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THE $D_p^q(\Delta^{+r})$-STATISTICAL CONVERGENCE

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Abstract. Let $p(n)$ and $q(n)$ be nondecreasing sequences of positive integers such that $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = \infty$ hold. Firstly, in this paper $D_p^q(\Delta^{+r})$-statistical convergence of $x = (x_n)$ where Δ^{+r} is r-th difference of the sequence $x = (x_n)$ for any $r \in \mathbb{Z}^+$ has been defined whereas the results are given under some restrictions on the sequence $p(n)$ and $q(n)$. Secondly, it has been determined that the sets of sequences A and B of the form $[D_p^q]_\alpha^0$ satisfy $A \subset [D_p^q]_0(\Delta^{+r}) \subset B$ and the sets C and D of the form $[D_p^q]_\alpha$ satisfy $C \leq [D_p^q]_\infty(\Delta^{+r}) \leq D$.
Keywords: $D_p^q(\Delta^{+r})$-statistical convergence; summability methods; Deferred Cesaro mean; sequence space.

1. Introduction and main definitions

One of the main problems of the analysis is to determine the set of convergent sequences of the space with considered method. Over the years, this problem has been examined by taking into consideration different summability methods. In recent years, this kind of works have been gained further momentum especially by using the concept of natural density in positive integers.

The concept of statistical convergence was introduced by [16] and [9] independently in the same year. The notion was associated with summability theory by [2] ,[10], [12, 13] and many others.

In this study, the results from [3] were extended and some new results were obtained using Deferred Cesaro mean defined by [1] in as follows:

$$(D_p^q x)_n := \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k, \tag{1.1}$$

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where $(p(n))$ and $(q(n))$ are sequences of nondecreasing integers satisfying

$$p(n) < q(n) \quad \text{and} \quad \lim_{n\to\infty} q(n) = \infty \ . \tag{1.2}$$

The set of all real valued sequences will be denoted by s and U^+ is denoted by

$$U^+ := \{(u_n) \in s : \ u_n > 0, \text{ for all } n \in \mathbb{N}\}.$$

For any $\alpha = (\alpha_n) \in U^+$ a new set of sequences can be defined as follows:

$$\alpha \circledast E := \left\{ x \in s : \left(\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} \right) \in E \right\}$$

where E is any sequence space. So, we get

$$\alpha \circledast E := \left\{ \begin{array}{lll} [D_p^q]_\alpha^0 & if & E = c_0, \\ [D_p^q]_\alpha^c & if & E = c, \\ [D_p^q]_\alpha & if & E = l_\infty, \\ [D_p^q]_\alpha^t & if & E = l_p. \end{array} \right.$$

where

$$[D_p^q]_\alpha^0 := \left\{ x \in s : \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} = 0 \right\},$$

$$[D_p^q]_\alpha := \left\{ x \in s : \sup_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} < \infty \right\},$$

$$[D_p^q]_\alpha^t := \left\{ x \in s : \sum_{n=1}^{\infty} \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} \right|^t < \infty \right\}, 1 < t < \infty$$

and

$$[D_p^q]_\alpha^c := \left\{ x \in s : \exists L \in \mathbb{R} \text{ such that } \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} = L \right\}.$$

The idea of difference sequence space was defined by [11] and it was generalized by [6]. Later on, [7] improved this idea by considering any sequence space X as follows

$$\Delta^{+r}(X) := \{x = (x_k) : (\Delta^{+r} x_k) \in X\}$$

where $r \in \mathbb{N}$, $\Delta^0 x := (x_k)$, $\Delta^+ x_k := x_k - x_{k+1}$ and $\Delta^{+r} x_k := \sum_{j=0}^{r} (-1)^j \binom{r}{j} x_{k+j}$.

If $x \in \Delta^{+r}(X)$ then there exists one and only one sequence $y = (y_k) \in X$ such that $y_k = \Delta^{+r} x_k$ and

$$x_k = \sum_{j=1}^{k-r} (-1)^r \binom{k-j-1}{r-1} y_j = \sum_{j=1}^{k} (-1)^r \binom{k+r-j-1}{r-1} y_{j-r},$$

where $y_{1-r} = y_{2-r} = ... = y_0 = 0$ for sufficiently large k, for instance $k > 2m$ (see more info in [4], [8]).
We can define following sets of sequences for any $r \geq 1$ as:

$$[D_p^q]_0(\Delta^{+r}) := \left\{ x \in s : \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |\Delta^{+r} x_k| = 0 \right\},$$

$$[D_p^q]_\infty(\Delta^{+r}) := \left\{ x \in s : \sup_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |\Delta^{+r} x_k| < \infty \right\},$$

$$[D_p^q]_t(\Delta^{+r}) := \left\{ x \in s : \sum_{n=1}^{\infty} \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |\Delta^{+r} x_k| \right|^t < \infty \right\}, 1 < t < \infty$$

and

$$[D_p^q]_c(\Delta^{+r}) := \left\{ x \in s : \exists L \in \mathbb{R} \text{ such that } \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |\Delta^{+r} x_k - L| = 0 \right\}.$$

In the case when $q_n = n$ and $p_n = 0$, we will denote the previous sets by $[S]_0(\Delta^{+r})$, $[S]_\infty(\Delta^{+r})$, $[S]_t(\Delta^{+r})$ and $[S]_c(\Delta^{+r})$, respectively.

Now, let us define $[D_p^q]_\alpha(\Delta^{+r})$ -statistical convergence of sequence for any $r \geq 1$:

Definition 1.1. A sequence $x = (x_n)$ is said $[D_p^q]_\alpha(\Delta^{+r})$- statistical convergent to zero if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right| = 0 \tag{1.3}$$

holds. It is denoted by $x_k \to 0([D_p^q S]_\alpha(\Delta^{+r}))$.
The set of $[D_p^q]_\alpha(\Delta^{+r})$-statistical convergent sequence is also denoted by $[D_p^q S]_\alpha(\Delta^{+r})$.
Remark 1. It is clear that for any positive integer r, if

(i) $q(n) = n$ and $p(n) = 0$, then (1.3) coincides with convergence of $\Delta^{+r} x_k$.

