

## GENERALIZED BERNSTEIN-KANTOROVICH OPERATORS OF BLENDING TYPE

Arun Kajla

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**Abstract.** In this note, we derive some approximation properties of the generalized Bernstein-Kantorovich-type operators based on two nonnegative parameters considered by A. Kajla [Appl. Math. Comput. 2018]. We establish a Voronovskaja-type asymptotic theorem for these operators. The rate of convergence for differential functions whose derivatives are of bounded variation is also derived. Finally, we show the convergence of the operators to certain functions by illustrative graphics using Mathematica software.

**Keywords:** Approximation; Bernstein-Kantorovich type operators; convergence.

### 1. Introduction

For  $f \in C(I)$ , with  $I = [0, 1]$ , the classical Bernstein polynomials are defined as follows:

$$\mathcal{B}_n(f; x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where  $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$  is the Bernstein basis.

Also for  $f : I \rightarrow \mathbb{R}$  an integrable function, the classical Bernstein-Kantorovich operators are defined by

$$M_n(f; x) = n \sum_{k=0}^n p_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t) dt, \quad x \in [0, 1], \quad n \in \mathbb{N}.$$

The above operators  $M_n$  can also be written as follows:

$$(1.1) \quad M_n(f; x) = \sum_{k=0}^n p_{n,k}(x) \int_0^1 f\left(\frac{k+t}{n}\right) dt.$$

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Stancu [31] introduced the Bernstein-type operators involving two parameters  $r, s \in \mathbb{N} \cup \{0\}$ , as follows:

$$(1.2) \quad (S_{n,r,s})f(x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{k=0}^s p_{s,k}(x) f\left(\frac{\mu+kr}{n}\right).$$

For  $r = s = 0$ , these operators reduces to Bernstein operators  $\mathcal{B}_n(f; x)$ .

Abel and Heilmann [1] investigated the complete asymptotic expansion of Bernstein-Durrmeyer operators. Gonska and Paltanea [16] presented genuine Bernstein-Durrmeyer operators based on one parameter family of linear positive operators and study the simultaneous approximation for these operators. Cárdenas-Morales and Gupta [12] derived a two-parameter family of Bernstein-Durrmeyer-type operators based on the Polya distribution and gave a Voronovskaja-type asymptotic theorem. In [9], Agrawal et al. introduced the Kantorovich-type generalization of Luaps operators and obtained the local and global approximation properties of these operators. Abel et al. [2] considered the Durrmeyer-type modification of the operators (1.2) defined by

$$(1.3) \quad \mathcal{S}_{n,r,s}(f; x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{k=0}^s p_{s,k}(x) (n+1) \int_0^1 p_{n,\mu+kr}(t) f(t) dt.$$

The authors studied a complete asymptotic expansion and derived some basic approximation theorems for these operators. Gupta et al. [18] considered the Durrmeyer variant of Baskakov operators based on the inverse Pòlya-Eggenberger distribution and studied the local and global approximation properties. Many researchers have contributed to this area of approximation theory [cf. [3–8, 10, 11, 13–15, 17–20, 22, 24–30] etc.] and the references therein.

For  $f \in C(I)$ , Kajla [23] defined the following Stancu-Kantorovich-type operators based on two nonnegative parameters:

$$(1.4) \quad \mathcal{K}_{n,r,s}(f; x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{k=0}^s p_{s,k}(x) \int_0^1 f\left(\frac{\mu+kr+t}{n}\right) dt.$$

The approximation behaviour of  $\mathcal{K}_{n,r,s}$  was examined in the paper [23].

In this article, we prove the Voronovskaja-type asymptotic theorem for these operators. The rate of convergence for differential functions whose derivatives are of bounded variation is also obtained. Finally, we show the convergence of the operators by illustrative graphics in Mathematica software to certain functions.

