\|X_n(t,\omega)\big\|_{L^p}^r - \big\|X(t,\omega)\big\|_{L^p}^r\Big| \quad &\leq \quad 2^{r-1}\big\|X_n(t,\omega) - X(t,\omega)\big\|_{L^p}^r \\ &\quad +(2^{r-1}+1)\big\|X(t,\omega)\big\|_{L^p}^r.\end{aligned}$$

Now, Lemma 3 of [13] yields the result. □

Lemma 3.2. *Let $J(X_n)$, $J(X)$ denote the values of duality mapping at X_n and X, respectively. Then, after choosing a subsequence if it is necessary, we have*

$$\mathbb{E}\int_0^T \big\|J(X_n(s)) - J(X(s))\big\|_{L^q}^r ds \longrightarrow 0.$$

Proof. By Lemma 3.1, one can find a subsequence denoted by the same notation $\{X_n\}$, such that

$$\big\|X_n(s) - X(s)\big\|_{L^p} \longrightarrow 0 \ \ a.s., \qquad for \ \ all \ s \in [0,T].$$

Hence, from the continuity of the duality mapping we imply that

$$\big\|J(X_n(s)) - J(X(s))\big\|_{L^q}^r \longrightarrow 0 \ \ a.s., \qquad for \ \ all \ s \in [0,T].$$

On the other hand

$$\big\|J(X_n(s)) - J(X(s))\big\|_{L^q}^r \leq 2^r\Big(\big\|X_n(s)\big\|_{L^p}^r + \big\|X(s)\big\|_{L^p}^r\Big).$$

Now, applying Lemma 3 of [13] and Lemma 3.1 we obtain the desired result. □

Lemma 3.3. *$\{X_n\}$ is a $D(A)$-valued process, the process $\{AX_n\}$ has integrable paths almost surely and we have for all $t \in [0,T]$ that*

$$X_n(t) = X_0^n + \int_0^t AX_n(s)ds + \int_0^t f_n(s)ds + \int_0^t g_n(s)dW_H(s), \tag{3.2}$$

and

$$\begin{aligned}\|X_n(t)\|_{L^p}^r \leq \quad & \|X_0\|_{L^p}^r + r\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle f_n(s), J(X_n(s))\rangle ds \\ & +r\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle g_n(s), J(X_n(s))\rangle dW_H(s) \\ & +\tfrac{1}{2}r(r-1)\int_0^t \|X_n(s)\|_{L^p}^{r-2}\|g(s)\|_{\gamma(H,L^p)}^2 ds.\end{aligned} \tag{3.3}$$

Proof. Note that

$$\int_0^t AX_n(\theta)d\theta = \underbrace{\int_0^t AS(\theta)X_0^n d\theta}_{T_1} + \underbrace{\int_0^t A\Big(\int_0^\theta S(\theta-s)f_n(s)ds\Big)d\theta}_{T_2} + \underbrace{\int_0^t A\Big(\int_0^\theta S(\theta-s)g_n(s)dW_H(s)\Big)d\theta}_{T_3}.$$

Furthermore, we have

$$T_1 = S(t)X_0^n - X_0^n,$$

and by the Fubini theorem,

$$T_2 = \int_0^t S(t-s)f_n(s)ds - \int_0^t f_n(s)ds.$$

Also, by the Fubini theorem for stochastic integrals in UMD Banach spaces [21], we have

$$T_3 = \int_0^t S(t-s)g_n(s)dW_H(s) - \int_0^t g_n(s)dW_H(s).$$

Hence (3.2) is obtained. Now we apply Itô formula (Theorem 2.1) to $h(X_n(\cdot))$ where $h(x) = \|x\|_{L^p}^r$. We find

$$\begin{aligned}\|X_n(t)\|_{L^p}^r = \quad & \|X_0^n\|_{L^p}^r + r\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle f_n(s), J(X_n(s))\rangle ds \\ & + r\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle AX_n(s), J(X_n(s))\rangle ds \\ & + r\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle g_n(s), J(X_n(s))\rangle dW_H(s) \\ & + \frac{1}{2}\int_0^t Tr_{g_n(s)}\big(D^2(\|X_n(s)\|_{L^p}^r)\big)ds,\end{aligned}$$

where $J(X_n(s))$ denotes the value of duality mapping at $X_n(s)$. Here we have used the first and second Fréchet derivatives of $\|\cdot\|_{L^p}^r$. Since $\|X_0^n\|_{L^p}^r \le \|X_0\|_{L^p}^r$, $\langle Ax, J(x)\rangle \le 0$ for all $x \in D(A)$, and

$$\|g_n(s)\|_{\gamma(H,L^p)} \le \|g(s)\|_{\gamma(H,L^p)},$$

we can apply the inequalities (2.1) and (2.2) to conclude the result. □

Proof of Theorem 3.2. It is enough to prove that the right hand side of (3.3) (after choosing a subsequence) converges term by term to that of (3.1), in

probability. We prove this in three steps:
Step 1: Note that

$$\Big|\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle f_n(s), J(X_n(s))\rangle ds - \int_0^t \|X(s)\|_{L^p}^{r-2}\langle f(s), J(X(s))\rangle ds\Big|$$

$$\leq \underbrace{\Big|\int_0^t \big(\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\big)\langle f_n(s), J(X_n(s))\rangle ds\Big|}_{A_n(t)}$$

$$+ \underbrace{\Big|\int_0^t \|X(s)\|_{L^p}^{r-2}\langle f_n(s) - f(s), J(X_n(s))\rangle ds\Big|}_{B_n(t)}$$

$$+ \underbrace{\Big|\int_0^t \|X(s)\|_{L^p}^{r-2}\langle f(s), J(X_n(s)) - J(X(s))\rangle ds\Big|}_{C_n(t)}.$$

By Hölder's inequality and elementary inequality $|a-b|^k \leq |a^k - b^k|$ which is true for all non-negative numbers a, b and all $k \geq 1$, we obtain

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} A_n(t)\Big] \leq \mathbb{E}\int_0^T \Big|\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\Big|\Big\|f(s)\Big\|_{L^p}\Big\|X_n(s)\Big\|_{L^p} ds$$

$$\leq \Big[\mathbb{E}\int_0^T \Big|\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\Big|^{\frac{r}{r-2}} ds\Big]^{\frac{r-2}{r}} \times$$

$$\Big[\mathbb{E}\int_0^T \Big\|f(s)\Big\|_{L^p}^r ds\Big]^{\frac{1}{r}}\Big[\mathbb{E}\int_0^T \Big\|X_n(s)\Big\|_{L^p}^r ds\Big]^{\frac{1}{r}}$$

$$\leq \Big[\mathbb{E}\int_0^T \Big|\|X_n(s)\|_{L^p}^{r} - \|X(s)\|_{L^p}^{r}\Big| ds\Big]^{\frac{r-2}{r}}\Big[\mathbb{E}\int_0^T \|f(s)\|_{L^p}^r ds\Big]^{\frac{1}{r}}\Big[T\mathbb{E}\|X_n\|_\infty^r\Big]^{\frac{1}{r}}.$$

The second and third terms on the right, are bounded and according to Lemma 3.1, after choosing a subsequence, the first term tends to zero. So, for this subsequence we have $\sup\limits_{0\leq t\leq T} A_n(t) \longrightarrow 0$ in L^1 and hence in probability. Also, the Hölder inequality implies that

$$\sup_{0\leq t\leq T} B_n(t) \leq \int_0^T \|X(s)\|_{L^p}^{r-2}\big\|f_n(s) - f(s)\big\|_{L^p}\big\|X_n(s)\big\|_{L^p} ds$$

$$\leq \big(T\|X\|_\infty^r\big)^{1-\frac{2}{r}}\Big(\int_0^T \|f_n(s) - f(s)\|_{L^p}^r ds\Big)^{\frac{1}{r}}\big(T\|X_n\|_\infty^r\big)^{\frac{1}{r}}.$$

