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# ON CAPABLE GROUPS OF ORDER $p^4$ *

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**Abstract.** A group $H$ is said to be capable, if there exists another group $G$ such that $\frac{G}{Z(G)} \cong H$, where $Z(G)$ denotes the center of $G$. In a recent paper [5], the authors considered the problem of capability of five non-abelian $p-$groups of order $p^4$ into account. In this paper, we try to solve the problem of capability by considering three other groups of order $p^4$. It is proved that the group

$$H_6 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yx = x^{p+1}y, zx = xyz, yz = zy\rangle$$

is not capable. Moreover, if $p > 3$ is a prime number and $d \not\equiv 0, 1 \ (mod\ p)$ then the following groups are not capable:

$$\begin{aligned}
H_7^1 &= \langle x, y, z \mid x^9 = y^3 = 1, z^3 = x^3, yx = x^4y, zx = xyz, zy = yz\rangle,\\
H_7^2 &= \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yx = x^{p+1}y, zx = x^{p+1}yz, zy = x^pyz\rangle,\\
H_8^1 &= \langle x, y, z \mid x^9 = y^3 = 1, z^3 = x^{-3}, yx = x^4y, zx = xyz, zy = yz\rangle,\\
H_8^2 &= \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yx = x^{p+1}y, zx = x^{dp+1}yz, zy = x^{dp}yz\rangle.
\end{aligned}$$

**Keywords:** Capable group; $p-$group; non-abelian $p-$groups; center.

## 1. Introduction

A group $H$ is said to be capable if there exists another group $G$ such that $\frac{G}{Z(G)} \cong H$, or equivalently $H$ can be represented as the inner automorphism group of a given group $G$. The capability of groups was first studied by Baer [1] who was asked the question “which conditions a group $H$ must fulfill in order to be the group of inner automorphisms of a group $G$?”. In the mentioned paper, he determined all capable groups which are direct products of cyclic groups. Since the time that

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Hall and Senior published their inovating work [3], such groups are called capable. It is well-known that the classification of capable groups is the first step towards the classification of prime power order groups [4]. The following theorem of Baer is well-known in the context of capable groups.

**Theorem 1.1.** *Let $A$ be a finite abelian group written as $A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ such that each integer $n_{i+1}$ is divisible by $n_i$. Then $A$ is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.*

Burnside [2] was classified all $p-$groups of order $p^4$ which $p$ is an odd prime number. This classification is expressed in the following theorem:

**Theorem 1.2.** *Suppose $p$ is an odd prime number and $d \not\equiv 0, 1$ (mod $p$). Then there are fifteen different groups of order $p^4$ up to isomorphisms. Five of those are abelian and the non-abelian groups are in the list below.*

$$H_1 = \langle x, y \mid x^{p^3} = y^p = 1, yxy^{-1} = x^{p^2+1}\rangle,$$
$$H_2 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, xy = yx, xz = zx, zyz^{-1} = x^p y\rangle,$$
$$H_3 = \langle x, y \mid x^{p^2} = y^{p^2} = 1, yxy^{-1} = x^{p+1}\rangle,$$
$$H_4 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, xy = yx, yz = zy, zxz^{-1} = x^{p+1}\rangle,$$
$$H_5 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, xy = yx, yz = zy, zxz^{-1} = xy\rangle,$$
$$H_6 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yxy^{-1} = x^{p+1}, zxz^{-1} = xy, yz = zy\rangle,$$
$$H_7^1 = \langle x, y, z \mid x^9 = y^3 = 1, [y, z] = 1, z^3 = x^3, y^{-1}xy = x^4, z^{-1}xz = xy^{-1}\rangle,$$
$$H_7^2 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yxy^{-1} = x^{p+1}, zxz^{-1} = x^{p+1}y, zyz^{-1} = x^p y\rangle \qquad p > 3,$$
$$H_8^1 = \langle x, y, z \mid x^9 = y^3 = 1, [y, z] = 1, z^3 = x^{-3}, y^{-1}xy = x^4, z^{-1}xz = xy^{-1}\rangle,$$
$$H_8^2 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yxy^{-1} = x^{p+1}, zxz^{-1} = x^{dp+1}y, zyz^{-1} = x^{dp}y\rangle \qquad p > 3,$$
$$H_9 = \langle x, y, z, t \mid x^p = y^p = z^p = t^p = [x, y] = [x, z] = [x, t] = [y, z] = [y, t] = 1, tzt^{-1} = xz\rangle,$$
$$H_{10}^1 = \langle x, y, z \mid x^9 = y^3 = z^3 = 1, xy = yx, z^{-1}xz = xy, z^{-1}yz = x^{-3}y\rangle,$$
$$H_{10}^2 = \langle x, y, z, t \mid x^p = y^p = z^p = t^p = [x, y] = [x, z] = [x, t] = [y, z] = [t, y]x^{-1} = [t, z]y^{-1} = 1\rangle \qquad p > 3.$$

