FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 34, No 3 (2019), 537–552
https://doi.org/10.22190/FUMI1903537A

QUASI MAPPING SINGULARITIES

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Abstract. We obtain a list of all simple classes of singularities of curves (irreducible and reducible) in real spaces of any dimension with respect to the quasi equivalence relation.
Keywords: Singularities; curve; quasi equivalence relation.

1. Introduction

Motivated by the importance of the locus of points on a hypersurface where a given vector field is not transversal to it, Vladimir Zakalyukin introduced a new equivalence relation on projections of hypersurfaces which he named quasi equivalence [9]. The relation is more rough than the standard group of diffeomorphisms preserving a given projection [8]. The difference between the $\mathcal{A}$-equivalence relation and the quasi relation is illustrated as follows: Let Λ be the graph of a map f from $\mathbb{R}^m$ to $\mathbb{R}^n$ and let π be a trivial fibration structure. If p_1 and p_2 are two points on Λ lying on the same fibre of the projection then they are mapped by π to the same image. This property persists for the $\mathcal{A}$-equivalent maps f_i,$i = 1, 2$. However, this is not the case for the quasi equivalence as p_1 and p_2 might be mapped by a diffeomorphism to different fibres and hence they are mapped by π to different images.

The locus of the points on the hypersurface where a given vector field is not transversal to it is of importance. One of the possible and interesting applications for the quasi-projection equivalence relation is used in partial differential equations (PDE) with boundary value problems. Consider the characteristic method solving the simplest Cauchy problem for first order linear PDE: $\sum a_i(x)\frac{\partial u}{\partial x_i} = 0$, where $u(x)$ is an unknown function with $x \in \mathbb{R}^m$ and $a_i(x)$ are given functions. The problem includes the boundary hypersurface $S \subset \mathbb{R}^m$ and the boundary values

Received December 27, 2018; accepted February 28, 2019
2010 *Mathematics Subject Classification.* 58K05.58K40.53A15.

$U|_S = U_0$. Generically, the characteristic vector field $v = a_i \frac{\partial}{\partial x_i}$ is tangent to S at some points which are called characteristic. Outside the set K of characteristic points, the problem has a unique local solution. So the geometry of the set K is an essential feature of the problem. If we rectify the vector field getting, say $\frac{\partial}{\partial x_1}$, then the problem of classifying K is exactly to find critical points of the projection of S along parallel rays. Similarly, in many other complicated PDE boundary value problems, mainly in continuum mechanics, the generalisation of the Neumann boundary condition is used.

In [3], the first steps in the study of the quasi-equivalence of projections of graphs of maps were taken within the approach similar to the one introduced by Zakalyukin [9]. Two cases were investigated there: maps from $\mathbb{R}$ to $\mathbb{R}^2$ and maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ (see [6] and [8] for the corresponding results for the $\mathcal{A}$-equivalence). In the current paper, we consider irreducible and reducible curve singularities in a linear real space of any dimension and give the list of stably simple classes with respect to the quasi equivalence (see [2] and [5] for the corresponding results for the $\mathcal{A}$-equivalence).

The paper is organized as follows. In Section 2 we review the definition of the quasi-equivalence relation of the projections of hypersurfaces and recall the main results from [9] which are needed in the next sections. In Section 3 we introduce the main definition of the quasi-equivalence of maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ and derive an algebraic expression for the respective tangent space to a quasi class of mapping. Then, we recall the classification of quasi-simple singularities of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ from [3], giving detailed proofs. After that, we classify quasi-stably simple classes of irreducible curves in $\mathbb{R}^n$. Finally, in Section 4 we classify stably simple reducible curve singularities with respect to the quasi-equivalence relation.

2. Quasi projections of hypersurfaces

Consider germs of subvarieties V in the space $\mathbb{R}^p = \{(x, y) : x \in \mathbb{R}^m, y \in \mathbb{R}^n\}$, equipped with the trivial fibration structure, given by the projection $\pi : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(x, y) \mapsto y$. When the distinction between x and y is not crucial, we will be using the notation $w = (x, y)$ for the whole set of coordinates on $\mathbb{R}^p$.

Consider germs of $\mathbf{C}^\infty$ functions $f : (\mathbb{R}^p, 0) \to \mathbb{R}$ and denote by $\mathbb{C}_w$ the ring of all such germs at the origin and by $\mathbb{M}_w$ the maximal ideal in $\mathbb{C}_w$.

Definition 2.1. [9] A point $b \in V$ is called *critical* if the fiber containing b is not transversal to V at b. In particular, b can be a singular point of V.

Definition 2.2. [9] Two subvarieties V_0 and V_1 in $\mathbb{R}^p$ are called *pseudo equivalent* if there exists a diffeomorphism $\Phi : \mathbb{R}^p \to \mathbb{R}^p$, such that:

1. $\Phi(V_1) = V_0$.

2. The set of critical points of V_1 is mapped by Φ onto the set of critical points of V_0.
3. The derivative of Φ at any critical point of V_1 maps the direction of the projection to that at the image of the point.

In the current section we consider only the case of analytic hypersurfaces V given by a single equation $f = 0$. Also, we assume the fibers are one dimensional $x \in \mathbb{R},\ m = 1$.

Now, suppose that all germs of hypersurfaces in a smooth family $V_t = \{f_t = 0\}$ are pseudo-equivalent to $V_0 = \{f_0 = 0\}$, $h_t(f_t \circ \theta_t) = f_0,\ t \in [0,1]$, with respect to a smooth family $\Phi_t : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ of germs of diffeomorphisms such that $\Phi_0 = id_{\mathbb{R}^p}$, $h_0 = 1$ and $t \in [0,1]$. Therefore, the respective homological equation is

$$-\frac{\partial f_t}{\partial t} = f_t A_t + \frac{\partial f_t}{\partial x}\dot{X}(t) + \sum_{i=1}^{n} \frac{\partial f_t}{\partial y_i}\dot{Y}_i(t),$$

where the vector field

$$v_t = \dot{X}(t)\frac{\partial}{\partial x} + \sum_{i=1}^{n} \dot{Y}_i(t)\frac{\partial}{\partial y_i}$$

generates the phase flow Φ_t and $A_t \in \mathbb{C}_w$.