(ii) $q(n) = n$ and $p(n) = n - 1$, then (1.3) coincides with $s_\alpha(\Delta^{+r})$, where

$$s_\alpha(\Delta^{+r}) := \{x \in s : \sup_k |\frac{\Delta^{+r} x_k}{\alpha_k}| < \infty\}.$$

(iii) $q(n) = \lambda_n$ and $p(n) = 0$ where λ_n is a strictly increasing sequence of natural numbers such that $\lim_{n\to\infty} \lambda_n = \infty$, then (1.3) coincides with λ-statistical convergence of sequences which is given by [15].

(iv) $q(n) = n$ and $p(n) = n - \lambda_n$ where (λ_n) is a nondecrasing sequence of natural numbers such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$ holds then (1.3) coincides with the $\lambda^{+r}(\mu)$-statistical convergence defined by [3] and with the definition of λ^m-statistcal convergence defined by [5].

(v) $q(n) = k_n$ and $p(n) = k_{n-1}$, where (k_n) is a lacunary sequence of nonnegative integers with $k_n - k_{n-1} \to \infty$ as $n \to \infty$ then $[D_p^q]_\alpha(\Delta^{+r})$- statistical convergence coincides with Δ^r- lacunary statistical convergence defined by [17].

2. Main results

2.1 Comparasion of $[D_p^q]^0_\alpha(\Delta^{+r})$ and $[D_p^q S]_\alpha(\Delta^{+r})$ when $r \geq 1$.

Theorem 2.1. *Let $r \geq 1$ be an integer. Then,*

(a) $[D_p^q]^0_\alpha(\Delta^{+r}) \subset [D_p^q S]_\alpha(\Delta^{+r})$ *holds and this inclusion is proper,*

(b) if $x \in l^\alpha_\infty(\Delta^{+r})$ *then* $[D_p^q S]_\alpha(\Delta^{+r}) \subset [D_p^q]^0_\alpha(\Delta^{+r})$, *where* $l^\alpha_\infty(\Delta^{+r}) := \{x \in s : \sup_k |\frac{\Delta^{+r} x_k}{\alpha_k}| < \infty\}$.

Proof. (a) Let us assume that $x \in [D_p^q]^0_\alpha(\Delta^{+r})$. So, for any $\epsilon > 0$, the following inequality

$$
\begin{aligned}
\frac{1}{q(n)-p(n)} & \sum_{k=p(n)+1}^{q(n)} \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| = \\
&= \frac{1}{q(n)-p(n)}\left[\sum_{\substack{k=p(n)+1\\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\geq\epsilon}}^{q(n)} + \sum_{\substack{k=p(n)+1\\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|<\epsilon}}^{q(n)}\right]\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \\
&\geq \frac{1}{q(n)-p(n)} \sum_{\substack{k=p(n)+1\\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\geq\epsilon}}^{q(n)} \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \\
&\geq \epsilon.\frac{1}{q(n)-p(n)} \sum_{\substack{k=p(n)+1\\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\geq\epsilon}}^{q(n)} 1 \\
&\geq \epsilon.\frac{1}{q(n)-p(n)}\left|\left\{p(n)<k\leq q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| > \epsilon\right\}\right| \geq 0
\end{aligned}
$$

holds. Since $x \in [D_p^q S]_\alpha(\Delta^{+r})$, then desired result is obtained.
The following example shows that this inclusion is proper. To see this, let a sequence $x = (x_n)$ as follows:

$$
\frac{\Delta^{+r}x_k}{\alpha_k} := \begin{cases} k, & q(n)-[|\sqrt{q(n)}|]+1<k\leq q(n), \\ 0, & \text{otherwise.} \end{cases}
$$

If we consider the method $[D_p^q]_\alpha^0(\Delta^{+r})$ for the sequence $p(n)$ satisfying

$$
0 < p(n) \leq q(n) - [|\sqrt{q(n)}|] + 1,
$$

then, for an arbitrary $\epsilon > 0$ we have

$$
\frac{1}{q(n)-p(n)}\left|\left\{q(n)-[|\sqrt{q(n)}|]+1<k\leq q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon\right\}\right| = \frac{[|\sqrt{q(n)}|]}{q(n)-p(n)} \to 0
$$

when $n \to \infty$. This calculation shows that $x \in [D_p^q S]_\alpha(\Delta^{+r})$.
But, it is clear that the sequence

$$
\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|
$$

is not convergent to zero. That is, $x \notin [D_p^q]_\alpha^0(\Delta^{+r})$. (b) Let $x \in l_\infty^\alpha(\Delta^{+r})$. Then, there exists $M > 0$ such that $\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \leq M$ holds for all k. Then, for any $\epsilon > 0$, the following inequality

$$\begin{aligned}
\frac{1}{q(n)-p(n)} & \sum_{k=p(n)+1}^{q(n)} \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| = \\
&= \frac{1}{q(n)-p(n)} \left[\sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon}}^{q(n)} + \sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| < \epsilon}}^{q(n)} \right] \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \\
&\leq \frac{1}{q(n)-p(n)} \left[M. \sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon}}^{q(n)} 1 + \epsilon. \sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| < \epsilon}}^{q(n)} 1 \right] \\
&\leq \frac{M}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon \right\} \right| \\
&+ \frac{\epsilon}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| < \epsilon \right\} \right| \\
&\leq \frac{M}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon \right\} \right| + \epsilon
\end{aligned}$$

holds. By taking limit when $n \to \infty$ in above inequality we obtain $x \in [D_p^q S]_\alpha(\Delta^{+r})$ because of $x \in [D_p^q]_\alpha^0(\Delta^{+r})$. So, proof is complated. □

2.2. Comparasion of $S_\alpha(\Delta^{+r})$ and $[D_p^q S]_\alpha(\Delta^{+r})$ when $r \geq 1$.

Let us denote the set of sequences $x = (x_n)$ by $S_\alpha(\Delta^{+r})$ for any fixed $\alpha \in U^+$ such that

$$\left(\frac{\Delta^{+r}x_k}{\alpha_k}\right) \in S.$$

In this section, the set of sequences $S_\alpha(\Delta^{+r})$ and $[D_p^q S]_\alpha(\Delta^{+r})$ will be compared under some restriction on $p(n)$ and $q(n)$.