Zainal et al. [5] examined the capability of five groups out of ten non-abelian groups of order $p^4$ and proved that among first five groups the previous theorem, only the group number 3 is capable. We record this result in the following theorem:

**Theorem 1.3.** (*See [5]*) *The groups $H_i$, $1 \leq i \leq 5$, is capable if and only if $i = 3$.*

## 2. Main Results

Our aim in this section is to prove the groups numbers $6, 7$ and $8$ in Theorem 1.2 are not capable.

**Theorem 2.1.** *The group $H_6$ is not capable.*

*Proof.* By definition of $H_6$ and some calculations we have the following equations,

$$y^j x^i = x^{ijp+i} y^j \tag{2.1}$$

$$z^k x^i = x^{\frac{i(i-1)}{2}kp+i} y^{ik} z^k \tag{2.2}$$

We put $i = p$ and $j = k = 1$ in Equations 2.1 and 2.2. Since $p$ is odd and $x^{p^2} = y^p = 1$, $yx^p = x^p y$ and $zx^p = x^p z$. Thus $\langle x^p \rangle \leq Z(H_6)$ and $|Z(H_6)| = p$ or $p^2$. Suppose $|Z(H_6)| = p^2$. Then for every $h \in H_6 \setminus Z(H_6)$, $Z(H_6)\langle C_{H_6}(h)\rangle \leq H_6$ and so $|C_{H_6}(h)| = p^3$. This proves that the conjugacy class $h^{H_6}$ has size $p$. Choose $j, k$ with this condition that $0 \leq j, k \leq p-1$. Since $x$ is not central and by Equations 2.1 and 2.2, $y^j x y^{-j} = x^{jp+1}$ and $z^k x z^{-k} = xy^k$, we find that $|x^{H_6}| > p$ which is not possible. Therefore $|Z(H_6)| = p$ and $Z(H_6) = \langle x^p \rangle$.

If $H_6$ is capable then there exists a non-abelian group $G$ with center $Z$ such that $H_6 \cong \frac{G}{Z}$. Since $G$ is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

$$\frac{G}{Z} = \left\langle \begin{array}{c} aZ, bZ, cZ \mid (aZ)^{p^2} = (bZ)^p = (cZ)^p = 1, (bZ)(aZ) = (aZ)^{p+1}(bZ), \\ (cZ)(aZ) = (aZ)(bZ)(cZ), (bZ)(cZ) = (cZ)(bZ) \end{array} \right\rangle.$$

By definition, $a^{p^2}, b^p, c^p \in Z$ and by Equation 2.1 one can see the following equation:

$$ba^p = a^p b. \tag{2.3}$$

By Equation 2.2 and some calculations, we have:

$$(aZcZ)^n = (aZ)^{t_n p}(aZ)^n (bZ)^{\frac{n(n-1)}{2}}(cZ)^n \tag{2.4}$$

in which $t_n = \frac{n(n-1)(n-2)}{6}$. By substituting $n = p$ in Equation 2.4, we obtain the following equality:

$$(aZcZ)^p = (aZ)^{t_p p}(aZ)^p. \tag{2.5}$$

We now consider two cases that $p = 3$ or $p > 3$.

1. $p > 3$. Then $p \mid t_p$ and so by Equation 2.5 and this fact that $a^{p^2} \in Z$,

$$\begin{aligned} (ac)^p Z &= (acZ)^p \\ &= (aZcZ)^p \\ &= (aZ)^{t_p p}(aZ)^p \\ &= (aZ)^p \\ &= a^p Z. \end{aligned}$$

   Hence there exists $z \in Z$ such that $(ac)^p = a^p z$ and so $ca^p = a^p c$. Finally, we apply Equation 2.3 to conclude that $a^p \in Z$ which is a contradiction.

2. $p = 3$. Then $t_p = 1$ and by Equation 2.5, $(ac)^3Z = (aZcZ)^3 = (aZ)^3(aZ)^3 = (aZ)^6 = a^6Z$. Hence there exists $z \in Z$ such that $(ac)^3 = a^6z$ and so $ca^6 = a^6c$. By these equations and and Equation 2.3, we conclude that $a^6 \in Z$ which is our final contradiction.