Let J_{f_t} be the ideal in $\mathbb{C}_w$ generated by $\frac{\partial f_t}{\partial x}$ and f_t. Denote by $Rad(J_{f_t})$ the *radical* of J_{f_t}. Recall that the radical of an ideal is the set of all elements in $\mathbb{C}_w$, vanishing on the set of common zeros of germs from that ideal. Denote by IJ_{f_t} and $IRad(J_{f_t})$ the *integral* of J_{f_t} and $Rad(J_{f_t})$, consisting of all function germs φ such that the partial derivative of φ with respect to x belongs to J_{f_t} and $Rad(J_{f_t})$, respectively.

Proposition 2.3. [9] *The components of v_t satisfy the following*

$$\dot{X}(t) \in \mathbb{C}_w \qquad \textit{and} \qquad \dot{Y}_i(t) \in IRad(J_{f_t}).$$

In general, the radical of an ideal behaves badly when the ideal depends on a parameter (see [4]). Therefore, we modify the pseudo-equivalence relation since it does not satisfy the properties of a geometrical subgroup of equivalences in J. Damon sense and hence the versatility theorem can fail [7]. Namely, we replace $Rad(J_{f_t})$ by the ideal J_{f_t} itself in the equivalence definition.

Definition 2.4. [9] Two subvarieties $V_0 = \{f_0 = 0\}$ and $V_1 = \{f_1 = 0\}$ in $\mathbb{R}^p$ are called *quasi equivalent* if there is a family of smooth functions h_t which depends continuously on parameter $t \in [0,1]$ and a continuous piece-wise smooth family of diffeomorphisms $\Phi_t : \mathbb{R}^p \to \mathbb{R}^p$ also depending on $t \in [0,1]$ such that:

1. $h_t(f_t \circ \Phi_t) = f_0$, $\Phi_0 = id_{\mathbb{R}^p}$, $h_0 = 1$.
2. The set of critical points of V_t is mapped by Φ_t onto the set of critical points of V_0.
3. The components of the vector field v_t generating Φ_t on each segment of smoothness satisfy the following: $\dot{X}(t) \in \mathbb{C}_w$ and $\dot{Y}_i(t) \in IJ_{f_t}$.

Remarks 2.5.

1. The module IJ_{f_t} is defined precisely as the set of elements of the form
$$e_i + \int_0^x (f_t a_i + \frac{\partial f_t}{\partial x} b_i) dx,$$
where $a_i, b_i \in \mathbb{C}_{x,y}$ and $e_i \in \mathbb{C}_y$.
2. If two subvarieties are equivalent with respect to the standard projection equivalence then they are quasi-equivalent, since functions independent of x are in IJ_f for any f.

The classification of simple classes of quasi-projections of hypersurfaces in low dimensions is given by the following theorems, the proof of which is based on the classification of V.V. Goryunov [8].

Theorem 2.6. [9] *For $n = 1$ the list of simple classes is the same as for the standard group of foliation-preserving diffeomorphisms of the plane acting on the germs of curves:*

$$\begin{array}{lll} A_k: & f = x^{k+1} + y, & k \geqslant 0, \\ B_k: & f = x^2 \pm y^k, & k \geqslant 2, \\ C_k: & f = xy + x^k, & k \geqslant 3, \\ F_4: & f = x^3 + y^2. & \end{array}$$

Theorem 2.7. [9] *For $n = 2$ the list of simple quasi-projections of regular hypersurface singularities consists of*

$$\begin{array}{lll} \widetilde{A}_k: & f = x^{k+1} + y_1 x + y_2, & k \geq 0, \\ \widetilde{B}_k: & f = x^3 + y_1^k x + y_2, & k \geq 2, \\ \widetilde{C}_k: & f = x^{k+1} + y_1^2 x + y_2, & k \geq 3, \\ \widetilde{F}_4: & f = x^4 + y_1^2 x + y_2. & \end{array}$$

The list of simple quasi projections of singular hypersurfaces is

$$A_k^*,\ k \geqslant 0, \quad D_\ell^*,\ \ell \geqslant 4, \quad E_s^*,\ s = 6, 7, 8: \qquad f = x^2 + g(y_1, y_2)$$

where g is one of the standard simple ADE function germs in y,

$$\begin{array}{ll} A_2^{**}: & f = x^3 + y_1 x + y_2^2, \\ A_2^{(k)}: & f = x^3 + y_1^k x + y_1^2 + y_2^2,\ k \geq 2. \end{array}$$

3. Quasi equivalence relation of maps from $\mathbb{R}^m$ to $\mathbb{R}^n$

Consider a $\mathbf{C}^\infty$ map germ $F : (\mathbb{R}^m, 0) \to \mathbb{R}^n$, $x = (x_1, \ldots, x_m) \mapsto y = (y_1, \ldots, y_n)$, $y_i = f_i(x)$, where $f_i : (\mathbb{R}^m, 0) \to \mathbb{R}$ is a smooth function-germ. Denote by $\mathbb{C}^m_n$ the space of all such maps. Since $\mathbb{C}^m_n$ is a vector space, sometimes its elements will be written as column vectors:

$$f = (f_1, f_2, \ldots, f_n)^t = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Let Λ_F be the graph of F, that is $\Lambda_F = \{(x, y) : y_i = f_i(x), i = 1, 2, \ldots, n\} \subset \mathbb{R}^p$.