Let  $e_i(x) = x^i, i = 0, 1, 2 \dots$

**Lemma 1.1.** [23] For the operators  $\mathcal{K}_{n,r,s}(f; x)$ , we have

$$(i) \quad \mathcal{K}_{n,r,s}(e_0; x) = 1;$$

$$\begin{aligned}
 (ii) \quad & \mathcal{K}_{n,r,s}(e_1; x) = x + \frac{1}{2n}; \\
 (iii) \quad & \mathcal{K}_{n,r,s}(e_2; x) = x^2 + \frac{x(1-x)}{n} \left( 1 + \frac{sr(r-1)}{n} \right) + \frac{x}{n} + \frac{1}{3n^2}; \\
 (iv) \quad & \mathcal{K}_{n,r,s}(e_3; x) = x^3 + \frac{3x(3-2x)}{2n} + \frac{x(7-9x) + 6rsx^2(r-1)(1-x) + 4x^3}{2n^2} \\
 & + \frac{1 - 2rsx \left( (5 + 9x - 4x^2) - 3r(1-x^2) - 2r^2(1-3x+8x^2) \right)}{4n^3}; \\
 (v) \quad & \mathcal{K}_{n,r,s}(e_4; x) = x^4 + \frac{x^4}{5n^4} \left[ 55n^2 - 30n^3 + 30n^2rs - 30(-1+r)rs - 30n^2r^2s + \right. \\
 & \left. 15(r-1)r^2(s-2)s - 15(r-1)r^2s(s+2) + 10n(-3+(r-1)r(7+4r)s) \right] + \\
 & \frac{x^3}{5n^4} \left[ 40n^3 - 80n - 120n^2 - 30n^2rs + 80(r-1)rs + 30n^2r^2s + 50(r-1)r^2s - \right. \\
 & \left. 30n(r-1)rs(2r+5) - 30r^3s(r-1)(s-2) + 15r^2s^2(r-1) + 15r^2s^2(r-1)(s+2) \right] \\
 & + \frac{x^2}{5n^4} \left[ 75n^2 - 75n - 75rs(r-1) - 65r^2s(r-1) - 5r^3s(r-1) + 20nrs(r-1) + \right. \\
 & \left. 15r^3s(s-2) - 15r^2s^2(r-1) \right] \\
 & + \frac{x}{5n^4} \left[ 30n + 25rs(r-1) + 15r^2s(r-1) + 5r^3s(r-1) \right] + \frac{1}{5n^4}.
 \end{aligned}$$

Let  $e_i^x(t) = (t-x)^i, i = 1, 2, 4$ .

**Lemma 1.2.** [23] For the operators  $\mathcal{K}_{n,r,s}(f; x)$ , we get

$$\begin{aligned}
 (i) \quad & \mathcal{K}_{n,r,s}(e_1^x(t); x) = \frac{1}{2n}; \\
 (ii) \quad & \mathcal{K}_{n,r,s}(e_2^x(t); x) = \frac{x(1-x)}{n} \left( 1 + \frac{sr(r-1)}{n} \right) + \frac{1}{3n^2}.
 \end{aligned}$$

**Lemma 1.3.** [23] For  $f \in C(I)$ , we have

$$\|\mathcal{K}_{n,r,s}(f; x)\| \leq \|f\|.$$

**Remark 1.1.** For every  $x \in I$ , we have

$$\begin{aligned}
 \lim_{n \rightarrow \infty} n \mathcal{K}_{n,r,s}(e_1^x(t); x) &= \frac{1}{2}, \\
 \lim_{n \rightarrow \infty} n \mathcal{K}_{n,r,s}(e_2^x(t); x) &= x(1-x), \\
 \lim_{n \rightarrow \infty} n^2 \mathcal{K}_{n,r,s}(e_4^x(t); x) &= 3x^2(1-x)^2.
 \end{aligned}$$

**Lemma 1.4.** For  $n \in \mathbb{N}$ , we obtain

$$\mathcal{K}_{n,r,s}(e_2^x(t); x) \leq \frac{\mathcal{X}_{r,s} x(1-x)}{n},$$

where  $\mathcal{X}_{r,s}$  is a positive constant depending only on  $r, s$ .