The first and third terms on the right are bounded and the second term tends to zero almost surely by the dominated convergence theorem. Hence, $\sup_{0\leq t\leq T} B_n(t) \longrightarrow 0$ almost surely and so in probability. Moreover, by Hölder's inequality we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} C_n(t)\Big] \leq \mathbb{E}\int_0^T \Big\|X(s)\|_{L^p}^{r-2}\Big\|f(s)\Big\|_{L^p}\Big\|J(X_n(s)) - J(X(s))\Big\|_{L^q} ds$$

$$\leq \Big(T\mathbb{E}\|X\|_\infty^r\Big)^{1-\frac{2}{r}}\left(\mathbb{E}\int_0^T \|f(s)\|_{L^p}^r ds\right)^{\frac{1}{r}}\left(\mathbb{E}\int_0^T \|J(X_n(s)) - J(X(s))\|_{L^q}^r ds\right)^{\frac{1}{r}}.$$

By Lemma 3.2, after choosing a subsequence, the right hand side tends to zero. So, for this subsequence we get $\sup_{0\leq t\leq T} C_n(t) \longrightarrow 0$ in L^1 and hence in probability.
Step 2: We have

$$\begin{aligned}
&\left|\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle g_n(s), (X_n(s))\rangle dW_H(s) - \int_0^t \|X(s)\|_{L^p}^{r-2}\langle g(s), J(X(s))\rangle dW_H(s)\right|\\
&\leq \left|\int_0^t \underbrace{\Big(\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\Big)\langle g_n(s), J(X_n(s))\rangle}_{\varphi_n(s)} dW_H(s)\right|\\
&\quad + \left|\int_0^t \underbrace{\|X(s)\|_{L^p}^{r-2}\langle g_n(s) - g(s), J(X_n(s))\rangle}_{\psi_n(s)} dW_H(s)\right|\\
&\quad + \left|\int_0^t \underbrace{\|X(s)\|_{L^p}^{r-2}\langle g(s), J(X_n(s)) - J(X(s))\rangle}_{\rho_n(s)} dW_H(s)\right|\\
&= D_n(t) + E_n(t) + F_n(t).
\end{aligned}$$

Since the γ-radonifying norm and the operator norm are equal in finite dimensional spaces, By B.D.G inequality (Theorem 2.3) for $b = 1$ and the Hölder inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq t\leq T} D_n(t)\Big] \leq C\ \mathbb{E}\big\|\varphi_n\big\|_{\gamma(L^2(0,T;H),\mathbb{R})} = C\ \mathbb{E}\big\|\varphi_n\big\|\\
&= C\ \mathbb{E}\sup_{\|f\|\leq 1}\left(\int_0^T \big|\varphi_n(s)f(s)\big| ds\right)\\
&\leq C\mathbb{E}\sup_{\|f\|\leq 1}\left(\int_0^T \|\varphi_n(s)\|^2 ds\right)^{1/2}\|f\|_{L^2(0,T;H)}\\
&\leq C\ \mathbb{E}\bigg[\int_0^T \big|\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\big|^2\big\|X_n(s)\big\|_{L^p}^2\big\|g_n(s)\big\|_{\gamma(H,L^p)}^2 ds\bigg]^{1/2}\\
&\leq C\Big[\mathbb{E}\|X_n\|_\infty^2\big(\|X_n\|_\infty^{r-2} + \|X\|_\infty^{r-2}\big)\Big]^{1/2}\times\\
&\qquad \bigg[\mathbb{E}\int_0^T \big|\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\big|\|g(s)\|_{\gamma(H,L^p)}^2 ds\bigg]^{1/2}\\
&\leq C\Big[\mathbb{E}\|X_n\|_\infty^r + \big(\mathbb{E}\|X_n\|_\infty^r\big)^{\frac{2}{r}}\big(\mathbb{E}\|X\|_\infty^r\big)^{\frac{r-2}{r}}\Big]^{1/2}\times\\
&\quad \bigg[\mathbb{E}\int_0^T \big|\|X_n(s)\|_{L^p}^r - \|X(s)\|_{L^p}^r\big| ds\bigg]^{\frac{r-2}{2r}}\bigg[\mathbb{E}\int_0^T \|g(s)\|_{\gamma(H,L^p)}^r\bigg]^{\frac{1}{r}},
\end{aligned}$$

where C is the same constant as in (2.4). Since $\mathbb{E}\|X_n\|_\infty^r$, $\mathbb{E}\|X\|_\infty^r$ are bounded by a constant independent of n, Hypothesis 3.1(c) and Lemma 3.1 imply that the right hand side tends to zero along some subsequence. So, after choosing a subsequence if necessary, we get $\sup\limits_{0\le t\le T} D_n(t) \longrightarrow 0$ in L^1. Similarly, one can see that

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T} E_n(t)\Big] \quad &\le C\ \mathbb{E}\|\psi_n\|_{\gamma(L^2(0,T;H),\mathbb{R})} \le C\ \mathbb{E}\bigg(\int_0^T \|\psi_n(s)\|^2 ds\bigg)^{1/2}\\
&\le C\ \mathbb{E}\bigg[\int_0^T \|X(s)\|_{L^p}^{2r-2}\|g_n(s)-g(s)\|^2_{\gamma(H,L^p)}ds\bigg]^{1/2}\\
&\le C\ \mathbb{E}\bigg[\|X\|_\infty^{r-1}\Big(\int_0^T \|g_n(s)-g(s)\|^2_{\gamma(H,L^p)}ds\Big)^{1/2}\ \bigg]\\
&\le CT^{\frac{r-2}{2r}}\Big(\mathbb{E}\|X\|_\infty^r\Big)^{\frac{r-1}{r}}\bigg(\mathbb{E}\int_0^T \|g_n(s)-g(s)\|^r_{\gamma(H,L^p)}\bigg)^{\frac{1}{r}}.
\end{aligned}$$

But by the dominated convergence theorem,

$$\mathbb{E}\int_0^T \|g_n(s)-g(s)\|^r_{\gamma(H,L^p)} \longrightarrow 0.$$

Therefore, $\sup\limits_{0\le t\le T} E_n(t) \longrightarrow 0$ in L^1. Also by Hölder's inequality, we find

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T} F_n(t)\Big] \le C\mathbb{E}\bigg(\int_0^T \|\rho_n(s)\|^2 ds\bigg)^{1/2}\\
&\le C\mathbb{E}\bigg[\int_0^T \|X(s)\|_{L^p}^{2r-4}\|J(X_n(s))-J(X(s))\|^2_{L^q}\|g(s)\|^2_{\gamma(H,L^p)}ds\bigg]^{1/2}\\
&\le C\mathbb{E}\Big[\|X\|_\infty^{r-2}\big\|\|J(X_n(\cdot))-J(X(\cdot))\|_{L^q}\big\|_{L^2(0,T)}\big\|\|g(\cdot)\|_{\gamma(H,L^p)}\big\|_{L^2(0,T)}\Big]\\
&\le CK\mathbb{E}\Big[\|X\|_\infty^{r-2}\big\|\|J(X_n(\cdot))-J(X(\cdot))\|_{L^q}\big\|_{L^r(0,T)}\big\|\|g(\cdot)\|_{\gamma(H,L^p)}\big\|_{L^r(0,T)}\Big]\\
&\le CK\big(\mathbb{E}\|X\|_\infty^r\big)^{\frac{r-2}{r}}\times\\
&\qquad\bigg(\mathbb{E}\int_0^T \|J(X_n(s))-J(X(s))\|^r_{L^q}ds\bigg)^{\frac{1}{r}}\bigg(\mathbb{E}\int_0^T \|g(s)\|^r_{\gamma(H,L^p)}ds\bigg)^{\frac{1}{r}},
\end{aligned}$$

where K is a constant. By Lemma 3.2, the right hand side approaches zero after choosing a subsequence. Hence, $\sup_{0\le t\le T} F_n(t) \longrightarrow 0$ in L^1 along some subsequence.