Definition 3.1. Two map germs F_0 and F_1 are called *quasi equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$, such that $\Phi(\Lambda_{F_1}) = \Lambda_{F_0}$ and the derivative of Φ preserves the direction of the projection at the points which lie on Λ_{F_1}.

Remarks 3.2.

1. The quasi-equivalence is an equivalence relation.
2. Clearly, if two map germs F_0 and F_1 are $\mathcal{A}$-equivalent then they are quasi-equivalent.

Denote by Q_F the quasi-equivalence class of a map germ F and call it a *quasi orbit*. Then, the tangent space TQ_F to Q_F has the following description.

Lemma 3.3. *TQ_F is the set of all expressions of the form*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix} \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_m \end{pmatrix} + \begin{pmatrix} \dot{Y}_1 \\ \dot{Y}_2 \\ \vdots \\ \dot{Y}_n \end{pmatrix},$$

where

$$\frac{\partial \dot{Y}_i}{\partial x_j} = \sum_{r=1}^{n} A_{ir} \frac{\partial f_r}{\partial x_j}, \quad \text{and} \quad \dot{X}_1, \dot{X}_2, \ldots, \dot{X}_m \in \mathbb{C}_x,$$

with $A_{ir} \in \mathbb{C}_x$ for all i and j.

Proof. Introduce a family Φ_t of diffeomorphism germs depending on a parameter $t \in [0,1]$ of the form

$$\Phi_t : (\mathbb{R}^m \times \mathbb{R}^n, 0) \to (\mathbb{R}^m \times \mathbb{R}^n, 0), w \mapsto \Big(X_1(t), \dots, X_m(t), Y_1(t), \dots, Y_n(t)\Big),$$

such that $\Phi_0 = id_{\mathbb{R}^m \times \mathbb{R}^n}$. Let $V_t = \sum_{i=1}^{m} \dot{X}_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \dot{Y}_i \frac{\partial}{\partial y_i}$ be the vector field generating Φ_t, where $\dot{X}_i = \frac{\partial X_i}{\partial t}$ and $\dot{Y}_i = \frac{\partial Y_i}{\partial t}$. Let $a_1 = \frac{\partial}{\partial x_1}, a_2 = \frac{\partial}{\partial x_2}, \dots, a_m = \frac{\partial}{\partial x_m}$ be the basis of the vector space $\mathbb{R}^m$. Then, the family of the vector fields Φ_t^* preserves the direction of the projection if the following relation is satisfied

$$\Phi_t^*(a_i) = \sum_{j=1}^{m} \lambda_j a_j, \tag{3.1}$$

where $\lambda_j \in \mathbb{C}_w$ and also depending on $t \in [0,1]$. Let $V_0 = \sum_{i=1}^{m} \dot{X}_i(0) \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \dot{Y}_i(0) \frac{\partial}{\partial y_i}$, where $\dot{X}_i(0) = \frac{\partial X_i}{\partial t}\big|_{t=0}$ and $\dot{Y}_i(0) = \frac{\partial Y_i}{\partial t}\big|_{t=0}$.

If we differentiate (3.1) with respect to t and substitute $t = 0$ then we obtain

$$[V_0, a_i] = \sum_{j=1}^{m} \lambda(0)_j a_j, \tag{3.2}$$

where $[.,.]$ is the Lie bracket and $\lambda_i(0) = \frac{\partial \lambda_i}{\partial t}\big|_{t=0}$. In fact, (3.2) is equivalent to

$$-\left(\sum_{r=1}^{m} \frac{\partial \dot{X}_r(0)}{\partial x_i} \frac{\partial}{\partial x_r} + \sum_{s=1}^{n} \frac{\partial \dot{Y}_s(0)}{\partial x_i} \frac{\partial}{\partial y_s}\right) = \sum_{j=1}^{m} \lambda(0)_j a_j. \tag{3.3}$$

Therefore, (3.3) implies that $\dot{X}_r(0) \in \mathbb{C}_w$ and $\frac{\partial \dot{Y}_s(0)}{\partial x_i} = 0$, for all r and s.

Now assume that all map germs in a smooth family F_t depending on $t \in [0,1]$ are quasi equivalent to F_0, with respect to Φ_t. Then, from Definition 3.1 we see that derivatives $\frac{\partial \dot{Y}_s(0)}{\partial x_i}$ belong to the radical of the ideal defining the graph Λ_0 of F_0. Therefore,

$$\frac{\partial \dot{Y}_s(0)}{\partial x_i} \in Rad(I),$$

where I is the ideal generated by $y_j - f_j$, $j = 1, 2, \dots, n$. Note that $Rad(I) = I$. Hence, we have

$$\frac{\partial \dot{Y}_s(0)}{\partial x_i} = \sum_{j=1}^{n} (y_j - f_j) B_{sj}, \tag{3.4}$$

where $B_{sj} \in \mathbb{C}_w$.

Denote by I^2 the square of the ideal I. Using the Hadamard Lemma, we can always write

$$\dot{Y}_s(0) = \widetilde{Y}_s + \sum_{j=1}^{n}(y_j - f_j)A_{sj} + \psi, \tag{3.5}$$

where $\widetilde{Y}_s \in \mathbb{C}_x$, $A_{sj} \in \mathbb{C}_w$ and $\psi \in I^2$. Differentiation of (3.5) with respect to x_i and using (3.5) followed by the restriction of $\frac{\partial \dot{Y}_s(0)}{\partial x_i}$ to the surface by setting $y_j = f_j$ yield that

$$\frac{\partial \widetilde{Y}_s}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i} \widetilde{A}_{sj}$$

where $\widetilde{A}_{sj} \in \mathbb{C}_x$, as required. □

Following [1], we call a map germ $F : (\mathbb{R}^m, 0) \to \mathbb{R}^n$ *simple* if its sufficiently small neighbourhood in the space of all map germs from $(\mathbb{R}^m, 0)$ to $\mathbb{R}^n$ contains only a finite number of quasi-equivalence classes.