**Theorem 1.1.** [23] Let  $f \in C(I)$ . Then  $\lim_{n \rightarrow \infty} \mathcal{K}_{n,r,s}(f; x) = f(x)$ , uniformly in  $I$ .

## 2. Voronovskaja type theorem

The aim of this section, we prove the Voronovskaja-type theorem for the operators  $\mathcal{K}_{n,r,s}$ .

**Theorem 2.1.** Let  $f \in C(I)$ . If  $f''$  exists at a point  $x \in I$ , then we have

$$\lim_{n \rightarrow \infty} n [\mathcal{K}_{n,r,s}(f; x) - f(x)] = \frac{1}{2}f'(x) + \frac{x(1-x)}{2}f''(x).$$

*Proof.* By Taylor's formula of  $f$ , we get

$$(2.1) \quad f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varpi(t,x)(t-x)^2,$$

where  $\lim_{t \rightarrow x} \varpi(t,x) = 0$ . By applying the linearity of the operator  $\mathcal{K}_{n,r,s}$ , we obtain

$$\begin{aligned} \mathcal{K}_{n,r,s}(f; x) - f(x) &= \mathcal{K}_{n,r,s}((t-x); x)f'(x) + \frac{1}{2}\mathcal{K}_{n,r,s}((t-x)^2; x)f''(x) \\ &\quad + \mathcal{K}_{n,r,s}(\varpi(t,x)(t-x)^2; x). \end{aligned}$$

Now, applying the Cauchy-Schwarz property, we can get

$$n\mathcal{K}_{n,r,s}(\varpi(t,x)(t-x)^2; x) \leq \sqrt{\mathcal{K}_{n,r,s}(\varpi^2(t,x); x)} \sqrt{n^2\mathcal{K}_{n,r,s}((t-x)^4; x)}.$$

From Theorem 1.1, we have  $\lim_{n \rightarrow \infty} \mathcal{K}_{n,r,s}(\varpi^2(t,x); x) = \varpi^2(x,x) = 0$ , since  $\varpi(t,x) \rightarrow 0$  as  $t \rightarrow x$ , and Remark 1.1 for every  $x \in I$ , we may write

$$(2.2) \quad \lim_{n \rightarrow \infty} n^2\mathcal{K}_{n,r,s}((t-x)^4; x) = 3x^2(1-x)^2.$$

Hence,

$$n\mathcal{K}_{n,r,s}(\varpi(t,x)(t-x)^2; x) = 0.$$

Applying Remark 1.1, we get

$$(2.3) \quad \begin{aligned} \lim_{n \rightarrow \infty} n\mathcal{K}_{n,r,s}(t-x; x) &= \frac{1}{2}, \\ \lim_{n \rightarrow \infty} n\mathcal{K}_{n,r,s}((t-x)^2; x) &= x(1-x). \end{aligned}$$

Collecting the results from the above theorem is completed.  $\square$

### 3. Rate of convergence

$DBV(I)$  denotes the class of all absolutely continuous functions  $f$  defined on  $I$ , having on  $I$  a derivative  $f'$  equivalent to a function of bounded variation on  $I$ . We notice that the functions  $f \in DBV(I)$  possess a representation

$$f(x) = \int_0^x g(t)dt + f(0)$$

where  $g \in BV(I)$ , i.e.,  $g$  is a function of bounded variation on  $I$ .

The operators  $\mathcal{K}_{n,r,s}(f; x)$  also admit the integral representation

$$(3.1) \quad \mathcal{K}_{n,r,s}(f; x) = \int_0^1 \mathcal{W}_{n,r,s}(x, t)f(t)dt,$$

where the kernel  $\mathcal{W}_{n,r,s}(x, t)$  is given by

$$\mathcal{W}_{n,r,s}(x, t) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{k=0}^s p_{s,k}(x)\chi_{n,k}(t),$$

where  $\chi_{n,k}(t)$  is the characteristic function of the interval  $[k/n, (k + 1)/n]$  with respect to  $I$ .