Step 3: By Hölder's inequality,

$$\begin{aligned}
&\sup_{0\le t\le T}\bigg|\int_0^t \|X_n(s)\|_{L^p}^{r-2}\|g(s)\|^2_{\gamma(H,L^p)}ds - \int_0^t \|X(s)\|_{L^p}^{r-2}\|g(s)\|^2_{\gamma(H,L^p)}ds\bigg|\\
&\le \int_0^T \Big|\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\Big|\|g(s)\|^2_{\gamma(H,L^p)}ds
\end{aligned}$$

$$\leq \left[\int_0^T \Big| \|X_n(s)\|_{L^p}^r - \|X(s)\|_{L^p}^r \Big| ds\right]^{\frac{r-2}{r}} \left[\int_0^T \|g(s)\|_{\gamma(H,L^p)}^r ds\right]^{\frac{2}{r}}.$$

Lemma 3.1 implies that by passing to a subsequence, the right hand side tends to zero in L^1. □

4. Measurability of the solutions

In this section, we stablish the existence, uniqueness and measurability of the solution to the integral equation

$$X(t,\omega) = S(t-s)X_0(\omega) + \int_0^t S(t-s)f\big(s,\omega,X(s,\omega)\big)ds + V(t,\omega), \tag{4.1}$$

on $[0,T]$ with $X_0 : \Omega \longrightarrow L^p$. Suppose that $V : [0,T] \times \Omega \longrightarrow L^p$ satisfies the Carathéodory condition; i.e., $V(\cdot,\omega)$ is continuous on $[0,T]$ for each $\omega \in \Omega$ and $V(t,\cdot)$ is measurable on Ω into $(L^p, \mathcal{B})$ for all $t \in [0,T]$, where $\mathcal{B}$ denotes the Borel σ-field of subsets of L^p. Moreover, we assume that $V(0,\omega) = 0$ for all $\omega \in \Omega$. This equation appears in the next section when we use an iterative method to prove the existence of the mild solutions to semilinear SEE's. In fact, existence and measurability of the solution of (4.1) is necessary in each step of iteration. We proceed as in [15] and we use the method based on random fixed point theory.

We say that the mapping $h : [0,T] \times L^p \longrightarrow L^p$ is weakly closed as a Nemytskii operator, if whenever $x_n \rightharpoonup x$ weakly in $L^2(0,T;L^p)$ and $h(\cdot, x_n(\cdot)) \rightharpoonup \xi(\cdot)$ weakly in $L^2(0,T;L^p)$, then $\xi(\cdot) = h(\cdot, x(\cdot))$.

The following are the relevant hypotheses on nonlinear part f of (4.1).

Hypothesis 4.1. *(a) The function $f : [0,T] \times \Omega \times L^p \longrightarrow L^p$ is jointly measurable.*

(b) For each $\omega \in \Omega$, the mapping $(t,x) \longmapsto f(t,\omega,x)$ is weakly closed as a Nemytskii operator.

(c) There exists a nonnegative measurable function $M : \Omega \longrightarrow \mathbb{R}$ such that for each $t \in [0,T]$ and $\omega \in \Omega$, the function $x \longmapsto f(t,\omega,x)$ is semimonotone with parameter $M(\omega)$; i.e.,

$$\big\langle f(t,\omega,x) - f(t,\omega,y), J(x-y)\big\rangle \leq M(\omega)\|x-y\|_{L^p}^2.$$

(d) There exists a constant C such that $\|f(t,\omega,x)\|_{L^p} \leq C(1 + \|x\|_{L^p})$ for all $t \in [0,T]$, $\omega \in \Omega$ and $x \in L^p$.

We first consider (4.1) in finite dimensions. It is well-known that the space L^p has a Schauder basis (see, e.g., [9]); i.e., there exists a sequence $(a_n, x_n)_{n\geq 0}$ in $(L^p)^* \times L^p$ such that

$$x = \sum_{n=1}^{\infty} \langle x, a_n \rangle x_n \ ,$$

for all $x \in L^p$. Let $N \in \mathbb{N}$ and $E_N = \text{span}\{x_1, x_2, ..., x_N\}$. Then $\{E_N\}_{N=1}^{\infty}$ is an increasing sequence of finite dimensional subspaces of L^p such that $\bigcup_{N=1}^{\infty} E_N$ is dense in L^p. We recall that the natural projection $P_N : L^p \longrightarrow E_N$ is defined by

$$P_N(x) = \sum_{n=1}^{N} \langle x, a_n \rangle x_n \ .$$

Theorem 4.2. *If we substitute L^p by E_N, then under Hypothesis 4.1, the integral equation*

$$X_0 = 0, \ X(t,\omega) = \int_0^t f(s,\omega,X(s,\omega))ds \tag{4.2}$$

has a unique measurable solution.

Before proceeding in the proof, we recall two results. First we give the following simple but useful lamma, the proof of which is similar to that of Lemma 2 of [36].

Lemma 4.1. *If $a(\cdot)$ is an L^p-valued integrable function on $[0,T]$, $x \in L^p$ and $X(t) = x + \int_0^t a(s)ds$, then*

$$\|X(t)\|_{L^p}^2 = \|x\|_{L^p}^2 + 2\int_0^t \langle a(s), J(X(s))\rangle ds.$$

Theorem 4.3. [12] *Let K be a closed, convex and separable subset of a Banach space. Then any continuous compact random operator $h : \Omega \times K \longrightarrow K$ has a random fixed point.*

Proof of Theorem 4.2: Let

$$\mathcal{K} = \Big\{ x \in C\big([0,T], E_N\big) \Big| \ \|x(t)\| \leq e^{Ct} - 1 \quad for \ \ all \ t \in [0,T] \Big\},$$

where C is the constant appeared in Hypothesis 4.1. Define h on $\Omega \times \mathcal{K} \longrightarrow E_N$ by

$$h(\omega, x)(t) = \int_0^t f(s,\omega,x(s))ds\cdot$$

Then $\mathcal{K}$ is the closed and convex subset of the separable Banach space $C\big([0,T], E_N\big)$. Hypothesis 4.1(d) shows that h is a map into $\mathcal{K}$ and by Hypothesis 4.1(a), for each $x \in \mathcal{K}$, $h(\cdot, x)$ is measurable. Now fix $\omega \in \Omega$. We show that $h(\omega, \cdot)$ is a continuous and compact operator on $\mathcal{K}$. Let $(x_n) \subseteq \mathcal{K}$ be a sequence strongly convergent to x; i.e.,

$$\sup_{0\le t\le T} \|x_n(t) - x(t)\|_{E_N} \longrightarrow 0.$$

Then, there exists $M > 0$ such that

$$\sup_{0\le t\le T} \|x_n(t)\| \le M \qquad for \;\; all \;\; n \in \mathbb{N}.$$

Consider an arbitrary subsequence of $\{x_n\}$ which we denote it by the same symbol $\{x_n\}$. From Hypothesis 4.1 it follows that $f(\cdot,\omega,x_n(\cdot))$ is a bounded sequence in $L^2(0,T;E_N)$ and so it has a subsequence $f(\cdot,\omega,x_{n_k}(\cdot))$ which is weakly convergent in $L^2(0,T;E_N)$. Therefore, by Hypothesis 4.1(b) $f(\cdot,\omega,x_{n_k}(\cdot)) \rightharpoonup f(\cdot,\omega,x(\cdot))$ weakly in $L^2(0,T;E_N)$. Hence, the whole sequence $f(\cdot,\omega,x_n(\cdot))$ is in fact weakly convergent to $f(\cdot,\omega,x(\cdot))$ in $L^2(0,T;E_N)$. For each $t \in [0,T]$, since $f(t,\omega,x_n(t)) \rightharpoonup f(t,\omega,x(t))$ weakly in E_N and E_N is finite dimensional, $f(t,\omega,x_n(t)) \longrightarrow f(t,\omega,x(t))$ strongly in E_N. Now, by Hypothesis 4.1(d) and the dominated convergence theorem, we have

$$\sup_{0\le t\le T} \big\|h(\omega,x_n)(t) - h(\omega,x)(t)\big\| \le \int_0^T \big\|f(s,\omega,x_n(s)) - f(s,\omega,x(s))\big\|_{E_N} ds,$$

the right-hand side of which goes to zero as $n \to \infty$. Thus, $h(\omega,\cdot)$ is continuous. To prove the compactness of h, we note first that for each $x \in \mathcal{K}$ and all $t \in [0,T]$,

$$\big\|h(\omega,x)(t)\big\| \le \int_0^t \big\|f(s,\omega,x(s))\big\| ds \le C \int_0^t \big(1 + \|x(s)\|\big) ds \le (e^{Ct} - 1).$$