3.1. Classification of simple mappings

We start this subsection with recalling the classification of simple singularities of quasi-mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ from [3], giving details of proofs of main results. After that, we classify simple irreducible curve singularities in $\mathbb{R}^m$ with respect to the quasi-stably equivalence relation.

3.1.1. Simple quasi classes of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$

Classification of simple quasi-singularities of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ is as follows.

Theorem 3.4. [3] *Let a map germ $F : (\mathbb{R}^2, 0) \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (y_1, y_2)$, be simple with respect to the quasi-equivalence relation. Then, F is quasi-equivalent to one of the following:*

Notation	*Normal form*	*Restrictions*
$\widetilde{A}_k$	$(x_2, x_1^{k+1} + x_1x_2)$	$k \geq 0$,
$\widetilde{B}_k$	$(x_2, x_1^3 + x_2^k x_1)$	$k \geq 2$
$\widetilde{C}_k$	$(x_2, x_1^{k+1} + x_1^2 x_2)$	$k \geq 2$
$\widetilde{F}_4$	$(x_2, x_1^4 + x_2^2 x_1)$	
$\mathcal{A}_2^{\pm}$	$(x_1^2 \pm x_2^2, x_1x_2)$	
$\mathcal{A}_3$	$(x_1x_2, x_1^2 + x_2^3)$	

To prove Theorem 3.4, we need the following auxiliary results.

We first treat the case when the co-rank of F is one. In this case and up to the $\mathcal{A}$-equivalence relation, we will assume that F has the form $\big(x_2, f\big)$, where $f \in \mathbb{M}_x^2$.

Let $F_t : (\mathbb{R}^2, 0) \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (x_2, f_t)$, be a family of quasi-equivalent map germs at the origin, preserving the first component, where $t \in [0,1]$ and $f_t \in \mathbb{M}_x^2$. Consider the family of regular germs $V_t = \big\{(x_1, y_1, y_2) : y_1 = x_2, y_2 = f_t\big\}$, equipped with trivial fibration structure $\pi : \mathbb{R}_{x_1} \times \mathbb{R}_y^2 \to \mathbb{R}_y^2$.

Lemma 3.5. *The quasi classifications of (x_2, f_t) reduces to the classifications of (V_t, π) with respect to the quasi-equivalence relation, introduced in Definition 2.4.*

Proof. Note that the $\dot{Y}_i$ summands in TQ_{F_t} satisfy the following

$$\frac{\partial \dot{Y}_i}{\partial x_1} = \frac{\partial f_t}{\partial x_1} B_i \tag{3.6}$$

and

$$\frac{\partial \dot{Y}_i}{\partial x_2} = A_i + \frac{\partial f_t}{\partial x_2} B_i, \tag{3.7}$$

for some $A_i, B_i \in \mathbb{C}_x$ and $i \in \{1, 2\}$. Since A_i is an arbitrary smooth function, (3.6) and (3.7) imply

$$\dot{Y}_i = D_i + \int_0^{x_1} \frac{\partial f_t}{\partial x_1} B_i \ dx_1,$$

where $D_i \in \mathbb{C}_{x_2}$. On the other hand, from the first row of the homological equation $-\frac{\partial F_t}{\partial t} = M$, $M \in TQ_{F_t}$, we have $\dot{Y}_1 = -\dot{X}_2$, where $\dot{X}_2 \in \mathbb{C}_x$. Hence, the second row takes the form

$$-\frac{\partial f_t}{\partial t} = \frac{\partial f_t}{\partial x_1} \dot{X}_1 - \frac{\partial f_t}{\partial y_1} \dot{Y}_1 + \dot{Y}_2, \tag{3.8}$$

where $\dot{X}_1 \in \mathbb{C}_x$. Note that the elements on the right side of (3.8) are exactly those belonging to the tangent space $T\widetilde{Q}_{f_t}$ at the regular germs (V_t, π) with respect to the quasi-equivalence relation, and the result follows. □

Now assume that F has co-rank 2. Then, using the $\mathcal{A}$-equivalence relation, one can show the following.

Lemma 3.6. *The adjacency of the 2-jets of map germs F is*

$$I^{\pm} : (x_1^2 \pm x_2^2, x_1 x_2) \leftarrow II : (x_1 x_2, x_1^2) \leftarrow (III)^{\pm} : (x_1^2 \pm x_2^2, 0) \leftarrow V : (x_1^2, 0) \leftarrow IV : (0, 0).$$

Remark 3.1. Classes in Lemma 3.6 remain non-quasi-equivalent.

Lemma 3.7. *1. If the 2-jet of F is equivalent to $(x_1^2 \pm x_2^2, 0)$ then F is non-simple with respect to the quasi equivalence relation.*

2. If the 4-jet of F is equivalent to $(x_1x_2, x_1^2 + \alpha x_1 x_2^2 + \beta x_2^4)$, $\alpha \neq 0, \beta \neq 0$, then F is non-simple with respect to the quasi-equivalence relation.

Proof. For the first part of the Lemma, consider the homogenous mapping $F_3 = (x_1^2 \pm x_2^2, f_3)$ where $f_3 = x_1^3 + \alpha x_1^2 x_2 + \beta x_1 x_2^2 + \gamma x_2^3$. Then, TQ_{F_3} is the set of all expressions of the form

$$\begin{pmatrix} 2x_1\dot{X}_1 \pm 2x_2\dot{X}_2 + \dot{Y}_1 \\ \frac{\partial f_3}{\partial x_1}\dot{X}_1 + \frac{\partial f_3}{\partial x_2}\dot{X}_2 + \dot{Y}_2 \end{pmatrix}, \tag{$*$}$$

where $\dot{X}_1, \dot{X}_2 \in \mathbb{C}_x$ and the $\dot{Y}_i$ summands satisfy the following constraints