Therefore, the group $H_6$ is not capable. □

**Theorem 2.2.** *The group $H_7^1$ is not capable.*

*Proof.* By definition of $H_7^1$ and some tedious calculations, one can see that

$$y^jx^i = x^{3ij+i}y^j \tag{2.6}$$

$$z^kx^i = x^{3k\frac{i(i-1)}{2}+i}y^{ik}z^k \tag{2.7}$$

We put $i = 3$ and $j = k = 1$ in Equations 2.6 and 2.7. Since $x^9 = y^3 = 1$, $yx^3 = x^3y$ and $zx^3 = x^3z$ and so $\langle x^3\rangle \leq Z(H_7^1)$. On the other hand, $|H_7^1| = 3^4$ and hence $|Z(H_7^1)| = 3$ or 9. Suppose $|Z(H_7^1)| = 9$. Then for every $h \in H_7^1 \setminus Z(H_7^1)$, $Z(H_7^1)\langle C_{H_7^1}(h)\rangle \leq H_7^1$ which implies that $|C_{H_7^1}(h)| = 3^3$ or equivalently $|h^{H_7^1}| = 3$. Note that $x \in H_7^1 \setminus Z(H_7^1)$. Choose $j, k$ such that $0 \leq j, k \leq 2$. By Equations 2.6 and 2.7, $y^jxy^{-j} = x^{3j+1}$ and $z^kxz^{-k} = xy^k$ which shows that $|x^{H_7^1}| > 3$. This contradiction implies that $|Z(H_7^1)| = 3$ and $Z(H_7^1) = \langle x^3\rangle$. If $H_7^1$ is capable, there is a non-abelian group $G$ with center $Z$ such that $H_7^1 \cong \frac{G}{Z}$. Since $G$ is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

$$\frac{G}{Z} = \left\langle \begin{array}{c} aZ, bZ, cZ \mid (aZ)^9 = (bZ)^3 = 1, (cZ)^3 = (aZ)^3, (bZ)(aZ) = (aZ)^4(bZ), \\ (cZ)(aZ) = (aZ)(bZ)(cZ), (cZ)(bZ) = (bZ)(cZ) \end{array} \right\rangle.$$

Obviously $a^9, b^3, c^9 \in Z$ and by Equation 2.6,

$$(aZbZ)^n = (aZ)^{3\frac{n(n-1)}{2}}(aZ)^n(bZ)^n.$$

In above equation, we put $n = 3$. Since $a^9, b^3 \in Z$, $(ab)^3Z = (abZ)^3 = (aZbZ)^3 = (aZ)^9(aZ)^3(bZ)^3 = (aZ)^3 = a^3Z$ and so there exists $z \in Z$ such that $(ab)^3 = a^3z$. Therefore,

$$ba^3 = a^3b \tag{2.8}$$

On the other hand, $a^3Z = c^3Z$ and so there exists $z_1 \in Z$ such that

$$a^3 = c^3z_1 \tag{2.9}$$

Put $k = 1$ and $i = 3$ in Equation 2.7. Since $o(aZ) = 9$ and $o(bZ) = 3$,

$$\begin{aligned} ca^3Z &= (cZ)(aZ)^3 \\ &= (aZ)^9(aZ)^3(bZ)^3(cZ) \\ &= (aZ)^3(cZ) \\ &= a^3cZ. \end{aligned}$$

Thus there exists $z_2 \in Z$ such that

$$ca^3 = a^3 c z_2. \tag{2.10}$$

Now by inserting the Equation 2.9 in 2.10, $cc^3 z_1 = c^3 z_1 c z_2$ which shows that $z_2 = 1$. Apply again Equation 2.10 to conclude that $ca^3 = a^3 c$. Now by Equation 2.8 $a^3 \in Z$ and hence $(aZ)^3 = Z$ which is our final contradiction. □

**Theorem 2.3.** *The group $H_7^2$ is not capable.*

*Proof.* By presentation of $H_7^2$ and some tedious calculations one can see that

$$\begin{aligned} y^j x^i &= x^{ijp+i} y^j, && (2.11)\\ z^k x^i &= x^{\frac{i(i+1)}{2}kp + \frac{k(k-1)}{2}ip + i} y^{ik} z^k, && (2.12)\\ z^k y^j &= x^{jkp} y^j z^k. \end{aligned}$$