Hence, $h(\omega,\cdot)$ is uniformly bounded. Moreover, for $0 \le t_1 < t_2 \le T$ and $x \in \mathcal{K}$ we have

$$\big\|h(\omega,x)(t_2) - h(\omega,x)(t_1)\big\| \le \int_{t_1}^{t_2} \big\|f(s,\omega,x(s))\big\| ds \le (t_2 - t_1)(e^{Ct_2} - e^{Ct_1}).$$

So, $h(\omega,\cdot)$ is an equicontinuous family on $[0,T]$. Therefore, $h(\omega,\cdot)$ is a compact operator. Now by Theorem 4.3, there exists a measurable function $\xi : \Omega \longrightarrow K$ such that

$$\xi(\omega)(t) = \int_0^t f(s,\omega,\xi(\omega)(s)) ds, \qquad \forall t \in [0,T]\cdot$$

According to Proposition 5.1 of [15], if we define $X : [0,T]\times\Omega \longrightarrow E_N$ by $X(t,\omega) = \xi(\omega)(t)$, then X is jointly measurable and X is a solution of problem (4.2). It remains to show the uniqueness of the solution. Let X and Y be two solutions of (4.2). We have

$$X(t,\omega) - Y(t,\omega) = \int_0^t \Big(f\big(s,\omega,X(s,\omega)\big) - f\big(s,\omega,Y(s,\omega)\big)\Big) ds.$$

By Lemma 4.1 and Hypothesis 4.1(c), for each $\omega \in \Omega$ we obtain

$$\begin{aligned}
&\big\|X(t,\omega)-Y(t,\omega)\big\|^2\\
&=2\int_0^t \big\langle f(s,\omega,X(s,\omega))-f(s,\omega,Y(s,\omega)),J(X(s,\omega)-Y(s,\omega))\big\rangle ds\\
&\le 2M(\omega)\int_0^t \big\|X(s,\omega)-Y(s,\omega)\big\|^2 ds.
\end{aligned}$$

Hence,

$$\mathbb{E}\Big(\sup_{0\le s\le t}\Big\|X(s,\omega)-Y(s,\omega)\Big\|^2\Big)\le 2M(\omega)\int_0^t \mathbb{E}\Big(\sup_{0\le \theta\le s}\Big\|X(\theta,\omega)-Y(\theta,\omega)\Big\|^2\Big)ds.$$

Thus, by the Gronwall inequality

$$\mathbb{E}\Big(\sup_{0\le s\le t}\big\|X(s,\omega)-Y(s,\omega)\big\|^2\Big)=0 \qquad \forall t\in[0,T];$$

that is, $X=Y$.

Theorem 4.4. *Assume that f satisfies Hypothesis 4.1. Then the equation*

$$X(0)=0,\quad X(t,\omega)=\int_0^t f(s,\omega,X(s,\omega))ds$$

has a unique measurable solution.

Proof. The uniqueness follows as in the proof of previous theorem. Let $P_n : L^p \longrightarrow E_n$ be the natural projection of L^p onto E_n. By Theorem 4.2, for each $n \in \mathbb{N}$ and $\omega \in \Omega$, the equation

$$X_0=0,\quad X(t,\omega)=\int_0^t P_n f(s,\omega,X(s,\omega))ds$$

has a unique measurable solution $X_n(t,\omega)$. Due to Lemma 4.1 and Hypothesis 4.1(c), we obtain

$$\begin{aligned}
&\big\|X_n(t,\omega)\big\|_{L^p}^2 = 2\int_0^t \big\langle P_n f(s,\omega,X_n(s,\omega)),J(X_n(s,\omega))\big\rangle ds\\
&=2\int_0^t \big\langle P_n f(s,\omega,X_n(s,\omega))-P_n f(s,\omega,0),J(X_n(s,\omega))\big\rangle ds\\
&\quad+2\int_0^t \big\langle P_n f(s,\omega,0),J(X_n(s,\omega))\big\rangle ds\\
&\le 2M(\omega)\int_0^t \big\|X_n(s,\omega)\big\|_{L^p}^2 ds+2\int_0^t \big\|f(s,\omega,0)\big\|_{L^p}\big\|X_n(s,\omega)\big\|_{L^p} ds\\
&\le (2M(\omega)+1)\int_0^t \big\|X_n(s,\omega)\big\|_{L^p}^2 ds+\int_0^t \big\|f(s,\omega,0)\big\|_{L^p}^2 ds.
\end{aligned}$$

So, by the Gronwall inequality,

$$\sup_{0\le t\le T}\big\|X_n(t,\omega)\big\|_{L^p} \le e^{(2M(\omega)+1)T}\int_0^T \big\|f(s,\omega,0)\big\|_{L^p}^2 ds \le Te^{(2M(\omega)+1)T}.$$

Now fix $\omega\in\Omega$. The above inequality shows that $\{X_n(\cdot,\omega)\}$ is a bounded sequence in $L^2(0,T;L^p)$. Also by Hypothesis 4.1(c), the sequence $f(\cdot,\omega,X_n(\cdot,\omega))$ is bounded in $L^2(0,T;L^p)$. Therefore, there exists a subsequence, again denoted by $\big(X_n(\cdot,\omega)\big)$, such that $\big(X_n(\cdot,\omega)\big)$ and $f(\cdot,\omega,X_n(\cdot,\omega))$ are both weakly convergent in $L^2(0,T;L^p)$. Let $X(\cdot,\omega)$ be the weak limit of $\big(X_n(\cdot,\omega)\big)$. Then, by Hypothesis 4.1(b), $f(\cdot,\omega,X(\cdot,\omega))$ is the weak limit of $f(\cdot,\omega,X_n(\cdot,\omega))$. Since L^p is a reflexive Banach space, by Theorem 3.2.13 of [1], (a_n,x_n) is a shrinking basis; that is, for each $v\in L^q$, $\|P_n^*v-v\|\longrightarrow 0$ as $n\longrightarrow 0$. So, we have

$$\begin{aligned}
&\big\langle P_nf(t,\omega,X_n(t,\omega)),v\big\rangle\\
&=\big\langle P_nf(t,\omega,X_n(t,\omega))-f(t,\omega,X_n(t,\omega)),v\big\rangle+\big\langle f(t,\omega,X_n(t,\omega)),v\big\rangle\\
&=\big\langle f(t,\omega,X_n(t,\omega)),P_n^*v-v\big\rangle+\big\langle f(t,\omega,X_n(t,\omega)),v\big\rangle\\
&\ \longrightarrow\big\langle f(t,\omega,X(t,\omega)),v\big\rangle,\qquad as\quad n\to\infty.
\end{aligned}$$

Thus,

$$\big\langle X_n(t,\omega),v\big\rangle=\int_0^t\big\langle P_nf(t,\omega,X_n(s,\omega)),v\big\rangle ds\longrightarrow\Big\langle\int_0^t f(s,\omega,X(s,\omega))ds,v\Big\rangle,$$

and hence,

$$\big\langle X(t,\omega),v\big\rangle=\Big\langle\int_0^t f(s,\omega,X(s,\omega))ds,v\Big\rangle\cdot$$

Therefore,

$$X(t,\omega)=\int_0^t f(s,\omega,X(s,\omega))ds\cdot$$

It remains to show that $X(\cdot,\cdot)$ is measurable on $[0,T]\times\Omega$. For arbitrary $v\in L^q$, we see that

$$\int_0^t\big\langle X_n(s,\omega),v\big\rangle ds\longrightarrow\int_0^t\big\langle X(s,\omega),v\big\rangle ds,$$

and the function $(t,\omega)\longmapsto\int_0^t\big\langle X_n(s,\omega),v\big\rangle ds$ is measurable. Hence, the function $(t,\omega)\longmapsto\int_0^t\big\langle X(s,\omega),v\big\rangle ds$ is also measurable and $\big\langle X(\cdot,\omega),v\big\rangle$ is continuous. So, we can differentiate and obtain

$$\big\langle X(t,\omega),v\big\rangle=\frac{d}{dt}\int_0^t\big\langle X(s,\omega),v\big\rangle ds,$$

which shows that $\big\langle X(t,\omega),v\big\rangle$ is measurable in $(t,\omega)\in[0,T]\times\Omega$. By separability of L^p, this implies the measurability of $X(\cdot,\cdot)$ on $[0,T]\times\Omega$. $\square$

Now, we are ready to state and prove our main result in this section.