$$\frac{\partial \dot{Y}_i}{\partial x_1} = 2x_1 A_i + \frac{\partial f_3}{\partial x_1} B_i \quad \text{and} \quad \frac{\partial \dot{Y}_i}{\partial x_2} = \pm 2x_2 A_i + \frac{\partial f_3}{\partial x_2} B_i,$$

for some $A_i, B_i \in \mathbb{C}_x$. Notice that the 3-jet of $\dot{Y}_i$ is $a_i(x_1^2 \pm x_2^2) + b_i f_3$, where $a_i, b_i \in \mathbb{R}$. Therefore, the 3-jet of TQ_{F_3} is generated by the vectors:

$$\begin{aligned} &v_1 = (2x_1^2, x_1\frac{\partial f_3}{\partial x_1}), v_2 = (2x_1x_2, x_2\frac{\partial f_3}{\partial x_1}), v_3 = (\pm 2x_2^2, x_2\frac{\partial f_3}{\partial x_2}),\\ &v_4 = (\pm 2x_1x_2, x_1\frac{\partial f_3}{\partial x_2}), v_5 = (0, f_3), v_6 = (x_1^3, 0), v_7 = (x_1^2x_2, 0),\\ &v_8 = (x_1x_2^2, 0), v_9 = (x_2^3, 0), v_{10} = (x_1^2 \pm x_2^2, 0), v_{11} = (0, x_1^2 \pm x_2^2). \end{aligned}$$

These vectors form a subspace of dimension at most 11. The dimension of the space of the 3-jets of co-rank 2 mappings is 14 which is greater than the subspace dimension. This means that the germ F_3 is non-simple with respect to the quasi equivalence relation.

Similarly, we can prove the second part of the Lemma. □

Proof of Theorem 3.4. Firstly, suppose that the co-rank of F is one. Then, Lemma 3.5 and Theorem 2.7 imply that if F is simple then it is quasi equivalent to one of the following: $(x_2, x_1^{k+1} + x_1x_2)$, $k \geq 0$, $(x_2, x_1^3 + x_2^k x_1)$, $k \geq 2$, $(x_2, x_1^{k+1} + x_1^2 x_2), k \geq 2$ and $(x_2, x_1^4 + x_2^2 x_1)$.

Next, let the co-rank of F be two. Then, Lemma 3.6 and Lemma 3.7 yield that all simple quasi singularities are among map germs whose 2-jets are quasi equivalent to either $(x_1^2 \pm x_2^2, x_1x_2)$ or (x_1x_2, x_1^2). Using Arnold's spectral sequence method [1], one can easily prove the results below.

- If F is a map germ with the 2-jet $(x_1^2 \pm x_2^2, x_1x_2)$, then F is quasi equivalent to
$\mathcal{A}_2^{\pm} : (x_1^2 \pm x_2^2, x_1x_2)$.
- Let $F = \big(x_1x_2 + f, x_1^2 + g\big)$, where $f, g \in \mathbb{M}_x^3$. If g contains a term ax_2, then F is quasi equivalent to $\mathcal{A}_3 : \big(x_1x_2, x_1^2 + x_2^3\big)$. Otherwise, in the most general case, F is equivalent to a non-simple germ, by Lemma 3.7. This finishes the proof of Theorem. □

3.1.2. Quasi-stably simple classes of irreducible curves in $\mathbb{R}^n$

Recall that an irreducible curve at the origin in $\mathbb{R}^n$ can be described by a germ of an analytic map $F : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$, $x \mapsto y = \big(y_1 = f_1(x), y_2 = f_2(x), \ldots, y_n = f_n(x)\big)$. Following Arnold in [2], we introduce the following.

Definition 3.8. An irreducible curve is called *quasi-stably simple* if it is simple with respect to the quasi-equivalence relation and remains simple when the ambient space is embedded into a larger space. Two curves which are obtained one from the other by such embedding are called *quasi-stably equivalent.*

Remark 3.2. By the codimension here and below, we mean the codimension in the space of the Taylor series with zero constant terms.

The classification of quasi-stably simple classes is as follows.

Theorem 3.9. *Assume that the curve F is quasi-stably simple. Then, F is quasi-stably equivalent to one of the lines* $\mathbb{A}_k : \big(x^k, 0\big)$, $k \geq 1$.

Remarks 3.10.

1. Any irreducible curve is either quasi-stably simple (and hence is quasi-stably equivalent to one of lines, stated in the theorem) or belongs to the subset of infinite codimension in the space of all curves.
2. The codimension of the class $\mathbb{A}_k$ is $kn - 1$.

Proof of Theorem 3.9. Up to the $\mathcal{A}$-equivalence relation, we may assume that any irreducible curve has the form $F = \big(x^k, f_2, \ldots, f_n\big)$, where $k \geq 1$ and $f_i \in \mathbb{M}_x^{k+1}$. Notice that the derivatives of the $\dot{Y}_i$ summands in TQ_F with respect to x belong to the ideal generated by x^{k-1} and hence $\dot{Y}_i = x^k A_i$, for some $A_i \in \mathbb{C}_x$. By Arnold's spectral sequence method, one can easily show that F is quasi-stably equivalent to the germ $\mathbb{A}_k : \big(x^k, 0\big)$, $k \geq 1$.

4. The quasi classification of some multi-germs of curves in $\mathbb{R}^n$

We start with recalling the standard notions and basic definitions concerning multi-germs of curves from [5].

A reducible curve at the origin in $\mathbb{R}^n$ is determined by a collection of maps

$$(\mathbb{R},0) \to (\mathbb{R}^n,0), x \mapsto y = (y_1,\dots,y_n).$$

Definitions 4.1. A multi-germ of curves in $\mathbb{R}^n$ is a set $G = (F_1,\dots,F_r)$ of germs of analytic maps $F_i : (\mathbb{R},0) \to (\mathbb{R}^n,0)$, where $\mathrm{Im}(F_i) \cap \mathrm{Im}\ (F_j) = \{0\}$ for $i \neq j$ ($F_1, F_2, \dots$ and F_r are called components of the multi-germ G).