**Lemma 3.1.** For a fixed  $x \in (0, 1)$  and sufficiently large  $n$ , we have

$$(i) \quad \beta_{n,r,s}(x, y) = \int_0^y \mathcal{W}_{n,r,s}(x, t)dt \leq \frac{\mathcal{X}_{r,s} \ x(1-x)}{n(x-y)^2}, \quad 0 \leq y < x,$$

$$(ii) \quad 1 - \beta_{n,r,s}(x, z) = \int_z^1 \mathcal{W}_{n,r,s}(x, t)dt \leq \frac{\mathcal{X}_{r,s} \ x(1-x)}{n(x-y)^2}n(z-x)^2, \quad x < z < 1.$$

*Proof.* (i) Using Lemma 1.2 we get

$$\begin{aligned} \beta_{n,r,s}(x, y) &= \int_0^y \mathcal{W}_{n,r,s}(x, t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 \mathcal{W}_{n,r,s}(x, t)dt \\ &= \mathcal{K}_{n,r,s}((t-x)^2; x)(x-y)^{-2} \leq \frac{\mathcal{X}_{r,s} \ x(1-x)}{n(x-y)^2}. \end{aligned}$$

As the proof of (ii) is similar, the details are omitted.  $\square$

**Theorem 3.1.** Let  $f \in DBV(I)$ . Then for every  $x \in (0, 1)$  and sufficiently large

$n$ , we have

$$\begin{aligned}
 |D_n^{*(1/n)}(f; x) - f(x)| &\leq \frac{|f'(x+) + f'(x-)|}{4n} + \sqrt{\frac{\mathcal{X}_{r,s}}{n} \frac{x(1-x)}{n}} \frac{|f'(x+) - f'(x-)|}{2} \\
 &+ \frac{\mathcal{X}_{r,s}}{n} \frac{(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f'_x) + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f'_x) \\
 &+ \frac{\mathcal{X}_{r,s}}{n} \frac{x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+((1-x)/k)} (f'_x) + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f'_x),
 \end{aligned}$$

where  $\bigvee_a^b(f'_x)$  denotes the total variation of  $f'_x$  on  $[a, b]$  and  $f'_x$  is defined by

$$(3.2) \quad f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+), & x < t < 1. \end{cases}$$

*Proof.* Since  $\mathcal{K}_{n,r,s}(1; x) = 1$ , by using Lemma 1.1, for every  $x \in (0, 1)$  we get

$$\begin{aligned}
 \mathcal{K}_{n,r,s}(f; x) - f(x) &= \int_0^1 \mathcal{W}_{n,r,s}(x, t)(f(t) - f(x))dt \\
 (3.3) \quad &= \int_0^1 \mathcal{W}_{n,r,s}(x, t) \int_x^t f'(u)dudt.
 \end{aligned}$$

For any  $f \in DBV(I)$ , by (3.2) we can write

$$\begin{aligned}
 f'(u) &= f'_x(u) + \frac{1}{2}(f'(x+) + f'(x-)) + \frac{1}{2}(f'(x+) - f'(x-))sgn(u - x) \\
 (3.4) \quad &+ \delta_x(u)[f'(u) - \frac{1}{2}(f'(x+) + f'(x-))],
 \end{aligned}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x \\ 0, & u \neq x. \end{cases}$$

Obviously,

$$\int_0^1 \left( \int_x^t \left( f'(u) - \frac{1}{2}(f'(x+) + f'(x-)) \right) \delta_x(u)du \right) \mathcal{W}_{n,r,s}(x, t)dt = 0.$$