Theorem 4.5. *Assume that X_0 is an L^p-valued random variable and $\big(S(t)\big)$ is a C_0-semigroup on L^p satisfying an exponential growth condition with generator A. Let V satisfies the Carathéodory condition and $V(0,\omega)=0$ for all $\omega\in\Omega$. Furthermore, let Hypothesis 4.1 holds. Then (4.1) has a unique measurable solution.*

Proof. One can easily see that it suffices to prove Theorem 4.5 in the case that $\lambda=0$, $X_0=0$ and $V=0$.
Uniqueness. Assume that X and Y are two solutions of (4.1) and fix any $\omega\in\Omega$. Then using the Itô type inequality (Theorem 3.2) with $g=0$ and $r=2$, and Hypothesis 4.1(c), we obtain

$$\begin{aligned}
&\big\|X(t,\omega)-Y(t,\omega)\big\|^2_{L^p}\\
&\le 2\int_0^t \big\langle f(s,\omega,X(s,\omega))-f(s,\omega,Y(s,\omega)),J(X(s,\omega)-Y(s,\omega))\big\rangle ds\\
&\le 2M(\omega)\int_0^t \big\|X(s,\omega)-Y(s,\omega)\big\|^2_{L^p}ds.
\end{aligned}$$

So,

$$\mathbb{E}\bigg(\sup_{0\le s\le t}\Big\|X(s,\omega)-Y(s,\omega)\Big\|^2_{L^p}\bigg)\le 2M(\omega)\int_0^t\mathbb{E}\bigg(\sup_{0\le\theta\le s}\Big\|X(\theta,\omega)-Y(\theta,\omega)\Big\|^2_{L^p}\bigg)ds.$$

Hence, by the Gronwall inequality, we conclude that $X=Y$.
Existence. Consider the Yosida approximations

$$R_n:=n(nI-A)^{-1}:L^p\longrightarrow D(A),\quad A_n:=AR_n$$

and $f_n(t,\omega,x)=A_nx+f(t,\omega,x)$. First, let us show that f_n satisfies Hypothesis 4.1. It is clear that f_n is jointly measurable. Moreover, A_n is continuous and so weakly closed. Hence f_n is weakly closed as a Nemytskii operator. Since L^p is a reflexive and strictly convex Banach space and A is maximal monotone, A_n is a monotone operator. Thus, for all $x,y\in L^p$ we have

$$\begin{aligned}
\big\langle f_n(t,\omega,x)-f_n(t,\omega,y),J(x-y)\big\rangle &= \underbrace{\big\langle A_nx-A_ny,J(x-y)\big\rangle}_{\le 0}\\
&\quad+\big\langle f(t,\omega,x)-f(t,\omega,y),J(x-y)\big\rangle\\
&\le M(\omega)\|x-y\|^2_{L^p}
\end{aligned}$$

So, $f_n(\omega)$ is semimonotone. Note also that $\|A_n\|\le n$ and therefore,

$$\big\|f_n(t,\omega,x)\big\|_{L^p}\le n\|x\|_{L^p}+C(1+\|x\|_{L^p})\le(n+C)(1+\|x\|_{L^p})\cdot$$

Now, by Theorem 4.4, for each $n \in \mathbb{N}$ the integral equation

$$X_0 = 0, \quad X(t,\omega) = \int_0^t f_n(s,\omega,X(s,\omega))ds,$$

has a unique measurable solution $X_n(t,\omega)$. According to Lemma 4.1, Hypothesis 4.1(c) and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\big\|X_n(t,\omega)\big\|_{L^p}^2 = 2\int_0^t & \underbrace{\big\langle A_n X_n(s,\omega), J(X_n(s,\omega))\big\rangle}_{\leq 0} ds \\
&+ 2\int_0^t \big\langle f(s,\omega,X_n(s,\omega)), J(X_n(s,\omega))\big\rangle ds \\
\leq 2M(\omega)&\int_0^t \Big\|X_n(s,\omega)\Big\|_{L^p}^2 ds + 2\int_0^t \Big\|f(s,\omega,0)\Big\|_{L^p} \Big\|X_n(s,\omega)\Big\|_{L^p} ds \\
\leq (2M(\omega)+1)&\int_0^t \big\|X_n(s,\omega)\big\|_{L^p}^2 ds + \int_0^t \big\|f(s,\omega,0)\big\|_{L^p}^2 ds.
\end{aligned}$$

Thus, by Gronwall's inequality

$$\sup_{0\leq t\leq T} \big\|X_n(t,\omega)\big\|_{L^p}^2 \leq e^{(2M(\omega)+1)T} \int_0^T \big\|f(s,\omega,0)\big\|_{L^p}^2 ds\cdot \tag{4.3}$$

On the other hand, A_n is bounded and generates the uniformly continuous contraction semigroup $\big(S_n(t)\big)$. We claim that for each $x \in L^p$, $S_n(t)x \longrightarrow S(t)x$. Since $D(A)$ is dense in L^p, it suffices to prove this for $x \in D(A)$. For $x \in D(A)$, we have

$$\begin{aligned}
S(t)x - S_n(t)x \qquad &= \int_0^t \frac{d}{d\theta}\big(S_n(t-\theta)S(\theta)x\big)d\theta \\
&= \int_0^t S_n(t-\theta)[A_n - A]S(\theta)x d\theta,
\end{aligned}$$

and

$$\big\|S_n(t-\theta)[A_n - A]S(\theta)x\big\| \leq \big\|(A_n - A)S(\theta)x\big\|.$$

Also,

$$(A_n - A)S(\theta)x = \big[(I - n^{-1}A)^{-1} - I\big]AS(\theta)x = (R_n - I)AS(\theta)x \longrightarrow 0,$$

$\theta \longmapsto AS(\theta)x$ is continuous and so bounded on $[0,T]$, and $\|R_n - I\| \leq 2$. Hence, by the dominated convergence theorem, $S_n(t)x \longrightarrow S(t)x$. We claim that

$$X_n(t,\omega) = \int_0^t S_n(t-s)f(s,\omega,X_n(s,\omega))ds\cdot \tag{4.4}$$

In fact, for any fixed $\omega \in \Omega$, by Theorem 2.38 of [7], the problem

$$Y(0,\omega)=0, \quad \frac{dY}{dt}=A_nY(t,\omega)+f(t,\omega,X_n(t,\omega)),$$

has a unique solution

$$Y(t,\omega)=\int_0^t S_n(t-s)f(s,\omega,X_n(s,\omega))ds.$$

That is,

$$Y(t,\omega)=\int_0^t A_nY(s,\omega)ds+\int_0^t f(s,\omega,X_n(s,\omega))ds.$$

Thus,

$$X_n(t,\omega)-Y(t,\omega)=\int_0^t \big(A_nX_n(s,\omega)-A_nY(s,\omega)\big)ds.$$

and so, by Lemma 4.1 and monotonicity of A_n,

$$\Big\|X_n(t,\omega)-Y(t,\omega)\Big\|_{L^p}^2=2\int_0^t\Big\langle A_nX_n(s,\omega)-A_nY(s,\omega),J(X_n(s,\omega)-Y(s,\omega))\Big\rangle ds\le 0\cdot$$