By substituting $i = p$ and $j = k = 1$ in Equations 2.11 and 2.12 we have $yx^p = x^p y$ and $zx^p = x^p z$. Hence $\langle x^p \rangle \leq Z(H_7^2)$ and arguments similar to the proof of Theorem 2.1 show that $Z(H_7^2) = \langle x^p \rangle$. If $H_7^2$ is capable, there is a non-abelian group $G$ with center $Z$ such that and $H_7^2 \cong \frac{G}{Z}$. Since $G$ is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

$$\frac{G}{Z} = \left\langle \begin{array}{c} aZ, bZ, cZ \mid (aZ)^{p^2} = (bZ)^p = (cZ)^p = 1, (bZ)(aZ) = (aZ)^{p+1}(bZ), \\ (cZ)(aZ) = (aZ)^{p+1}(bZ)(cZ), (cZ)(bZ) = (aZ)^p(bZ)(cZ) \end{array} \right\rangle.$$

Thus $a^{p^2}, b^p, c^p \in Z$. Now by Equation 2.11 and a similar argument as Theorem 2.1,

$$ba^p = a^p b. \tag{2.13}$$

Apply Equation 2.12 to conclude that

$$(aZcZ)^n = (aZ)^{k_n p}(aZ)^n(bZ)^{\frac{n(n-1)}{2}}(cZ)^n$$

in which $k_n = \frac{n(n-1)(2n-1)}{6}$. Next we assume that $n = p$. Since $b^p, c^p$ are central,

$$\begin{aligned} (ac)^p Z &= (acZ)^p = (aZcZ)^p \\ &= (aZ)^{k_p p}(aZ)^p(bZ)^{\frac{p(p-1)}{2}}(cZ)^p \\ &= (aZ)^{(k_p+1)p} = a^{(k_p+1)p} Z. \end{aligned}$$

Hence there exists $z \in Z$ such that

$$(ac)^p = a^{(k_p+1)p} z. \tag{2.14}$$

It is clear that $p \mid 6k_p$. Since $p > 3$, $p \mid k_p$ and so $p \nmid k_p + 1$. Since $(ac)^p(ac) = (ac)(ac)^p$, Equation 2.14 implies that $ca^{(k_p+1)p} = a^{(k_p+1)p}c$ and by Equation 2.13, $a^{(k_p+1)p} \in Z$. So, $(aZ)^{(k_p+1)p} = Z$. But $o(aZ) = p^2$ and hence $p^2 \mid (k_p+1)p$ which implies that $p \mid k_p + 1$. This contradiction completes the proof. □

**Theorem 2.4.** *The group $H_8^1$ is not capable.*

*Proof.* By presentation of $H_8^1$ we have:

$$y^j x^i = x^{3ij+i} y^j, \tag{2.15}$$

$$z^k x^i = x^{3k\frac{i(i-1)}{2}+i} y^{ik} z^k. \tag{2.16}$$

Again substitute $i = 3$ and $j = k = 1$ in Equations 2.15 and 2.16. Since $x^9 = y^3 = 1$, $yx^3 = x^3y$ and $zx^3 = x^3z$. Thus $\langle x^3\rangle \leq Z(H_8^1)$. Similar to the proof of Theorem 2.2, $Z(H_8^1) = \langle x^3\rangle$. If $H_8^1$ is capable, there is a non-abelian group $G$ with center $Z$ such that $H_8^1 \cong \frac{G}{Z}$. Since $G$ is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

$$\frac{G}{Z} = \left\langle \begin{array}{c} aZ, bZ, cZ \mid (aZ)^9 = (bZ)^3 = 1, (cZ)^3 = (aZ)^{-3}, (bZ)(aZ) = (aZ)^4(bZ), \\ (cZ)(aZ) = (aZ)(bZ)(cZ), (cZ)(bZ) = (bZ)(cZ) \end{array} \right\rangle.$$