Theorem 2.2. *$S_\alpha(\Delta^{+r}) \subseteq [D_p^q S]_\alpha(\Delta^{+r})$ if and only if*

$$\liminf_{n\to\infty} \frac{q(n)-p(n)}{q(n)} > 0. \tag{2.1}$$

Proof. Let $x \in S_\alpha(\Delta^{+r})$ be an arbitrary sequence such that

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \ge \epsilon\right\}\right| = 0,$$

holds for every $\epsilon > 0$. Since the sequence $q(n)$ satisfies $\lim_{n\to\infty} q(n) = \infty$, then

$$\left\{\frac{1}{q(n)}\left|\left\{k \le q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \ge \epsilon\right\}\right|\right\}_{n\in\mathbb{N}}$$

is also convergent to zero because of (see in [12], Theorem 2.2.1). Hence, by a simple calculation we have the following inequality

$$\frac{1}{q(n)}\left|\left\{k \le q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \ge \epsilon\right\}\right| \ge \frac{1}{q(n)}\left|\left\{p(n) < k \le q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \ge \epsilon\right\}\right|$$

$$\ge \frac{q(n)-p(n)}{q(n)}.\frac{1}{q(n)-p(n)}\left|\left\{p(n) < k \le q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \ge \epsilon\right\}\right|.$$

Taking in to consider (2.1), if we take limit when $n \to \infty$ in the above inequality then,

$$x \in [D_p^q S]_\alpha(\Delta^{+r}).$$

Conversely, assume that

$$\liminf_{n\to\infty} \frac{q(n)-p(n)}{q(n)} = 0$$

holds. Now, let us choose a subsequence $(n(j))_{j\ge 1}$ such that $\frac{q(n_{(j)})-p(n_{(j)})}{q(n_{(j)})} < \frac{1}{j}$ holds for all $i \in \mathbb{N}$. Let a sequence $x = (x_n)$ such that

$$\frac{\Delta^{+r}x_k}{\alpha_k} := \begin{cases} 1, & p(n_{(j)}) + 1 < k \le q(n_{(j)}), \\ 0 & \text{otherwise,} \end{cases}$$

holds. Then, $x \in [S]_0(\Delta^{+r})$ and hence by Theorem 1 (a), we have $x \in S_\alpha(\Delta^{+r})$. But $x \notin [D_p^q]_\alpha^0(\Delta^{+r})$, and therefore by Theorem 1 (b), we have $x \notin ([D_p^q S]_\alpha(\Delta^{+r}))$. □

Corollary 2.1. *Let $\{q(n)\}_{n\in\mathbb{N}}$ be an arbitrary sequence with $q(n) < n$ for all $n \in \mathbb{N}$ and $\left\{\frac{n}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ be a bounded sequence. Then, $S_\alpha(\Delta^{+r}) \subset [D_p^q S]_\alpha(\Delta^{+r})$ for all $r \ge 1$.*

Theorem 2.3. *Let $q(n) = n$ for all $n \in \mathbb{N}$. Then, $[D_p^n S]_\alpha(\Delta^{+r}) \subset S_\alpha(\Delta^{+r})$ holds for all $r \ge 1$.*

Proof. Let us assume that $x \in [D_p^n S]_\alpha(\Delta^{+r})$. We shall apply the technique which was suggested by [1] and was also used in [10].
Let

$$p(n) = n^{(1)} > p(n^{(1)}) = n^{(2)} > p(n^{(2)}) = n^{(3)} > ...,$$

and we may write the set $\left\{k \leq n : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$ as

$$= \left\{k \leq n^{(1)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\} \cup \left\{n^{(1)} < k \leq n : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$$

and the set $\left\{1 < k \leq n^{(1)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$ as

$$= \left\{k \leq n^{(2)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\} \cup \left\{n^{(2)} < k \leq n^{(1)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\},$$

and the set $\left\{k \leq n^{(2)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$ as

$$= \left\{k \leq n^{(3)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\} \cup \left\{n^{(3)} < k \leq n^{(2)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}.$$

If we continue this operation consecutively, after the final step we have

$$\left\{k \leq n^{(h-1)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$$

$$= \left\{k \leq n^{(h)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\} \cup \left\{n^{(h)} < k \leq n^{(h-1)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$$

for a certain positive integer $h > 0$ depending on n such that $n^{(h)} \geq 1$ and $n^{(h+1)} = 0$.
By combining all the equalities obtained above we have

$$\frac{1}{n}\left|\left\{k \leq n : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}\right|$$

$$= \sum_{m=0}^{h} \frac{n^{(m)} - n^{(m+1)}}{n} . \frac{1}{n^{(m)} - n^{(m+1)}} \left|\left\{n^{(m+1)} < k \leq n^{(m)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}\right|.$$

As a result of this equality it can be said that statistical convergence of the sequence $(\frac{\Delta^{+r} x_k}{\alpha_k})$ is a linear combination of the following sequence

$$\left\{\frac{1}{n^{(m)} - n^{(m+1)}} \left|\left\{n^{(m+1)} < k \leq n^{(m)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}\right|\right\}_{m \in \mathbb{N}}.$$

Now, let us consider a matrix $A = (a_{nm})$ as

$$a_{n,m} := \begin{cases} \frac{n^{(m)} - n^{(m+1)}}{n}, & m = 0, 1, 2, ..., h, \\ 0, & \text{otherwise}, \end{cases}$$

It is clear that, where $n^{(0)} = n$.

The matrix $A=(a_{n,m})$ satisfied the Silverman Toeplitz Theorem (see in [14]). So, we have

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n:\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\ge\epsilon\right\}\right|=0$$

because of

$$\frac{1}{n^{(m)}-n^{(m+1)}}\left|\left\{n^{(m+1)}<k\le n^{(m)}:\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\ge\epsilon\right\}\right|\to 0,$$

when $n\to\infty$. This completes the proof. □

2.3. Comparasion of $[D_p^qS]_\alpha(\Delta^{+r})$ and $[D_r^sS]_\alpha(\Delta^{+r})$ for all $r\ge 1$.

In this section, the sequence spaces $[D_p^qS]_\alpha(\Delta^{+r})$ and $[D_r^sS]_\alpha(\Delta^{+r})$ will be compared under which for all $n\in\mathbb{N}$ in addition to (1.2),

$$p(n)\le r(n)<s(n)\le q(n) \tag{2.2}$$

holds.

Theorem 2.4. *Let $r(n)$ and $s(n)$ be sequences of positive natural numbers satisfying (2.2) in addition to (1.2) such that the sets*

$$\{k:p(n)<k\le r(n)\}\quad and\quad \{k:s(n)<k\le q(n)\}$$

are finite for all $n\in\mathbb{N}$. Then, $[D_r^sS]_\alpha(\Delta^{+r})\subset[D_p^qS]_\alpha(\Delta^{+r})$ holds.