Hence, $\sup_{0\le t\le T}\big\|X_n(t,\omega)-Y(t,\omega)\big\|_{L^p}^2=0$; i.e, $X_n=Y$ and we obtain (4.4). Now, we are going to use the method of the proof of Theorem 4.4. Fix $\omega\in\Omega$. From (4.3) and Hypothesis 4.1 (b) and (d), it follows that there exists a subsequence $\big(X_{n_k}(\cdot,\omega)\big)$ and an element $X(\cdot,\omega)\in L^2(0,T;L^p)$, such that $\big(X_{n_k}(\cdot,\omega)\big)$ and $f(\cdot,\omega,X_{n_k}(\cdot,\omega))$ are respectively weakly convergent to $X(\cdot,\omega)$ and $f(\cdot,\omega,X(\cdot,\omega))$ in $L^2(0,T;L^p)$. So, for each $v\in L^q$ we have as $n\to\infty$ that

$$\begin{aligned}&\big\langle S_n(t-s)f(s,\omega,X_n(s,\omega)),v\big\rangle\\&=\big\langle S_n(t-s)f(s,\omega,X_n(s,\omega))-S(t-s)f(s,\omega,X_n(s,\omega)),v\big\rangle\\&\ +\big\langle S(t-s)f(s,\omega,X_n(s,\omega)),v\big\rangle\longrightarrow\big\langle S(t-s)f(s,\omega,X(s,\omega)),v\big\rangle\end{aligned}$$

and thus,

$$\big\langle X_n(t,\omega),v\big\rangle=\int_0^t\big\langle S_n(t-s)f(s,\omega,X_n(s,\omega)),v\big\rangle ds$$

tends to $\int_0^t\big\langle S(t-s)f(s,\omega,X(s,\omega)),v\big\rangle ds$ as $n\to\infty$. Hence,

$$\big\langle X(t,\omega),v\big\rangle=\big\langle\int_0^t S(t-s)f(s,\omega,X(s,\omega))ds,v\big\rangle,$$

and therefore,

$$X(t,\omega)=\int_0^t S(t-s)f(s,\omega,X(s,\omega))ds\cdot$$

This finishes the proof of the existence.

Measurability. Similar to the proof of Theorem 4.4, one can see that $X(\cdot,\cdot)$ is measurable on $[0,T]\times\Omega$. □

5. Existence and uniqueness of mild solutions

In this section, we use semigroup theory to make an iterative method in order to prove the existence and uniqueness of the mild solutions of monotone-type semilinear SEE's. The Itô type inequality (Theorem 3.2) is a key tool to study both existence and uniqueness. Consider the following semilinear stochastic evolution equation on L^p ($p \geq 2$):

$$\begin{cases} dX(t) = AX(t)dt + f(t, X(t))dt + g(t, X(t))dW_H(t), \ t \in [0, T] \\ X(0) = X_0, \end{cases} \tag{5.1}$$

where the initial data X_0 is an L^p-valued $\mathcal{F}_0$-measurable random variable and $\mathbb{E}\|X_0\|_{L^p}^r < \infty$. Our hypotheses on A, g and nonlinear part f are as follows.

Hypothesis 5.1. *(a) $A : D(A) \subseteq L^p \longrightarrow L^p$ is the generator of a C_0-semigroup $\big(S(t)\big)_{t\geq 0}$ of linear operators satisfying an exponential growth condition; i.e., there exists $\lambda \geq 0$ such that*

$$\|S(t)\| \leq e^{\lambda t} \qquad \forall t \geq 0\cdot$$

(b) f satisfies Hypothesis 4.1 with the constant M which is independent of $\omega \in \Omega$.

(c) $g : [0, T] \times \Omega \times L^p \longrightarrow \gamma(H, L^p)$ is a progressively measurable process such that for all $t \in [0, T]$, $\omega \in \Omega$ and $x, y \in L^p$

$$\big\|g(t, \omega, x) - g(t, \omega, y)\big\|_{\gamma(H,L^p)} \leq C\|x - y\|_{L^p},$$

where C is the constant appeared in Hypothesis 4.1(d). Moreover,

$$\mathbb{E}\Big(\sup_{0\leq s\leq t} \|g(s, 0)\|^r_{\gamma(H,L^p)}\Big) < \infty, \quad \forall t \in [0, T]\cdot$$

Definition 5.2. *An adapted process $X : [0, T] \times \Omega \longrightarrow L^p$ is called a mild solution of (5.1) if it satisfies the integral equation*

$$X(t) = S(t)X_0 + \int_0^t S(t - s)f(s, X(s))ds + \int_0^t S(t - s)g(s, X(s))dW_H(s). \tag{5.2}$$

Theorem 5.3. *If Hypothesis 5.1 holds, then (5.1) has a unique continuous mild solution X such that*

$$\mathbb{E}\Big(\sup_{0\leq s\leq t} \|X(s)\|^r_{L^p}\Big) < \infty, \qquad r \geq 2, t \in [0, T]\cdot$$

Proof. One can easily see that it suffices to prove the theorem in the case that X_0 and λ are zero.

Uniqueness: Let $X(t)$ and $Y(t)$ be two continuous mild solutions of (5.1) with initial data $X(0) = Y(0) = 0$. Then we have

$$X(t) - Y(t) = \int_0^t S(t-s)\big(f(s,X(s)) - f(s,Y(s))\big)ds + \int_0^t S(t-s)\big(g(s,X(s)) - g(s,Y(s))\big)dW_H(s).$$

We can apply Itô-type inequality (Theorem 3.2) with $r = 2$ and find that

$$\begin{aligned}\big\|X(t) - Y(t)\big\|^2_{L^p} \quad \leq \quad & 2\int_0^t \big\langle f(s,X(s)) - f(s,Y(s)), J(X(s) - Y(s))\big\rangle ds \\ + \quad & 2\int_0^t \big\langle g(s,X(s)) - g(s,Y(s)), J(X(s) - Y(s))\big\rangle dW_H(s) \\ + \quad & \int_0^t \big\|g(s,X(s)) - g(s,Y(s))\big\|^2_{\gamma(H,L^p)} ds.\end{aligned}$$

By Hypothesis 5.1 (b) and (c), we have

$$\int_0^t \Big\langle f(s,X(s)) - f(s,Y(s)), J(X(s) - Y(s))\Big\rangle ds \leq M\int_0^t \Big\|X(s) - Y(s)\Big\|^2_{L^p} ds, \tag{5.3}$$

and

$$\int_0^t \Big\|g(s,X(s)) - g(s,Y(s))\Big\|^2_{\gamma(H,L^p)} ds \leq C^2\int_0^t \Big\|X(s) - Y(s)\Big\|^2_{L^p} ds. \tag{5.4}$$

Also, using B.D.G inequality (Theorem2.3) with $b = 1$, Hypothesis 5.1(c) and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}&\mathbb{E}\sup_{0\leq\rho\leq t}\left|\int_0^\rho \underbrace{\Big\langle g(s,X(s)) - g(s,Y(s)), J(X(s) - Y(s))\Big\rangle}_{\phi(s)} dW_H(s)\right| \\ &\leq C_1\mathbb{E}\Big\|\phi\Big\|_{\gamma(L^2(0,t;H),\mathbb{R})} = C_1\mathbb{E}\Big\|\phi\Big\| = C_1\sup_{\|f\|\leq 1}\Big[\phi, f\Big]_{L^2(0,t;H)} \\ &\leq C_2\mathbb{E}\left(\int_0^t \Big\|\phi(s)\Big\|^2 ds\right)^{1/2} \\ &\leq C_2\mathbb{E}\left[\sup_{0\leq s\leq t}\|X(s) - Y(s)\|_{L^p}\left(\int_0^t \|X(s) - Y(s)\|^2_{L^p} ds\right)^{1/2}\right] \\ &\leq C_2\left[\mathbb{E}\Big(\sup_{0\leq s\leq t}\|X(s) - Y(s)\|^2_{L^p}\Big)\right]^{1/2}\left[\mathbb{E}\int_0^t \|X(s) - Y(s)\|^2_{L^p} ds\right]^{1/2} \\ &\leq \frac{1}{4}\mathbb{E}\left(\sup_{0\leq s\leq t}\|X(s) - Y(s)\|^2_{L^p}\right) + 2{C_2}^2\ \mathbb{E}\int_0^t \|X(s) - Y(s)\|^2_{L^p} ds.\end{aligned} \tag{5.5}$$