By (3.1) and a straightforward calculation we have

$$\begin{aligned}
 \int_0^1 \left( \int_x^t \frac{1}{2}(f'(x+) + f'(x-))du \right) \mathcal{W}_{n,r,s}(x, t)dt &= \frac{1}{2}(f'(x+) + f'(x-)) \int_0^1 (t-x)\mathcal{W}_{n,r,s}(x, t)dt \\
 &= \frac{1}{2}(f'(x+) + f'(x-))\mathcal{K}_{n,r,s}((t-x); x)
 \end{aligned}$$

$$\begin{aligned}
 & \left| \int_0^1 \mathcal{W}_{n,r,s}(x,t) \left( \int_x^t \frac{1}{2} (f'(x+) - f'(x-)) \operatorname{sgn}(u-x) du \right) dt \right| \\
 & \leq \frac{1}{2} |f'(x+) - f'(x-)| \int_0^1 |t-x| \mathcal{W}_{n,r,s}(x,t) dt \\
 & \leq \frac{1}{2} |f'(x+) - f'(x-)| \mathcal{K}_{n,r,s}(|t-x|; x) \\
 & \leq \frac{1}{2} |f'(x+) - f'(x-)| \left( \mathcal{K}_{n,r,s}((t-x)^2; x) \right)^{1/2}.
 \end{aligned}$$

Applying the lemmas 1.2 and 1.4 and using (3.3),(3.4) we obtain the following estimate

$$\begin{aligned}
 |\mathcal{K}_{n,r,s}(f; x) - f(x)| & \leq \frac{1}{4n} |f'(x+) + f'(x-)| \\
 & \quad + \frac{1}{2} |f'(x+) - f'(x-)| \sqrt{\frac{\mathcal{X}_{r,s} x(1-x)}{n}} \\
 & \quad + \left| \int_0^x \left( \int_x^t f'_x(u) du \right) \mathcal{W}_{n,r,s}(x,t) dt \right. \\
 (3.5) \quad & \quad \left. + \int_x^1 \left( \int_x^t f'_x(u) du \right) \mathcal{W}_{n,r,s}(x,t) dt \right|.
 \end{aligned}$$

Let

$$\begin{aligned}
 A_{n,r,s}(f'_x, x) & = \int_0^x \left( \int_x^t f'_x(u) du \right) \mathcal{W}_{n,r,s}(x,t) dt, \\
 B_{n,r,s}(f'_x, x) & = \int_x^1 \left( \int_x^t f'_x(u) du \right) \mathcal{W}_{n,r,s}(x,t) dt.
 \end{aligned}$$

To complete the proof, it is sufficient to estimate the terms  $A_{n,r,s}(f'_x, x)$  and  $B_{n,r,s}(f'_x, x)$ . Since  $\int_a^b d_t \beta_{n,r,s}(x,t) \leq 1$  for all  $[a, b] \subseteq [0, 1]$ , using integration by parts and applying Lemma 3.1 with  $y = x - (x/\sqrt{n})$ , we have

$$\begin{aligned}
 |A_{n,r,s}(f'_x, x)| & = \left| \int_0^x \left( \int_x^t f'_x(u) du \right) d_t \beta_{n,r,s}(x,t) \right| \\
 & = \left| \int_0^x \beta_{n,r,s}(x,t) f'_x(t) dt \right| \\
 & \leq \left( \int_0^y + \int_y^x \right) |f'_x(t)| |\beta_{n,r,s}(x,t)| dt \\
 & \leq \frac{\mathcal{X}_{r,s} x(1-x)}{n} \int_0^y \bigvee_t^x (f'_x)(x-t)^{-2} dt + \int_y^x \bigvee_t^x (f'_x) dt \\
 & \leq \frac{\mathcal{X}_{r,s} x(1-x)}{n} \int_0^y \bigvee_t^x (f'_x)(x-t)^{-2} dt + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f'_x).
 \end{aligned}$$

