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FRACTIONAL HERMITE-HADAMARD INEQUALITIES THROUGH r-CONVEX FUNCTIONS via POWER MEANS*

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Abstract. In this paper, we firstly establish another important integral identity for twice differentiable mapping involving Riemann-Liouville fractional integrals. Secondly, we use this integral identity to derive several Riemann-Liouville fractional Hermite-Hadamard inequalities through r-convex functions via power means. Finally, some applications to quadrature formulas and special means of real numbers are given.

1. Introduction and Preliminaries

The concept of fractional calculus appeared in 1695 in a letter between L'Hospital and Leibniz. Since then, many mathematicians have further developed this area and we recommend the study of Riemann, Liouville, Caputo, and other famous mathematicians. Up to now, fractional calculus have played an important role in various fields such as electricity, biology, economics and signal and image processing.

For $f \in L[a,b]$, the Riemann-Liouville integrals [1] $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ x < b$$

respectively, where $\Gamma(\cdot)$ is the Gamma function and $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$.

Very recently, Sarikaya et al. [2] proved an interesting fractional version inequality for a differentiable mapping:

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Theorem 1.1. *(see Lemma 2, [2]) Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 \le a < b$. If f' is integrable, then the following equality for fractional integrals holds*

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)] \\ = & \frac{b-a}{2}\int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta+(1-t)b)dt. \end{aligned}$$

Next, Wang et al. [3] proved another interesting fractional version inequality for twice differentiable mapping:

Lemma 1.1. *(see Lemma 2.1, [3]) Let $f : [a,b] \to \mathbb{R}$ be twice differentiable mapping on (a,b) with $a < b$. If f'' is integrable, then the following equality for fractional integrals holds*

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)] \\ = & \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} f''(ta+(1-t)b)dt. \end{aligned}$$

For more recent studies on the fractional version Hermite-Hadamard inequality involving Riemann-Liouville and Hadamard fractional integrals, one can see [4, 5, 6, 7, 8, 9] and reference therein.

Note that Sarikaya and Aktan [10] presented a general interesting integral identity:

Lemma 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping such that f'' is integrable and $0 \le \lambda \le 1$. Then the following equality for fractional integrals holds:*

$$\begin{aligned} & (\lambda-1)f(\frac{a+b}{2}) - \lambda\frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(t)dt \\ = & (b-a)^2\int_0^1 \mu(t) f''(ta+(1-t)b)dt, \end{aligned} \tag{1.1}$$

where

$$\mu(t) = \begin{cases} \frac{t(t-\lambda)}{2}, & 0 \le t < \frac{1}{2}, \\ \frac{(1-t)(1-\lambda-t)}{2}, & \frac{1}{2} \le t \le 1. \end{cases}$$

By using (1.1), new inequalities of the Simpson and the Hermite-Hadamard type for convex functions are established.

As far as we known, fractional version equality corresponding to integral version equality (1.1) has not been reported. Motivated by the above papers, we establish another fractional version equality (see Lemma 2.1). Then, we use this new fractional equality to derive several inequalities of the Hermite-Hadamard type for r-convex functions. At last, we give some applications to special means of real numbers.

2. New integral identity involving Riemann-Liouville fractional integrals

We present an important fractional integral identity for twice differentiable mapping.

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping such that f'' is integrable. For $0\le\lambda\le 1$, the following equality for fractional integrals holds:*

$$\begin{aligned}(2.1)\qquad &\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right)-(1+\lambda)\frac{f(a)+f(b)}{2}-(1-\lambda)f(\frac{a+b}{2})\\ =\;& (b-a)^2\int_0^1 k(t,\lambda)f''(ta+(1-t)b)dt,\end{aligned}$$

where

$$(2.2)\qquad k(t,\lambda)=\begin{cases}\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}+\frac{(1-\lambda)t}{2}, & 0\le t<\frac{1}{2},\\ \frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}+\frac{(1-\lambda)(1-t)}{2}, & \frac{1}{2}\le t\le 1.\end{cases}$$