Here, C_1 is the constant appeared in inequality 2.4 and we have used the inequality $ab \leq \frac{1}{2}(\frac{1}{k}a^2 + kb^2)$ for any $a, b \in \mathbb{R}$ and any $k > 0$, with $k = 2C_2$. From (5.3), (5.4) and (5.5) we obtain

$$\frac{1}{2}\mathbb{E}\Big(\sup_{0\le s\le t}\|X(s)-Y(s)\|^2_{L^p}\Big)\le A\int_0^t \mathbb{E}\Big(\sup_{0\le \theta\le s}\|X(\theta)-Y(\theta)\|^2_{L^p}\Big)ds,$$

where $A=2M+C^2+4C_2^2$. Hence, by the Gronwall inequality

$$\mathbb{E}\Big(\sup_{0\le s\le t}\|X(s)-Y(s)\|^2_{L^p}\Big)=0 \quad for\ all\ t\in[0,T].$$

So, $X=Y$ on $[0,T]$ almost surly.
Existence: Let $X_1(t)=0$ and define $X_n(t)$ by induction. Assume $X_n(t)$ is defined. Theorem 4.5 implies that there exists a continuous adapted solution X_{n+1} of

$$X_{n+1}(t)=\int_0^t S(t-s)f(s,X_{n+1}(s))ds+V_n(t),$$

where

$$V_n(t)=\int_0^t S(t-s)g(s,X_n(s))dW_H(s)\cdot$$

We claim that

$$\mathbb{E}\Big(\sup_{0\le s\le t}\|X_n(s)\|^r_{L^p}\Big)<\infty,\quad \forall n\in\mathbb{N},\ \forall t\in[0,T], \tag{5.6}$$

the proof of which is by induction on n. By Hypothesis 5.1(b),

$$\|X_{n+1}(t)\|^2_{L^p}\le 4C^2\int_0^t \big(1+\|X_{n+1}(s)\|^2_{L^p}\big)ds+2\|V_n(t)\|^2_{L^p}.$$

Hence,

$$\sup_{0\le s\le t}\|X_{n+1}(s)\|^2_{L^p}\le 4C^2t+4C^2\int_0^t\sup_{0\le\theta\le s}\|X_{n+1}(\theta)\|^2_{L^p}ds+2\sup_{0\le s\le t}\|V_n(s)\|^2_{L^p}.$$

So, by Gronwall's inequality we obtain

$$\sup_{0\le s\le t}\|X_{n+1}(s)\|^2_{L^p}\le\Big[4C^2t+2\sup_{0\le s\le t}\|V_n(s)\|^2_{L^p}\Big]e^{4C^2t},$$

and thus,

$$\sup_{0\le s\le t}\|X_{n+1}(s)\|^r_{L^p}\le 2^{r/2}\Big[(4C^2t)^{r/2}+2^{r/2}\sup_{0\le s\le t}\|V_n(s)\|^r_{L^p}\Big]e^{2rC^2t}.$$

Therefore, to get (5.6) it suffices to prove that

$$\mathbb{E}\sup_{0\le s\le t}\|V_n(s)\|^r_{L^p}<\infty\cdot$$

By Theorem 2.2 there exists a constant K such that

$$\mathbb{E}\sup_{0\le s\le t}\|V_n(s)\|^r_{L^p}\le K^r\mathbb{E}\bigg[\int_0^T \big\|g(t,X_n(t))\big\|^2_{\gamma(H,L^p)}dt\bigg]^{r/2}.$$

By Hypothesis 5.1(c) and Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}\|V_n(s)\|^r_{L^p} &\le K^r\mathbb{E}\bigg[2C\int_0^T\|X_n(t)\|^2_{L^p}dt+2\int_0^T\|g(t,0)\|^2_{\gamma(H,L^p)}dt\bigg]^{r/2}\\ &\le 2^rK^r\bigg[C^{r/2}T\,\mathbb{E}\sup_{0\le t\le T}\|X_n(t)\|^r_{L^p}+\int_0^T\mathbb{E}\|g(t,0)\|^r_{\gamma(H,L^p)}dt\bigg],\end{aligned}$$

which is finite by induction. Next, we are going to prove the convergence of sequence $\{X_n\}$ to a mild solution of (5.1). Note that

$$\begin{aligned}X_{n+1}(t)-X_n(t) &= \int_0^t S(t-s)\big(f(s,X_{n+1}(s))-f(s,X_n(s))\big)ds\\ &+ \int_0^t S(t-s)\big(g(s,X_n(s))-g(s,X_{n-1}(s))\big)dW_H(s).\end{aligned}$$

Therefore, Itô type inequality (Theorem 3.2) implies that

$$\begin{aligned}&\|X_{n+1}(t)-X_n(t)\|^r_{L^p}\le\\ &r\int_0^t\|X_{n+1}(s)-X_n(s)\|^{r-2}_{L^p}\Big\langle f(s,X_{n+1}(s))-f(s,X_n(s)),J(X_{n+1}(s)-X_n(s))\Big\rangle ds\\ &+r\int_0^t\underbrace{\|X_{n+1}(s)-X_n(s)\|^{r-2}_{L^p}\Big\langle g(s,X_n(s))-g(s,X_{n-1}(s)),J(X_{n+1}(s)-X_n(s))\Big\rangle}_{\phi(s)}dW_H(s)\\ &+\frac{r(r-1)}{2}\int_0^t\|X_{n+1}(s)-X_n(s)\|^{r-2}_{L^p}\Big\|g(s,X_n(s))-g(s,X_{n-1}(s))\Big\|^2_{\gamma(H,L^p)}ds\\ &= A_n(t)+B_n(t)+C_n(t).\end{aligned}\tag{5.7}$$

Using Hypothesis 5.1(b) for the first term, $A_n(t)$, we find

$$A_n(t)\le rM\int_0^t\|X_{n+1}(s)-X_n(s)\|^r_{L^p}ds.\tag{5.8}$$

Moreover, using Theorem 2.2 for the second term, $B_n(t)$, yields us

$$\mathbb{E}\sup_{0\le\theta\le t}|B_n(\theta)|\le rD\mathbb{E}\|\phi\|_{\gamma(L^2(0,t;H),\mathbb{R})},$$

in which D is a constant. By an argument similar to that of the proof of Theorem 3.2 (Step 2), one can see that

$$\mathbb{E}\sup_{0\le\theta\le t}|B_n(\theta)|\le rD\mathbb{E}\bigg(\int_0^t\|\phi(s)\|^2ds\bigg)^{1/2}.$$

From Hypothesis 5.1(c), we obtain that the right hand side is

$$
\begin{aligned}
&\leq rD\mathbb{E}\Big[\int_0^t \big\|X_{n+1}(s)-X_n(s)\big\|_{L^p}^{2r-2}\big\|X_n(s)-X_{n-1}(s)\big\|_{L^p}^{2}ds\Big]^{1/2}\\
&\leq rD\mathbb{E}\Big[\sup_{0\leq s\leq t}\|X_{n+1}(s)-X_n(s)\|_{L^p}^{r/2}\times\\
&\qquad\Big(\int_0^t \|X_{n+1}(s)-X_n(s)\|_{L^p}^{r-2}\|X_n(s)-X_{n-1}(s)\|_{L^p}^{2}ds\Big)^{1/2}\ \Big]
\end{aligned}
$$

Using the elementary inequality $ab \leq \frac{1}{2}(k^{-1}a^2 + kb^2)$ which is true for any $a, b \in \mathbb{R}$ and $k > 0$ with $k = rD$, we obtain

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq\theta\leq t}|B_n(\theta)| &\leq \frac{1}{2}\ \mathbb{E}\sup_{0\leq s\leq t}\big\|X_{n+1}(s)-X_n(s)\big\|_{L^p}^{r}\\
+\quad \frac{r^2D^2}{2}\ &\mathbb{E}\int_0^t \big\|X_{n+1}(s)-X_n(s)\big\|_{L^p}^{r-2}\big\|X_n(s)-X_{n-1}(s)\big\|_{L^p}^{2}ds.
\end{aligned}
$$