By the substitution of  $u = x/(x - t)$ , we obtain

$$\begin{aligned} \frac{\mathcal{X}_{r,s}}{n} \frac{x(1-x)}{n} \int_0^{x-(x/\sqrt{n})} (x-t)^{-2} \bigvee_t^x (f'_x) dt &= \frac{\mathcal{X}_{r,s}}{n} \frac{(1-x)}{n} \int_1^{\sqrt{n}} \bigvee_{x-(x/u)}^x (f'_x) du \\ &\leq \frac{\mathcal{X}_{r,s}}{n} \frac{(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-(x/k)}^x (f'_x) du \\ &\leq \frac{\mathcal{X}_{r,s}}{n} \frac{(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f'_x). \end{aligned}$$

Thus,

$$(3.6) \quad |A_{n,r,s}(f'_x, x)| \leq \frac{\mathcal{X}_{r,s}}{n} \frac{(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f'_x) + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f'_x).$$

Using integration by parts and applying Lemma 3.1 with  $z = x + ((1-x)/\sqrt{n})$ , we have

$$|B_{n,r,s}(f'_x, x)|$$

$$\begin{aligned} &= \left| \int_x^1 \left( \int_x^t f'_x(u) du \right) \mathcal{W}_{n,r,s}(x, t) dt \right| \\ &= \left| \int_x^z \left( \int_x^t f'_x(u) du \right) dt (1 - \beta_{n,r,s}(x, t)) + \int_z^1 \left( \int_x^t f'_x(u) du \right) dt (1 - \beta_{n,r,s}(x, t)) \right| \\ &= \left| \left[ \int_x^t f'_x(u) (1 - \beta_{n,r,s}(x, t)) du \right]_x^z - \int_x^z f'_x(t) (1 - \beta_{n,r,s}(x, t)) dt \right. \\ &\quad \left. + \int_z^1 \left( \int_x^t f'_x(u) du \right) dt (1 - \beta_{n,r,s}(x, t)) \right| \\ &= \left| \int_x^z f'_x(u) du (1 - \beta_{n,r,s}(x, z)) - \int_x^z f'_x(t) (1 - \beta_{n,r,s}(x, t)) dt \right. \\ &\quad \left. + \left[ \int_x^t f'_x(u) du (1 - \beta_{n,r,s}(x, t)) \right]_z^1 - \int_z^1 f'_x(t) (1 - \beta_{n,r,s}(x, t)) dt \right| \\ &= \left| \int_x^z f'_x(t) (1 - \beta_{n,r,s}(x, t)) dt + \int_z^1 f'_x(t) (1 - \beta_{n,r,s}(x, t)) dt \right| \\ &\leq \frac{\mathcal{X}_{r,s}}{n} \frac{x(1-x)}{n} \int_z^1 \bigvee_x^t (f'_x) (t-x)^{-2} dt + \int_x^z \bigvee_x^t (f'_x) dt \\ &= \frac{\mathcal{X}_{r,s}}{n} \frac{x(1-x)}{n} \int_{x+((1-x)/\sqrt{n})}^1 \bigvee_x^t (f'_x) (t-x)^{-2} dt + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f'_x). \end{aligned}$$



By the substitution of  $v = (1 - x)/(t - x)$ , we get

$$\begin{aligned}
 |B_{n,r,s}(f'_x, x)| &\leq \frac{\mathcal{X}_{r,s} x(1-x)}{n} \int_1^{\sqrt{n}x + ((1-x)/v)} \bigvee_x (f'_x)(1-x)^{-1} dv + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x + ((1-x)/\sqrt{n})} (f'_x) \\
 &\leq \frac{\mathcal{X}_{r,s} x}{n} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_x^{x + ((1-x)/v)} (f'_x) dv + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x + ((1-x)/\sqrt{n})} (f'_x) \\
 (3.7) \quad &= \frac{\mathcal{X}_{r,s} x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x + ((1-x)/k)} (f'_x) + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x + ((1-x)/\sqrt{n})} (f'_x).
 \end{aligned}$$