Proof. Let us consider a sequence $x=(x_n)$ such that $x\in[D_r^sS]_\alpha(\Delta^{+r})$. For an arbitrary $\epsilon>0$ the equality

$$\left\{p(n)<k\le q(n):\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\ge\epsilon\right\}=\left\{p(n)<k\le r(n):\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\ge\epsilon\right\}$$

$$\cup\left\{r(n)<k\le s(n):\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\ge\epsilon\right\}\cup\left\{s(n)<k\le q(n):\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\ge\epsilon\right\}$$

holds. So, the following inequality

$$\begin{aligned}&\frac{1}{q(n)-p(n)}\left|\left\{p(n)<k\le q(n):\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\ge\epsilon\right\}\right|\\ \le\ &\frac{1}{s(n)-r(n)}\left|\left\{p(n)<k\le r(n):\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\ge\epsilon\right\}\right|\\ +\ &\frac{1}{s(n)-r(n)}\left|\left\{r(n)<k\le s(n):\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\ge\epsilon\right\}\right|\\ +\ &\frac{1}{s(n)-r(n)}\left|\left\{s(n)<k\le q(n):\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|\ge\epsilon\right\}\right|\end{aligned}$$

holds. By taking limit of each side in the above inequality when $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \left| \left\{ p(n) < k \le q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \ge \epsilon \right\} \right| = 0.$$

This implies that $x \in [D_p^q S]_\alpha(\Delta^{+r})$. So, the proof is completed. □

Theorem 2.5. *Let $p = p(n)$, $q = q(n)$ and $r = r(n)$, $s = s(n)$ be sequences of positive natural numbers satisfying* (1.2) *and* (2.2) *such that*

$$\liminf_{n\to\infty} \frac{s(n)-r(n)}{q(n)-p(n)} > 0$$

holds. Then, $[D_p^q S]_\alpha(\Delta^{+r}) \subset [D_r^s S]_\alpha(\Delta^{+r})$ holds.

Proof. It is easy to see from (2.2) and (1.2) that the following inclusion

$$\left\{ r(n) < k \le s(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \ge \epsilon \right\}$$
$$\subset \left\{ p(n) < k \le q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \ge \epsilon \right\}$$

and the following inequality

$$\left| \left\{ r(n) < k \le s(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \ge \epsilon \right\} \right|$$
$$\le \left| \left\{ p(n) < k \le q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \ge \epsilon \right\} \right|$$

hold. So, the last inequality gives that

$$\frac{s(n)-r(n)}{q(n)-p(n)} . \frac{1}{s(n)-r(n)} \left| \left\{ r(n) < k \le s(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \ge \epsilon \right\} \right|$$
$$\le \frac{1}{q(n)-p(n)} \left| \left\{ p(n) < k \le q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \ge \epsilon \right\} \right| .$$

Therefore, by taking the limit of each side in above inequality when $n \to \infty$, a desired result is obtained. □

2.4. Some properties of the set $D_{p,q}^{\sim}$. Now, define the set $D_{p,q}^{\sim}$ of sequence $\alpha \in U^+$ satisfying the condition

$$sup_n \left(\frac{1}{\alpha_{q(n)}} \sum_{k=p(n)+1}^{q(n)} \alpha_k \right) < \infty.$$

Let Δ be the well known operator defined by $\Delta x_n = x_n - x_{n-1}$ for all n, with $x_0 = 0$.

Lemma 2.1. *Let $\alpha \in U^+$. The following statements are equivalent:*

(i) $\alpha \in D_{p,q}^{\sim}$ *and* $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$,

(ii) the operator Δ *is bijective from* $[D_p^q]_\alpha$ *to itself,*

(iii) the operator Δ *is bijective from* $[D_p^q]_\alpha^0$ *to itself.*

Proof. Firstly, let us show that (i) implies (ii). Let $x = (x_n)$ and $y = (y_n) \in [D_p^q]_\alpha$ be arbitrary sequences and assume that $\Delta x = \Delta y$. It means that

$$(x_1, x_2 - x_1, ..., x_n - x_{n-1}, ...) = (y_1, y_2 - y_1, ..., y_n - y_{n-1}, ...)$$

holds. From this assumption we have, $x_n - x_{n-1} = y_n - y_{n-1}$ for all $n \geq 1$. This calculation gives that $x_n = y_n$ holds for all $n \in \mathbb{N}$. Hence, Δ is an injective function from $[D_p^q]_\alpha$ to itself.

Now, let $y \in [D_p^q]_\alpha$ be an arbitrary sequence. We must find a sequence $x \in [D_p^q]_\alpha$ such that $\Delta x_n = y_n$ holds. That is,

$$(y_1, y_2, ..., y_n, ...) = (x_1, x_2 - x_1, ..., x_n - x_{n-1}, ...)$$

holds. Therefore, the sequence $x = (x_n)$ must be as $x_n := \sum_{k=1}^{n} y_k$, for all $n \in \mathbb{N}$ Now, let us check that $x \in [D_p^q]_\alpha$. By using the method in [1], we have

$$\begin{aligned} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k}{\alpha_k}\right| &= \left[-\frac{p(n)}{q(n)-p(n)}\right] \left[\frac{1}{p(n)} \left(\left|\frac{x_1}{\alpha_1}\right| + \left|\frac{x_2}{\alpha_2}\right| + ... + \left|\frac{x_{p(n)}}{\alpha_{p(n)}}\right|\right)\right] \\ &+ \left[\frac{q(n)}{q(n)-p(n)}\right] \left[\frac{1}{q(n)} \left(\left|\frac{x_1}{\alpha_1}\right| + \left|\frac{x_2}{\alpha_2}\right| + ... + \left|\frac{x_{q(n)}}{\alpha_{q(n)}}\right|\right)\right] \\ &= \left[-\frac{p(n)}{q(n)-p(n)}\right] \left[\frac{1}{p(n)} \left(\left|\frac{y_1}{\alpha_1}\right| + \left|\frac{y_1+y_2}{\alpha_2}\right| + ... + \left|\frac{y_1+y_2+...+y_{p(n)}}{\alpha_{p(n)}}\right|\right)\right] \\ &+ \left[\frac{q(n)}{q(n)-p(n)}\right] \left[\frac{1}{q(n)} \left(\left|\frac{y_1}{\alpha_1}\right| + \left|\frac{y_1+y_2}{\alpha_2}\right| + ... + \left|\frac{y_1+y_2+...+y_{q(n)}}{\alpha_{q(n)}}\right|\right)\right] \\ &= \frac{1}{q(n)-p(n)} \left(\left|\frac{y_{p(n)+1}}{\alpha_{p(n)+1}}\right| + \left|\frac{y_{p(n)+1}+y_{p(n)+2}}{\alpha_{p(n)+2}}\right| + ... + \left|\frac{y_{p(n)+1}+...+y_{q(n)}}{\alpha_{q(n)}}\right|\right) \end{aligned}$$