Obviously, $a^9, b^3, c^9 \in Z$ and by Equation 2.15,

$$(aZbZ)^n = (aZ)^{3\frac{n(n-1)}{2}}(aZ)^n(bZ)^n.$$

Put $n = 3$. Since $a^9, b^3 \in Z$,

$$(ab)^3Z = (abZ)^3 = (aZbZ)^3 = (aZ)^9(aZ)^3(bZ)^3 = (aZ)^3 = a^3Z.$$

Hence there exists $z \in Z$ such that $(ab)^3 = a^3z$ and so

$$ba^3 = a^3b. \tag{2.17}$$

On the other hand, $c^3Z = a^{-3}Z$ and so there exists $z_1 \in Z$ such that

$$a^3 = c^{-3}z_1. \tag{2.18}$$

Since $o(aZ) = 9$ and $o(bZ) = 3$, by Equation 2.16 and substituting $k = 1$ and $i = 3$, we can see that

$$\begin{aligned} ca^3Z &= (cZ)(aZ)^3 \\ &= (aZ)^9(aZ)^3(bZ)^3(cZ) \\ &= (aZ)^3(cZ) = a^3cZ \end{aligned}$$

and so there exists $z_2 \in Z$ such that

$$ca^3 = a^3cz_2. \tag{2.19}$$

We now insert Equation 2.18 in our last equation to deduce that $cc^{-3}z_1 = c^{-3}z_1cz_2$. Thus $z_2 = 1$ and by Equation 2.19, $ca^3 = a^3c$. Therefore, $a^3 \in Z$ and hence $9 = o(aZ) \mid 3$, which is impossible. This completes the proof. □

**Theorem 2.5.** *The group $H_8^2$ is not capable.*

*Proof.* By presentation of $H_8^2$ and some tedious calculations, we have

$$y^j x^i = x^{ijp+i} y^j, \tag{2.20}$$

$$z^k x^i = x^{\frac{i(i-1)}{2}kp+\frac{k(k+1)}{2}idp+i} y^{ik} z^k, \tag{2.21}$$
$$z^k y^j = x^{jkdp} y^j z^k.$$

In Equations 2.20 and 2.21, we insert $i = p$ and $j = k = 1$. It is clear that $yx^p = x^p y$ and $zx^p = x^p z$ and so $\langle x^p \rangle \leq Z(H_8^2)$. Similar to Theorem 2.1, we can see that $Z(H_8^2) = \langle x^p \rangle$. If $H_8^2$ is capable, there is a non-abelian group $G$ with center $Z$ such that $H_8^2 \cong \frac{G}{Z}$. Since $G$ is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

$$\frac{G}{Z} = \left\langle \begin{array}{r} aZ, bZ, cZ \mid (aZ)^{p^2} = (bZ)^p = (cZ)^p = 1, (bZ)(aZ) = (aZ)^{p+1}(bZ), \\ (cZ)(aZ) = (aZ)^{dp+1}(bZ)(cZ), (cZ)(bZ) = (aZ)^{dp}(bZ)(cZ) \end{array} \right\rangle,$$

where $d \not\equiv 0, 1 (mod\ p)$. It is obvious that $a^{p^2}, b^p, c^p \in Z$ and by Equations 2.20 and a similar argument used in the proof of the Theorem 2.1,

$$ba^p = a^p b. \tag{2.22}$$

Moreover, by Equation 2.21,

$$(aZcZ)^n = (aZ)^{s_n dp}(aZ)^{t_n p}(aZ)^n(bZ)^{\frac{n(n-1)}{2}}(cZ)^n \tag{2.23}$$

in which $s_n = \frac{n(n-1)(n+1)}{6}$ and $t_n = \frac{n(n-1)(n-2)}{6}$. It is easy to see that $p \mid s_p$ and $p \mid t_p$. Also by inserting $n = 1$ in Equation 2.23,

$$\begin{aligned} (ac)^p Z &= (acZ)^p = (aZcZ)^p \\ &= (aZ)^{s_p dp}(aZ)^{t_p p}(aZ)^p(bZ)^{\frac{p(p-1)}{2}}(cZ)^p \\ &= (aZ)^p = a^p Z. \end{aligned}$$

Hence there exists $z \in Z$ such that $(ac)^p = a^p z$ and so $ca^p = a^p c$. This implies that $a^p \in Z$ and therefore $p^2 = o(aZ) \mid p$, which is our final contradiction. □

## 3. Concluding Remarks

In this paper the authors continued a recently published paper of Zainal et al. [5] in investigating finite $p-$groups of order $p^4$. It is proved that three non-abelian groups of this order are not capable. By results of [5] and our results to complete the classification of capable group of order $p^4$ it is enough to investigate the groups $H_9$ and $H_{10}$ in Theorem 1.2. Our calculations with computer algebra software GAP in working with small groups of order $p^4$ suggests the following conjecture:

**Conjecture 3.1.** *The groups $H_9$ and $H_{10}$ are not capable.*

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