Collecting the estimates (3.5)-(3.7), we get the required result. This completes the proof of the theorem.  $\square$

#### 4. Numerical Examples.

**Example 4.1.** In Figure 1, for  $n = 10, r = 1, s = 1$ , the comparison of convergence of  $\mathcal{K}_{n,r,s}(f; x)$  (yellow) and Bernstein-Kantorovich  $M_n(f; x)$  (blue) operators to  $f(x) = e^{x^3}$  (red) is illustrated. It is observed that the  $\mathcal{K}_{n,r,s}(f; x)$  gives a better approximation to  $f(x)$  than Bernstein-Kantorovich  $M_n(f; x)$  operators for  $n = 10, r = 1, s = 1$ .

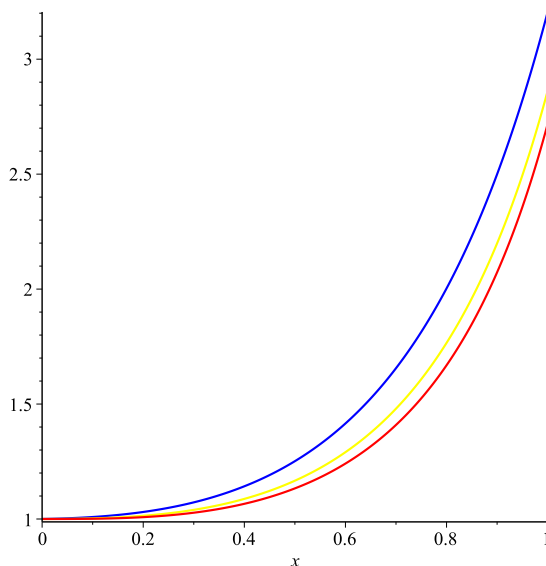


Figure 1. The convergence of  $M_{10}(f; x)$  and  $\mathcal{K}_{10,1,1}(f; x)$  to  $f(x)$

**Example 4.2.** In Figure 2, for  $n = 50, r = 1, s = 1$ , the comparison of convergence of  $\mathcal{K}_{n,r,s}(f; x)$  (yellow) and Bernstein-Kantorovich  $M_n(f; x)$  (blue) operators to  $f(x) = x^2 \sin(\frac{2x}{\pi})$  (red) is illustrated. It is observed that the  $\mathcal{K}_{n,r,s}(f; x)$  gives a better approximation to  $f(x)$  than Bernstein-Kantorovich  $M_n(f; x)$  operators for  $n = 50, r = 1, s = 1$ .

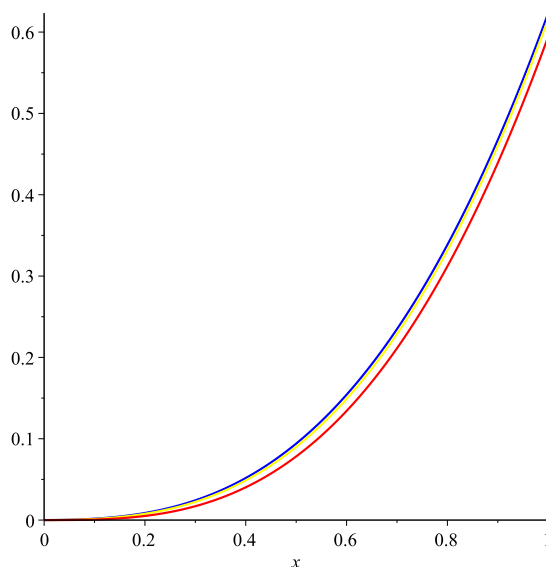


Figure 2. The convergence of  $M_{50}(f; x)$  and  $K_{50,1,1}(f; x)$  to  $f(x)$

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Arun Kajla  
Department of Mathematics  
Central University of Haryana  
Haryana-123031, India  
[rachitkajla47@gmail.com](mailto:rachitkajla47@gmail.com)