Also, with a simple calculation, the following inequality

$$\left|\frac{y_{p(n)+1}+y_{p(n)+2}}{\alpha_{p(n)+2}}\right| \leq \left|\frac{y_{p(n)+1}}{\alpha_{p(n)+1}}\right| \frac{\alpha_{p(n)+1}}{\alpha_{p(n)+2}} + \left|\frac{y_{p(n)+2}}{\alpha_{p(n)+2}}\right|$$

and

$$\left|\frac{y_{p(n)+1}+...+y_{q(n)}}{\alpha_{q(n)}}\right| \leq \left|\frac{y_{p(n)+1}}{\alpha_{q(n)}}\right| + \left|\frac{y_{p(n)+2}}{\alpha_{q(n)}}\right| + ... + \left|\frac{y_{q(n)}}{\alpha_{q(n)}}\right|$$

$$\leq \left|\frac{y_{p(n)+1}}{\alpha_{p(n)+1}}\right| \frac{\alpha_{p(n)+1}}{\alpha_{q(n)}} + \left|\frac{y_{p(n)+2}}{\alpha_{p(n)+2}}\right| \frac{\alpha_{p(n)+2}}{\alpha_{q(n)}} + ... + \left|\frac{y_{q(n)}}{\alpha_{q(n)}}\right| \frac{\alpha_{q(n)}}{\alpha_{q(n)}}$$

$$\leq K. \left(\frac{\alpha_{p(n)+1} + \alpha_{p(n)+2} + ... + \alpha_{q(n)}}{\alpha_{q(n)}}\right)$$

holds for a positive K. Consequently, we conclude that $x \in [D_p^q]_\alpha$ for $\alpha \in D_{p,q}^\sim$ and $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$. Similarly, it can be proved that $\Delta : [D_p^q]_\alpha^0 \to [D_p^q]_\alpha^0$ is a bijective function. $\square$

It can easily be deduced that if $\alpha \in D_{p,q}^\sim$, then for any given integer $r \geq 1$ the operator Δ^r is a bijective function from $[D_p^q]_\alpha$ to itself. So, $[D_p^q]_\alpha(\Delta^r) = [D_p^q]_\alpha$. It is the same for the operator Δ considered as an operator from $[D_p^q]_\alpha$ to itself.

Lemma 2.2. *Let $r \geq 1$ be an integer and $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$. The following statements are equivalent:*

(i) $\alpha \in D_{p,q}^\sim$,

(ii) $[D_p^q]_\alpha(\Delta) = [D_p^q]_\alpha$,

(iii) $[D_p^q]_\alpha(\Delta^r) = [D_p^q]_\alpha$.

Proof. First show that (i) implies (ii). Let $x \in [D_p^q]_\alpha$ be an arbitrary sequence. Then, the following inequality

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k - x_{k-1}}{\alpha_k}\right| \leq \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left(\left|\frac{x_k}{\alpha_k}\right| + \left|\frac{x_{k-1}}{\alpha_k}\right|\right)$$

$$\leq \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k}{\alpha_k}\right| + \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_{k-1}}{\alpha_{k-1}}\right| . \left|\frac{\alpha_{k-1}}{\alpha_k}\right|$$

holds. It gives that

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k - x_{k-1}}{\alpha_k}\right| < \infty.$$

So, $x \in [D_p^q]_\alpha(\Delta)$. Conversely, let $x \in [D_p^q]_\alpha(\Delta)$. This implies that $b := (\Delta x) \in [D_p^q]_\alpha$ for every n. So, we have

$$x_n = u + \sum_{k=1}^{n} b_k$$

for $u \in \mathbb{C}$ (see in [3] Lemma 2.2). Then, $b = (b_n) \in [D_p^q]_\alpha$.
So, if we take $u = 0$, then we obtain

$$\begin{aligned}
\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k}{\alpha_k}\right| &= \left[-\frac{p(n)}{q(n)-p(n)}\right]\left[\frac{1}{p(n)}\left(\left|\frac{x_1}{\alpha_1}\right| + \left|\frac{x_2}{\alpha_2}\right| + ... + \left|\frac{x_{p(n)}}{\alpha_{p(n)}}\right|\right)\right] \\
&+ \left[\frac{q(n)}{q(n)-p(n)}\right]\left[\frac{1}{q(n)}\left(\left|\frac{x_1}{\alpha_1}\right| + \left|\frac{x_2}{\alpha_2}\right| + ... + \left|\frac{x_{q(n)}}{\alpha_{q(n)}}\right|\right)\right] \\
&= \left[-\frac{p(n)}{q(n)-p(n)}\right]\left[\frac{1}{p(n)}\left(\left|\frac{b_1}{\alpha_1}\right| + \left|\frac{b_1+b_2}{\alpha_2}\right| + ... + \left|\frac{b_1+b_2+...+b_{p(n)}}{\alpha_{p(n)}}\right|\right)\right] \\
&+ \left[\frac{q(n)}{q(n)-p(n)}\right]\left[\frac{1}{q(n)}\left(\left|\frac{b_1}{\alpha_1}\right| + \left|\frac{b_1+b_2}{\alpha_2}\right| + ... + \left|\frac{b_1+b_2+...+b_{q(n)}}{\alpha_{q(n)}}\right|\right)\right] \\
&= \frac{1}{q(n)-p(n)}\left(\left|\frac{b_{p(n)+1}}{\alpha_{p(n)+1}}\right| + \left|\frac{b_{p(n)+1}+b_{p(n)+2}}{\alpha_{p(n)+2}}\right| + ... + \left|\frac{b_{p(n)+1}+...+b_{q(n)}}{\alpha_{q(n)}}\right|\right).
\end{aligned}$$

So, from Lemma 1, we have $[D_p^q]_\alpha(\Delta) = [D_p^q]_\alpha$.