The group of $\mathcal{A}$-equivalences $\mathcal{A} = \mathcal{L} \times \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_r$, where $\mathcal{R}_i$ is the i-th copy of the group of the standard right equivalences, acts on the space of multi-germs $G = (F_1,\dots,F_r)$ by the formula

$$(\phi,\varphi_1,\dots,\varphi_r).(F_1,\dots,F_r) = (\phi \circ F_1 \circ \varphi_1^{-1},\dots,\phi \circ F_r \circ \varphi_r^{-1}),$$

where $\phi \in \mathcal{L}$ and $\varphi_i \in \mathcal{R}_i$.

Definitions 4.2. A multi-germ G is called *simple* if there exists a neighbourhood of G in the space of multi-germs which intersects only the finite number of $\mathcal{A}$-orbits. It is *stably simple*, if it remains simple when the ambient space is immersed in a larger space.

Definitions 4.3. Two multi-germs G and $\widetilde{G}$ in $\mathbb{R}^n$ are equivalent if they lie in one orbit of the $\mathcal{A}$-action.

The tangent space $T\mathcal{A}.G$ to the orbit $\mathcal{A}.G$ is equal to $T\mathcal{R}.G + T\mathcal{L}.G$. The first set is the direct sum $\bigoplus_{i=1}^{r} \mathbb{M}_x(\frac{\partial F_i}{\partial x})$ and its elements denoted by matrices where the i-th column of which corresponds to an element of $T\mathcal{R}.F_i$. On the other hand, $T\mathcal{L}.G$ is the set of matrices of the form

$$\begin{bmatrix} \dot{Y}_{11} & \dot{Y}_{12} & \dots & \dot{Y}_{1r} \\ \dot{Y}_{21} & \dot{Y}_{22} & \dots & \dot{Y}_{2r} \\ \vdots & \vdots & \dots & \vdots \\ \dot{Y}_{n1} & \dot{Y}_{n2} & \dots & \dot{Y}_{nr} \end{bmatrix},$$

where $\dot{Y}_{ij} = U_i \circ F_j$ and $U_i \in \mathbb{M}_y$.

The quasi-equivalence of multi-germs of curves is defined as follows.

Let $F_j : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$, $x \mapsto y = (y_1, \ldots, y_n)$, $y_i = f_{ij}(x)$, $i = 1, \ldots, n$ and denote by Λ_j its graph.

Definition 4.4. Two multi-germs $G = (F_1, \ldots, F_r)$ and $\widetilde{G} = (\widetilde{F}_1, \ldots, \widetilde{F}_r)$ in $\mathbb{R}^n$ are called *quasi equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R} \times \mathbb{R}^n, 0)$, such that $\Phi(\Lambda_j) = \widetilde{\Lambda}_j$, for all j, and the derivative of Φ preserves the direction of the projection at the points which lie on Λ_j.

Obviously, the quasi-equivalence of multi-germs of curves is an equivalence relation. By similar consideration and technique which were used in the proof of Lemma 3.3, we obtain the following description of the tangent space $TQ.G$ to the quasi class $Q.G$ of a multi-germ G.

Lemma 4.5. *$TQ.G = T\mathcal{R}.G + T\mathcal{Q}.G$, where $T\mathcal{R}.G = \bigoplus_{i=1}^{r} \mathbb{M}_x(\frac{\partial F_i}{\partial x})$ and $T\mathcal{Q}.G$ is the set of matrices of the form*

$$\begin{bmatrix} \dot{Y}_{11} & \dot{Y}_{12} & \ldots & \dot{Y}_{1r} \\ \dot{Y}_{21} & Y_{22} & \ldots & \dot{Y}_{2r} \\ \vdots & \vdots & \ldots & \vdots \\ \dot{Y}_{n1} & \dot{Y}_{n2} & \ldots & \dot{Y}_{nr} \end{bmatrix}.$$

which satisfy the following

$$\begin{bmatrix} \dot{Y}'_{11} & \dot{Y}'_{12} & \ldots & \dot{Y}'_{1r} \\ \dot{Y}'_{21} & \dot{Y}'_{22} & \ldots & \dot{Y}'_{2r} \\ \vdots & \vdots & \ldots & \vdots \\ \dot{Y}'_{n1} & \dot{Y}'_{n2} & \ldots & \dot{Y}'_{nr} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{bmatrix} \begin{bmatrix} f'_{11} & f'_{12} & \ldots & f'_{1r} \\ f'_{21} & f'_{22} & \ldots & f'_{2r} \\ \vdots & \vdots & \ldots & \vdots \\ f'_{n1} & f'_{n2} & \ldots & f'_{nr} \end{bmatrix}.$$

where $A_{ij} \in \mathbb{C}_x$, $f'_{ij} = \frac{df_{ij}}{dx}$ and $\dot{Y}'_{ij} = \frac{d\dot{Y}_{ij}}{dx}$.