Applying the inequality $u^{1-\alpha}v^{\alpha} \leq (1-\alpha)u + \alpha v$ which holds for all $u, v \geq 0$ and $0 \leq \alpha \leq 1$, we deduce that

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq\theta\leq t}|B_n(\theta)| \quad&\leq\quad \frac{1}{2}\mathbb{E}\Big(\sup_{0\leq s\leq t}\big\|X_{n+1}(s)-X_n(s)\big\|_{L^p}^{r}\Big)\\
&+\quad \frac{r(r-2)D^2}{2}\int_0^t \mathbb{E}\Big(\sup_{0\leq\theta\leq s}\big\|X_{n+1}(\theta)-X_n(\theta)\big\|_{L^p}^{r}\Big)ds\\
&+\quad rD^2\int_0^t \mathbb{E}\Big(\sup_{0\leq\theta\leq s}\big\|X_n(\theta)-X_{n-1}(\theta)\big\|_{L^p}^{r}\Big)ds.
\end{aligned}
\tag{5.9}
$$

Similarly, by Hypothesis 5.1(c) one can show that

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq\theta\leq t}|C_n(\theta)| \quad&\leq\quad \frac{(r-1)(r-2)}{2}\ C^2\int_0^t \mathbb{E}\Big(\sup_{0\leq\theta\leq s}\Big\|X_{n+1}(\theta)-X_n(\theta)\Big\|_{L^p}^{r}\Big)ds\\
&+\quad (r-1)C^2\int_0^t \mathbb{E}\Big(\sup_{0\leq\theta\leq s}\Big\|X_n(\theta)-X_{n-1}(\theta)\Big\|_{L^p}^{r}\Big)ds.
\end{aligned}
\tag{5.10}
$$

Now, we define

$$
h_n(t) = \mathbb{E}\Big(\sup_{0\leq s\leq t}\big\|X_{n+1}(s)-X_n(s)\big\|_{L^p}^{r}\Big), \quad t \in [0, T].
$$

Note that $h_n(t) < \infty$ for all $t \in [0, T]$ and hence by substituting (5.8), (5.9) and (5.10) in the right hand side of (5.7), we obtain

$$
h_n(t) \leq \alpha\int_0^t h_n(s)ds + \beta\int_0^t h_{n-1}(s)ds,
$$

where

$$
\alpha = 2rM + r(r-2)D^2 + (r-1)(r-2)C^2,
$$

and

$$\beta = 2rD^2 + 2(r-1)C^2.$$

Therefore, by Gronwall's inequality

$$h_n(t) \leq \beta e^{\alpha t} \int_0^t h_{n-1}(s)ds\cdot$$

We know $h_0(t) \leq h_0(T) = \mathbb{E} \sup_{0\leq s\leq T} \|X_1(s)\|_{L^p}^r < \infty$. Thus, if $\gamma = h_0(T)$ it follows inductively that

$$h_n(t) \leq \gamma \, \frac{(\beta e^{\alpha T} t)^n}{n!}, \quad n \geq 1.$$

Hence, $\{X_n\}$ is a Cauchy sequence in $L^r(\Omega, C(0,T;L^p))$ and so there exists a continuous adapted process $X(t)$ with

$$\mathbb{E}\Big(\sup_{0\leq s\leq t} \big\|X(s)\big\|_{L^p}^r \Big) < \infty,$$

such that $\mathbb{E}\Big(\sup_{0\leq s\leq t} \big\|X_n(s) - X(s)\big\|_{L^p}^r \Big) \longrightarrow 0$. To complete the proof, we show that $X(t)$ is the mild solution of (5.1). Consider

$$R(t) = X(t) - \int_0^t S(t-s)f(s,X(s))ds - \int_0^t S(t-s)g(s,X(s))dW_H(s),$$

and

$$R_n(t) = X_{n+1}(t) - \int_0^t S(t-s)f(s,X_{n+1}(s))ds - \int_0^t S(t-s)g(s,X_n(s))dW_H(s).$$

We know that $R_n(t) = 0$ for all $t \in [0,T]$. Let $x \in L^q$ and $t \in [0,T]$. We show that $\langle R(t), x\rangle = 0$ *a.s.*, which implies $R(t) = 0$ *a.s.* Then letting t ranges over all rational numbers and using continuity of R, it follows that $R(t) = 0$ for all $t \in [0,T]$ w.p.1. First, by passing to a subsequence if necessary, we may assume that

$$\sup_{0\leq s\leq T} \big\|X_n(s) - X(s)\big\|_{L^p} \longrightarrow 0, \quad a.s.$$

Consequently

(5.11) $$\langle X_{n+1}(t), x\rangle \longrightarrow \langle X(t), x\rangle, \quad a.s.$$

Now, since

$$\int_0^T \|X_{n+1}(s) - X(s)\|_{L^p}^2 ds \leq T \sup_{0\leq s\leq T} \|X_{n+1}(s) - X_n(s)\|_{L^p}^2 \longrightarrow 0, \quad a.s.$$

we have

$$X_{n+1} \longrightarrow X \qquad in \;\; L^2(0,T;L^p) \quad a.s.$$

Moreover, by Hypothesis 5.1(b),

$$\begin{aligned}\int_0^T \big\|f(s, X_{n+1}(s))\big\|_{L^p}^2 ds &\le TC\big(1 + \sup_{0\le s\le T} \|X_{n+1}(s)\|_{L^p}\big)^2 \\ &\le TC\big(2 + \sup_{0\le s\le T} \|X(s)\|_{L^p}\big)^2 < \infty,\end{aligned}$$

for large enough n. This shows that $f(\cdot, X_{n+1}(\cdot))$ is a bounded sequence in $L^2(0,T;L^p)$. So, by passing to a subsequence, we may assume that $f(\cdot, X_{n+1}(\cdot))$ is weakly convergence in $L^2(0,T;L^p)$. Hence, it follows from weakly closedness of f (Hypothesis 5.1(b)) that

$$f\big(\cdot, X_{n+1}(\cdot)\big) \rightharpoonup f(\cdot, X(\cdot))$$

weakly in $L^2(0,T;L^p)$. Therefore,

$$\int_0^T \big\langle f(s, X_{n+1}(s)) - f(s, X(s)), v(s)\big\rangle \longrightarrow 0,$$

for all $v \in L^2(0,T;L^p)$. Thus,

$$\begin{aligned}&\int_0^t \Big\langle S(t-s)\Big(f(s, X_{n+1}(s)) - f(s, X(s))\Big), x\Big\rangle ds \\ &= \int_0^T \Big\langle f(s, X_{n+1}(s)) - f(s, X(s)), S^*(t-s)x1_{[0,t]}(s)\Big\rangle ds \longrightarrow 0. \qquad (5.12)\end{aligned}$$

At last, by Theorem 2.2 and Hypothesis 5.1(c), there exist constants K and C such that

$$\begin{aligned}&\mathbb{E}\Big\| \int_0^t S(t-s)\big(g(s, X_n(s)) - g(s, X(s))\big) dW_H(s)\Big\|_{L^p}^r \\ &\le K\mathbb{E}\Big(\int_0^T \big\|g(s, X_n(s)) - g(s, X(s))\big\|_{\gamma(H,L^p)}^2\Big)^{r/2} \\ &\le KC\ \mathbb{E}\Big(\sup_{0\le s\le T} \|X_n(s) - X(s)\|_{L^p}^r\Big) \longrightarrow 0.\end{aligned}$$

Consequently, after choosing a subsequence, we have

$$\Big\langle \int_0^t S(t-s)\big(g(s, X_n(s)) - g(s, X(s))\big) dW_H(s), x\Big\rangle \longrightarrow 0. \qquad (5.13)$$

From (5.11) , (5.12) and (5.13), we get that

$$\langle R(t), x\rangle = \lim_{n\to\infty} \langle R_n(t), x\rangle = 0 \cdot$$

The proof is now complete. □

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