Now, let us show that (ii) implies (iii). Hence, Δ is bijective function and so does the composition Δ^r. In that case $[D_p^q]_\alpha(\Delta^r) = [D_p^q]_\alpha$. We obtain that

$$[D_p^q]_\alpha = [D_p^q]_\alpha(\Delta) = ... = [D_p^q]_\alpha(\Delta^r) = [D_p^q]_\alpha(\Delta^{r+1}).$$

On the contrary, let $[D_p^q]_\alpha(\Delta^r) = [D_p^q]_\alpha$. Therefore, (iii) must imply (i) to achieve this equality. □

Lemma 2.3. *Let $\alpha, \beta \in U^+$. Then, $[D_p^q]_\alpha^0 = [D_p^q]_\beta^0$ if and only if there exists $M_1, M_2 > 0$ such that for all $n \in \mathbb{N}$,*

$$M_1 \leq \frac{\alpha_n}{\beta_n} \leq M_2 \tag{2.3}$$

holds.

Proof. It is easy to see from (2.3) that

$$M_1 \frac{x_k}{\alpha_k} \leq \frac{x_k}{\beta_k} \leq M_2 \frac{x_k}{\alpha_k}$$

holds. Also, this inequality implies that

$$M_1 \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} \leq \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\beta_k} \leq \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k}$$

holds. Then, if we take limit in the above inequality when $n \to \infty$, a desired implication will be obtained. □

Theorem 2.6. *Let $\alpha \in U^+$ and $r \geq 1$ be arbitrary integer. Then, the following statements are true:*

(i) $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$ *if and only if* $[D_p^q]^0_{(\alpha_{n-1})} \subset [D_p^q]^0_\alpha(\Delta^+)$.

(ii) the following statements are equivalent:

(a) $\alpha \in D^{\sim}_{p,q}$ *and* $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$ *,*

(b) $[D_p^q]^0_\alpha(\Delta^+) = [D_p^q]^0_{(\alpha_{n-1})}$

(c) $[D_p^q]^0_\alpha(\Delta^+) \subset [D_p^q]^0_{(\alpha_{n-1})}$.

(iii) (a) $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$ *if and only if, for any given integer* $r \geq 1$ *,*

$$[D_p^q]^0_\alpha \subset [D_p^q]^0_\alpha(\Delta^+) \subset ... \subset [D_p^q]^0_\alpha(\Delta^{+r});$$

(b) If $\alpha \in D^{\sim}_{p,q}$ *and* $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$*, then* $[D_p^q]^0_\alpha = [D_p^q]^0_\alpha(\Delta^+)$.

Proof. (i) Assume that $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$ and let $x \in [D_p^q]^0_{(\alpha_{n-1})}$ be an arbitrary sequence. Then,

$$\begin{aligned} \frac{1}{q(n)-p(n)} \quad & \sum_{k=p(n)+1}^{q(n)} \left| \frac{x_k - x_{k+1}}{\alpha_k} \right| \\ \leq \quad & \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left(\left| \frac{x_k}{\alpha_k} \right| . \left| \frac{\alpha_{k-1}}{\alpha_{k-1}} \right| + \left| \frac{x_{k+1}}{\alpha_k} \right| \right) \\ \leq \quad & \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left| \frac{x_k}{\alpha_{k-1}} \right| . \left| \frac{\alpha_{k-1}}{\alpha_k} \right| \\ + \quad & \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left| \frac{x_{k+1}}{\alpha_k} \right| . \end{aligned}$$

under assumptions, the above inequality implies that $x \in [D_p^q]^0_\alpha(\Delta^+)$ when $n \to \infty$. This gives $[D_p^q]^0_{(\alpha_{n-1})} \subset [D_p^q]^0_\alpha(\Delta^+)$.

Conversely, assume that $[D_p^q]^0_{(\alpha_{n-1})} \subset [D_p^q]^0_\alpha(\Delta^+)$. Therefore, it is clear that $\alpha \in D^{\sim}_{p,q}$ and $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$.

(ii) Let us show that (a) implies (b). Now $\alpha \in D^{\sim}_{p,q}$ implies that $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$ and by (i), $[D_p^q]^0_{(\alpha_{n-1})} \subset [D_p^q]^0_\alpha(\Delta^+)$.

Conversely, $x \in [D_p^q]^0_\alpha(\Delta^+)$ implies that $b = \Delta^+ x \in [D_p^q]^0_\alpha$. So, for every n, we have $x_n = -\sum_{k=1}^{n-1} b_k$ for $x_1 = 0$ (see in [3], Lemma 2.2).

Then, $b = (b_n) \in [D_p^q]^0_\alpha$. Since $b = \Delta^+ x$, then $(b_1, b_2, ..., b_n ...) = (x_1 - x_2, x_2 - x_3, ..., x_{n-1} - x_n, ...)$, for all $n \in \mathbb{N}$. Therefore,

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left| \frac{x_k}{\alpha_{k-1}} \right| == \left[-\frac{p(n)}{q(n)-p(n)} \right] \left[\frac{1}{p(n)} \left(\left| \frac{x_2}{\alpha_1} \right| + \left| \frac{x_3}{\alpha_2} \right| + ... + \left| \frac{x_{p(n)}}{\alpha_{p(n)-1}} \right| \right) \right]$$

$$+\left[\frac{q(n)}{q(n)-p(n)}\right]\left[\frac{1}{q(n)}\left(\left|\frac{x_2}{\alpha_1}\right|+\left|\frac{x_3}{\alpha_2}\right|+...+\left|\frac{x_{q(n)}}{\alpha_{q(n)-1}}\right|\right)\right]$$

$$=\left[-\frac{p(n)}{q(n)-p(n)}\right]\left[\frac{1}{p(n)}\left(\left|\frac{-b_1}{\alpha_1}\right|+\left|\frac{-(b_1+b_2)}{\alpha_2}\right|+...+\left|\frac{-(b_1+b_2+...+b_{p(n)-1})}{\alpha_{p(n)-1}}\right|\right)\right]$$

$$+\left[\frac{q(n)}{q(n)-p(n)}\right]\left[\frac{1}{q(n)}\left(\left|\frac{-b_1}{\alpha_1}\right|+\left|\frac{-(b_1+b_2)}{\alpha_2}\right|+...+\left|\frac{-(b_1+b_2+...+b_{q(n)-1})}{\alpha_{q(n)-1}}\right|\right)\right]$$