Proposition 4.6. *$T\mathcal{A}.G \subset TQ.G$.*

Proof. Let $V \in T\mathcal{A}.G$. Then, we can write $V = V_1 + V_2$, where $V_1 \in T\mathcal{R}.G$ and $V_2 \in T\mathcal{L}.G$. Hence, $V_2 = (\dot{Y}_{ij})$, where $\dot{Y}_{ij} = U_i \circ F_j$ and $U_i \in \mathbb{M}_y$. Notice that $\frac{d\dot{Y}_{ij}}{dx} = \sum_{k=1}^{n} \frac{df_{kj}}{dx}\frac{\partial U_i}{\partial y_k}$. Moreover, if we let $F'_j = (f'_{1j}, f'_{2j}, \ldots, f'_{2n})$, where $f'_{ij} = \frac{df_{ij}}{dx}$ and denote by $(F'_j)^T$ the transpose of F'_j, then we have

$$\sum_{k=1}^{n} \frac{df_{kj}}{dx}\frac{\partial U_i}{\partial y_k} = A_i(F'_j)^T,$$

where $A_i = (A_{i1}, A_{i2}, \dots, A_{in})$ with $A_{ik} = \frac{\partial U_i}{\partial y_k}$, $f'_{kj} = \frac{df_{kj}}{dx}$ and the result follows. $\square$

Remark 4.1. For the standard $\mathcal{A}$-equivalences of multi-germs, we are free to change the coordinates about each point independently of the associated branch in the source, whereas in the target the same coordinate change must be applied to each branch. On the other hand, for the quasi-equivalence, we are still free to change the coordinates in the source about each point independently of the associated branch, but in the target if a quasi-change of the coordinates Y_{ij} occurs on a certain branch F_j and the derivative of $\dot{Y}_{ij}$ is equal to $A_i(F'_j)^T$, then the same factor A_i must be applied to all quasi-changes of the coordinates on other branches.

Definition 4.7. A multi-germ G is called *simple* with respect to the quasi-equivalence relation if there exists a neighbourhood of G in the space of multi-germs which intersects only finite number of quasi-classes. Moreover, it is called *quasi-stably simple* if it remains simple when the ambient space is immersed in a larger space.

We will only consider bi-germs (multi-germs with two components) of curves and give the beginning of the classifications with respect to the quasi-equivalence relation.

Theorem 4.8. *Let G be a quasi-stably simple bi-germ. Then, up to permutation of curves, G is quasi-equivalent to one of the bi-germs $\big(F_1, F_2\big)$, described in the following table.*

Notation	F_1	F_2	*Restrictions*
$\mathcal{A}_k$	$(x, 0)$	$(0, x^k)$	$k \geq 1$
$\mathcal{B}_{k,l}$	$(x, 0)$	(x^k, x^l)	$l > k \geq 1$
$\mathcal{C}_2$	$(x^2, 0)$	$(0, x^2)$	
$\mathcal{C}_3$	$(x^2, 0)$	$(0, 0, x^3)$	
$\mathcal{D}_{2,3}$	$(x^2, 0)$	$(x^2, 0, x^3)$	

To prove Theorem 4.8, we use the spectral sequence method [1] together with the following auxiliary results.

Consider a pair G of curves with a regular first component which will be written in the normal form $(x, 0, \dots, 0)$ or equivalently as $(x, 0)$. Introduce a family of quasi-equivalent pairs $G_t = \big((x, 0), F_2(t)\big)$, preserving the first component, where $F_2(t) = \big(f_1(t), f_2(t), \dots, f_m(t)\big)$, $f_i \in \mathbb{C}_x$ and $t \in [0, t]$ such that $G_0 = G$. Let $f'_i = \frac{df_i}{dx}$ and denote by Ω the ideal generated by $f'_1, f'_2, f'_3, \dots, f'_n$, and by $\widetilde{\Omega}$ the ideal generated by $f'_2, f'_3, \dots, f'_n$.

Lemma 4.9. *The homological equation of G_t is*

$$\begin{bmatrix} 0 & \dot{f}_1 \\ 0 & \dot{f}_2 \\ \vdots & \vdots \\ 0 & \dot{f}_n \end{bmatrix} = \begin{bmatrix} H_1 & f_1' H_2 \\ 0 & f_2' H_2 \\ \vdots & \vdots \\ 0 & f_n' H_2 \end{bmatrix} + \begin{bmatrix} \dot{Y}_{11} & \dot{Y}_{12} \\ \dot{Y}_{21} & \dot{Y}_{22} \\ \vdots & \vdots \\ \dot{Y}_{n1} & \dot{Y}_{n2} \end{bmatrix},$$

such that $\dot{Y}_{11} \in \mathbb{M}_x$, $\dot{Y}_{i1} = 0$, $\dot{Y}_{12}' \in \Omega$, and $\dot{Y}_{i2}' \in \widetilde{\Omega}$ for all $i \in \{2, 3, \ldots, n\}$. Here, $\dot{f}_i = \frac{df_i}{dt}$ and $H_1, H_2 \in \mathbb{M}_x$.

Proof. By differentiating G_t with respect to t, we obtain the homological equation described in Lemma. Moreover, Lemma 4.5 implies that

$$\begin{bmatrix} \dot{Y}_{11}' & \dot{Y}_{12}' \\ \dot{Y}_{21}' & \dot{Y}_{22}' \\ \vdots & \vdots \\ \dot{Y}_{n1}' & \dot{Y}_{n2}' \end{bmatrix} = \begin{bmatrix} A_{11} & \sum_{k=1}^{n} A_{1k} f_k' \\ A_{21} & \sum_{k=1}^{n} A_{2k} f_k' \\ \vdots & \vdots \\ A_{n1} & \sum_{k=1}^{n} A_{nk} f_k' \end{bmatrix}. \tag{4.1}$$

Comparing the columns of the homological equation and (4.1) yields that $\dot{Y}_{11} = -H_1$, $\dot{Y}_{i1} = 0$, and therefore $A_{11} = -\frac{dH_1}{dx}$, $A_{i1} = 0$ for all $i \in \{2, 3, \ldots, m\}$. As H_1 is an arbitrary germ, we have that $\dot{Y}_{12}' \in \Omega$ and $\dot{Y}_{i2}' \in \widetilde{\Omega}$ for all $i \in \{2, 3, \ldots, n\}$, as required. □

Now suppose that both components of G are singular. Then,

Lemma 4.10. [5] *The 2-jet of G is $\mathcal{A}$-equivalent to either $\big((x^2, 0), (0, x^2)\big)$ or $\big((x^2, 0), (x^2, 0)\big)$.*