$$=\frac{1}{q(n)-p(n)}\left(\left|\frac{b_{p(n)+1}}{\alpha_{p(n)+1}}\right|+\left|\frac{b_{p(n)+1}+b_{p(n)+2}}{\alpha_{p(n)+2}}\right|+...+\left|\frac{b_{p(n)+1}+...+b_{q(n)-1}}{\alpha_{q(n)-1}}\right|\right)$$

holds. From here, the following inequalities

$$\left|\frac{b_{p(n)+1}+b_{p(n)+2}}{\alpha_{p(n)+2}}\right|\leq\left|\frac{b_{p(n)+1}}{\alpha_{p(n)+1}}\right|\frac{\alpha_{p(n)+1}}{\alpha_{p(n)+2}}+\left|\frac{b_{p(n)+2}}{\alpha_{p(n)+2}}\right|$$

and

$$\left|\frac{b_{p(n)+1}+b_{p(n)+2}+...+b_{q(n)-1}}{\alpha_{q(n)-1}}\right|\leq\left|\frac{b_{p(n)+1}}{\alpha_{q(n)-1}}\right|+\left|\frac{b_{p(n)+2}}{\alpha_{q(n)-1}}\right|+...+\left|\frac{b_{q(n)-1}}{\alpha_{q(n)-1}}\right|$$

$$\leq\left|\frac{b_{p(n)+1}}{\alpha_{p(n)+1}}\right|\frac{\alpha_{p(n)+1}}{\alpha_{q(n)-1}}+\left|\frac{b_{p(n)+2}}{\alpha_{p(n)+2}}\right|\frac{\alpha_{p(n)+2}}{\alpha_{q(n)-1}}+...+\left|\frac{b_{q(n)-1}}{\alpha_{q(n)-1}}\right|\frac{\alpha_{q(n)-1}}{\alpha_{q(n)-1}}$$

$$\leq K.\left(\frac{\alpha_{p(n)+1}+\alpha_{p(n)+2}+...+\alpha_{q(n)-1}}{\alpha_{q(n)-1}}\right)$$

hold for any $K>0$. Then, $x\in[D_p^q]^0_{(\alpha_{n-1})}$ and we conclude that $[D_p^q]^0_\alpha(\Delta^+)\subset[D_p^q]^0_{(\alpha_{n-1})}$.
So, $[D_p^q]^0_\alpha(\Delta^+)=[D_p^q]^0_{(\alpha_{n-1})}$ and we have shown that (a) implies (b). Consequantly, conclude that (b) implies (c).

(iii) (a) Let $(\alpha_{p(n)}/\alpha_{p(n)-1})\in l_\infty$. Since $[D_p^q]^0_\alpha\subset[D_p^q]^0_\alpha(\Delta^+)$, then for all $x\in[D_p^q]^0_\alpha$ we have

$$\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\left|\frac{x_k-x_{k+1}}{\alpha_k}\right|$$

$$\leq\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\left(\left|\frac{x_k}{\alpha_k}\right|+\left|\frac{x_{k+1}}{\alpha_k}\right|.\left|\frac{\alpha_{k+1}}{\alpha_{k+1}}\right|\right)$$

$$\leq\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\left|\frac{x_k}{\alpha_k}\right|+\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\left|\frac{x_{k+1}}{\alpha_{k+1}}\right|.\left|\frac{\alpha_{k+1}}{\alpha_k}\right|.$$

under assumption, this inequality implies that $[D_p^q]^0_\alpha\subset[D_p^q]^0_\alpha(\Delta^+)$ when $n\to\infty$.

Now, from the mathematical induction method for any given integer $r\geq1$ and $x\in[D_p^q]^0_\alpha(\Delta^{+r})$; then $\Delta^{+r}x\in[D_p^q]^0_\alpha$ and with $[D_p^q]^0_\alpha\subset[D_p^q]^0_\alpha(\Delta^+)$ holds because of

$\Delta^{+r}x \in [D_p^q]_\alpha^0(\Delta^+)$ and $x \in [D_p^q]_\alpha^0(\Delta^{+(r+1)})$.
So, we have $[D_p^q]_\alpha^0(\Delta^{+r}) \subset [D_p^q]_\alpha^0(\Delta^{+(r+1)})$.
(b) Now, $\alpha \in D_{p,q}^{\sim}$ implies that $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$ and by (iii) (a), we have $[D_p^q]_\alpha^0 \subset [D_p^q]_\alpha^0(\Delta^+)$.

Conversely, let $x \in [D_p^q]_\alpha^0(\Delta^+)$ implies that $b = \Delta^+x \in [D_p^q]_\alpha^0$ and for every n, we have $x_n = -\sum_{k=1}^{n-1} b_k$ for $x_1 = 0$ Theorem 6 (ii). Then $b \in [D_p^q]_\alpha^0$.

$x \in [D_p^q]_\alpha^0(\Delta^+)$ for $x \in [D_p^q]_\alpha^0$ when shown similarly with (ii). Therefore, the conditions $\alpha \in D_{p,q}^{\sim}$ and $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$ are equivalent to $[D_p^q]_\alpha^0 = [D_p^q]_\alpha^0(\Delta^+)$. □

Corollary 2.2. *Let $r \geq 1$ be an integer and assume that $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$. Then, $[D_p^q]_\alpha^0(\Delta^{+r}) \subset [D_p^q]_\alpha^0$ implies that $[D_p^q]_\alpha^0(\Delta^{+r}) = [D_p^q]_\alpha^0$.*

Proof. By Theorem 3 (iii) (a), the condition $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$ implies that $[D_p^q]_\alpha^0 \subset [D_p^q]_\alpha^0(\Delta^{+r})$. Since $[D_p^q]_\alpha^0(\Delta^{+r}) \subset [D_p^q]_\alpha^0$ then,

$$[D_p^q]_\alpha^0(\Delta^{+r}) = [D_p^q]_\alpha^0$$

holds. □

Remark 2. In Theorem 3, the conditions $\alpha \in D_{p,q}^{\sim}$ and $(\alpha_{p(n)}/\alpha_{p(n)-1})_n \in l_\infty$ are equivalent to $[D_p^q]_\alpha^0(\Delta^+) = [D_p^q]_{(\alpha_{n-1})}^0 = [D_p^q]_\alpha^0$.

Proof. If $\alpha \in D_{p,q}^{\sim}$ and $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$, then there are $K_1, K_2 > 0$ such that

$$K_1 \leq \frac{\alpha_{p(n)}}{\alpha_{p(n)-1}} \leq K_2$$

holds for all $n \in \mathbb{N}$. Then by Lemma 3, we have $[D_p^q]_{(\alpha_{n-1})}^0 = [D_p^q]_\alpha^0$. By Theorem 6 (ii), we conclude that the condition $\alpha \in D_{p,q}^{\sim}$ implies that $[D_p^q]_\alpha^0(\Delta^+) = [D_p^q]_{(\alpha_{n-1})}^0 = [D_p^q]_\alpha^0$. □

Corollary 2.3. *Let $\alpha \in U^+$ and $r \geq 1$ be an integer. Then, the condition $\alpha \in D_{p,q}^{\sim}$ implies $[D_p^q]_\alpha^0(\Delta^{+r}) = [D_p^q]_{(\alpha_{n-r})}^0$.*

Proof. The condition $\alpha \in D_{p,q}^{\sim}$ implies by Theorem 6 (ii) $[D_p^q]_\alpha^0(\Delta^+) = [D_p^q]_{(\alpha_{n-1})}^0$. Now let $r \geq 1$ be an integer and assume that

$$[D_p^q]_\alpha^0(\Delta^{+r}) = [D_p^q]_{(\alpha_{n-r})}^0.$$

Then, $x \in [D_p^q]_\alpha^0(\Delta^{+(r+1)})$ if and only if $(\Delta^{+(r+1)})x \in [D_p^q]_\alpha^0$, which in turn is

$$\Delta^+x \in [D_p^q]_\alpha^0(\Delta^{+r}) = [D_p^q]_{(\alpha_{n-r})}^0.$$

So, $[D_p^q]_\alpha^0(\Delta^{+(r+1)}) = [D_p^q]_{(\alpha_{n-r})}^0(\Delta^+)$ since $\alpha \in D_{p,q}^{\sim}$, then $(\alpha_{n-r}) \in D_{p,q}^{\sim}$ and

$$[D_p^q]_{(\alpha_{n-r})}^0(\Delta^+) = [D_p^q]_\alpha^0(\Delta^{+(r+1)}) = [D_p^q]_{(\alpha_{n-(r+1)})}^0.$$

This shows (i). □

Theorem 2.7. *Let $\alpha \in D_{p,q}^{\sim}$ and $r \geq 1$ be an integer. Then, $(\alpha_{p(n)}) \in l_\infty$ implies that*

$$[D_p^q]_\alpha^0 \subset [D_p^q]_0(\Delta^{+r}) \quad \textit{and} \quad [D_p^q]_\alpha \subset [D_p^q]_\infty(\Delta^{+r})$$

holds.

Proof. Let $(\alpha_{p(n)}) \in l_\infty$ and $x \in [D_p^q]_\alpha$. Then, the following inequality

$$\begin{aligned}\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - x_{k+1}| &\leq \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k| \\ &+ \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |x_{k+1}| \\ &\leq \frac{1}{q(n)-p(n)} \left[\sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k}{\alpha_k}\right| .\alpha_k + \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_{k+1}}{\alpha_{k+1}}\right| .\alpha_{k+1} \right]\end{aligned}$$

holds. Hence, $x \in [D_p^q]_\infty(\Delta^+)$, because of $[D_p^q]_\alpha \subset [D_p^q]_\infty(\Delta^{+r})$. Similarly $[D_p^q]_\alpha^0 \subset [D_p^q]_0(\Delta^{+r})$ is satisfied. □

Theorem 2.8. *Let $\alpha \in U^+$ and $r \geq 1$ be an integer. Assume that $(1/\alpha_{p(n)}) \in l_\infty$. Then, we have*

$$[D_p^q]_0(\Delta^{+r}) \subset [D_p^q]_\alpha^0 \quad \textit{and} \quad [D_p^q]_\infty(\Delta^{+r}) \subset [D_p^q]_\alpha.$$

Proof. Assume that $\alpha \in D_{p,q}^{\sim}$ and let $(1/\alpha_{p(n)}) \in l_\infty$. Let $x \in [D_p^q]_0(\Delta^+)$ implies that $b = \Delta^+ x \in [D_p^q]_0$ and for every n, we have from Theorem 6 that $x_n = -\sum_{k=1}^{n-1} b_k$.

$$\begin{aligned}\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k}{\alpha_k}\right| &= \left[-\frac{p(n)}{q(n)-p(n)}\right] \left[\frac{1}{p(n)} \left(\left|\frac{x_1}{\alpha_1}\right| + \left|\frac{x_2}{\alpha_2}\right| + ... + \left|\frac{x_{p(n)}}{\alpha_{p(n)}}\right|\right)\right] \\ &+ \left[\frac{q(n)}{q(n)-p(n)}\right] \left[\frac{1}{q(n)} \left(\left|\frac{x_1}{\alpha_1}\right| + \left|\frac{x_2}{\alpha_2}\right| + ... + \left|\frac{x_{q(n)}}{\alpha_{q(n)}}\right|\right)\right] \\ &= \left[-\frac{p(n)}{q(n)-p(n)}\right] \left[\frac{1}{p(n)} \left(\left|\frac{-b_1}{\alpha_2}\right| + \left|\frac{-(b_1+b_2)}{\alpha_3}\right| + ... + \left|\frac{-(b_1+b_2+...+b_{p(n)})}{\alpha_{p(n)}}\right|\right)\right] \\ &+ \left[\frac{q(n)}{q(n)-p(n)}\right] \left[\frac{1}{q(n)} \left(\left|\frac{-b_1}{\alpha_2}\right| + \left|\frac{-(b_1+b_2)}{\alpha_3}\right| + ... + \left|\frac{-(b_1+b_2+...+b_{q(n)})}{\alpha_{q(n)}}\right|\right)\right] \\ &= \frac{1}{q(n)-p(n)} \left(\left|\frac{b_{p(n)+1}}{\alpha_{p(n)+2}}\right| + \left|\frac{b_{p(n)+1}+b_{p(n)+2}}{\alpha_{p(n)+3}}\right| + ... + \left|\frac{b_{p(n)+1}+...+b_{q(n)-1}}{\alpha_{q(n)}}\right|\right) \\ &= \frac{M}{q(n)-p(n)} \left[\left|b_{p(n)+1}\right| + \left|b_{p(n)+1}+b_{p(n)+2}\right| + ... + \left|b_{p(n)+1}+...+b_{q(n)-1}\right|\right]\end{aligned}$$

the inequality is provided. So, $x \in [D_p^q]_\alpha^0$. □

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