Moreover,

Lemma 4.11. *A pair of curves with the 3-jet $\big((x^2, x^3), (x^2, \alpha x^3)\big)$, where $\alpha \neq 1$, is not simple with respect to quasi-equivalence.*

Proof. Let $G_\alpha = \big((x^2, x^3), (x^2, \alpha x^3)\big)$. Then, the 3-jet in $TQ.G_\alpha$ is generated by the following 10 vectors: $v_1 = \big((2x^2, 3x^3), (0, 0)\big)$, $v_2 = \big((0, 0), (2x^2, 3\alpha x^3)\big)$, $v_3 = \big((2x^3, 0), (0, 0)\big)$, $v_4 = \big((0, 0), (2x^3, 0)\big)$, $v_5 = \big((x^2, 0), (x^2, 0)\big)$, $v_6 = \big((0, x^2), (0, x^2)\big)$, $v_7 = \big((2x^3, 0), (2x^3, 0)\big)$, $v_8 = \big((0, 2x^3), (0, 2x^3)\big)$, $v_9 = \big((x^3, 0), (\alpha x^3, 0)\big)$, $v_{10} = \big((0, x^3), (0, \alpha x^3)\big)$. Notice that $v_3 + v_4 = v_7$, $2av_1 + 2v_2 - 4\alpha v_5 = 3\alpha v_8$ and $v_1 + v_2 - 2v_5 = 3v_9$. Therefore, the vectors v_7, v_8 and v_9 can be removed from the list above. The remaining vectors form a subspace of dimension at most 7. The dimension of the space of all 3-jets of bi-germs with two singular components is 8 which is greater than the subspace dimension. This means that the germ G_α is non-simple. □

4.1. Proof of the main Theorem 4.8

We distinguish the following cases.

1. Pairs of curves with a regular first component. In this case the pair takes the form $G = ((x,0),F)$. Therefore, we classify the second component using Lemma 4.9 as follows.
   - Assume the 1-jet of F is nontrivial and equal to $(\alpha x, \beta x)$, with $\alpha, \beta \in \mathbb{R}$, and hence is equivalent to either $(0,x)$ or $(x,0)$. Consider the first case. Then, G is quasi equivalent to $\mathcal{A}_1 : \big((x,0),(0,x)\big)$. Next, if k be the minimal number such that the k-jet of F is not $(x,0)$ then G is quasi-equivalent to $\mathcal{B}_{1,k} : \big((x,0),(x,x^k)\big)$ where $k \geq 2$.
   - Consider the case when F is singular and its multiplicity is k. Then, the k-jet of F is equivalent to either $(0,x^k)$ or $(x^k,0)$. Suppose that l is the minimal number such that the l-jet of F is not $(x^k,0)$ then G is quasi equivalent to $\mathcal{B}_{k,l} : \big((x,0),(x^k,x^l)\big)$ where $l > k \geq 2$. Next, if the k-jet of F is $(0,x^k)$ then G is quasi-equivalent to $\mathcal{A}_k : \big((x,0),(0,x^k)\big)$, with $k \geq 2$.
2. Pairs of curves with singular components. In this case the nontrivial 2-jet of G is equivalent to either $\big((x^2,0),(0,x^2)\big)$ or $\big((x^2,0),(x^2,0)\big)$.
   - Consider the case when the 2-jet is $\big((x^2,0),(0,x^2)\big)$. Then, G is quasi-equivalent $\mathcal{C}_2 : \big((x^2,0),(0,x^2)\big)$.
   - If the 2-jet is $\big((x^2,0),(x^2,0)\big)$ then Lemma 4.11 yields that all quasi-stably simple singularities are among pairs with the 3-jet is either $\big((x^2,x^3,0),(x^2,0,x^3)\big)$ or $\big((x^2,x^3,0),(0,0,x^3)\big)$. In such cases, we obtain $\mathcal{C}_3 : \big((x^2,0),(0,0,x^3)\big)$ and $\mathcal{D}_{2,3} : \big((x^2,0),(x^2,0,x^3)\big)$, respectively. Pairs from other cases are adjacent to the family $\big((x^2,x^3),(x^2,\alpha x^3)\big)$, where $\alpha \neq 1$.

REFERENCES

1. V. I. Arnold and S. M. Gusein-Zade and A. N. Varchenko: *Singularities of differentiable maps.* Vol. I. Monographs in Mathematics 82, BirkhäuserBoston, Boston, 1985.
2. V. I. Arnold: *Simple singularities of curves.* Trudy Mat. Inst. Steklov. 226 (1999), 27-35; English transl., Proc. Steklov Inst. Math
3. F. D. Alharbi: *Classification of singularities of functions and mappings via non standard equivalence relations.* PhD Thesis, University of Liverpool, 2011, 256pp.
4. F. D. Alharbi: *Quasi cusp singularities.* The Journal of Singularities, volume 12(2015), 1–18.
5. P. A. Kolgushkin and R. R. Sadykov: *Simple singularities of multigerms of curves.* Revista Matematica Complutense (2001) vol. XIV, num. 2, 311-344.

6. J. W. Bruce and T. J. Gaffney: *Simple singularities of mappings* $\mathbf{C}, 0 \to \mathbf{C}^2, 0$. Proceedings of the London Mathematical Society (2) **26** (1982), no. 3, 465–474.
7. J. Damon: *The unfolding and determinacy theorems for subgroups of A and K.* Proceedings of the American Mathematical Society **50** (1984), no. 306, 233-254.
8. V. V. Goryunov: *Singularities of projections of complete intersections.* (Russian) Current problems in mathematics, Vol. 22, Itogi Nauki i Tekhniki, VINITI, Moscow, 1983, 167–206; English translation in Journal of Soviet Mathematics **27** (1984), no. 3, 2785–2811.
9. V. M. Zakalyukin: *Quasi-projections.* Proceedings of the Steklov Institute of Mathematics 259 (2007), no. 1, 273–280.

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