Proof. Note the definition of $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ and using the method of changing variables, we have

$$\begin{aligned}&J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\\
=\;&\frac{1}{\Gamma(\alpha)}\Big[\int_0^1 t^{\alpha-1}(b-a)^\alpha f(ta+(1-t)b)dt\\
&+\int_0^1(1-t)^{\alpha-1}(b-a)^\alpha f(ta+(1-t)b)dt\Big]\\
=\;&\frac{(b-a)^\alpha}{\Gamma(\alpha)}\Big[\frac{1}{\alpha}f(a)-\frac{a-b}{\alpha}\int_0^1 t^\alpha f'(ta+(1-t)b)dt\\
&+\frac{1}{\alpha}f(b)+\frac{a-b}{\alpha}\int_0^1(1-t)^\alpha f'(ta+(1-t)b)dt\Big]\\
=\;&\frac{(b-a)^\alpha}{\Gamma(\alpha)}\Big[\frac{(a-b)^2}{\alpha(\alpha+1)}\int_0^1(t^{\alpha+1}+(1-t)^{\alpha+1})f''(ta+(1-t)b)dt\\
&+\frac{1}{\alpha}(f(a)+f(b))+\frac{a-b}{\alpha(\alpha+1)}(f'(b)-f'(a))\Big]\\
=\;&\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\Big[\frac{(b-a)^2}{\alpha+1}\int_0^1(t^{\alpha+1}+(1-t)^{\alpha+1}-1)f''(ta+(1-t)b)dt+(f(a)+f(b))\Big].\end{aligned}$$

From the above we obtain

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}(J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a))-(f(a)+f(b))\\
=\;&(b-a)^2\Big[\int_0^{\frac{1}{2}}\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}f''(ta+(1-t)b)dt\end{aligned}$$

$$+\int_{\frac{1}{2}}^{1}\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}f''(ta+(1-t)b)dt\Bigg]. \tag{2.3}$$

In addition,

$$\begin{aligned}&\int_0^{\frac{1}{2}} tf''(ta+(1-t)b)dt+\int_{\frac{1}{2}}^{1}(1-t)f''(ta+(1-t)b)dt\\ =\ &\frac{f(a)+f(b)}{(b-a)^2}-\frac{2f(\frac{a+b}{2})}{(b-a)^2}.\end{aligned} \tag{2.4}$$

For (2.4), by multiplying both sides by $\frac{1-\lambda}{2}(b-a)^2$ and exchange the left side with the right side of the equation we obtain

$$\begin{aligned}&\frac{1-\lambda}{2}(f(a)+f(b))-(1-\lambda)f(\frac{a+b}{2})\\ =\ &(b-a)^2\Bigg[\int_0^{\frac{1}{2}}\frac{1-\lambda}{2}tf''(ta+(1-t)b)dt\\ &+\int_{\frac{1}{2}}^{1}\frac{1-\lambda}{2}(1-t)f''(ta+(1-t)b)dt\Bigg].\end{aligned} \tag{2.5}$$

Combining (2.3) and (2.5), we obtain (2.1). □

Remark 2.1. In Lemma 2.1, if we put $\alpha = 1$ then the equality (2.1) becomes

$$\begin{aligned}&\frac{2}{b-a}\int_a^b f(t)dt-(1+\lambda)\frac{f(a)+f(b)}{2}-(1-\lambda)f(\frac{a+b}{2})\\ &=(b-a)^2\int_0^1 k(t,\lambda)f''(ta+(1-t)b)dt\end{aligned}$$

where $k(t,\lambda)$ is given by

$$k(t,\lambda)=\begin{cases}\frac{t^2+(1-t)^2-1}{2}+\frac{(1-\lambda)t}{2}, & 0\le t<\frac{1}{2},\\ \frac{t^2+(1-t)^2-1}{2}+\frac{(1-\lambda)(1-t)}{2}, & \frac{1}{2}\le t\le 1.\end{cases}$$

Here we derive another integral equality which is not same as the result in Lemma 1.2.

3. Main results

The following definitions will be used in this section.

Definition 3.1. (see [11]) The function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ is said to be r-convex, where $r \ge 0$, if for every $x, y \in I$ and $t \in [0, 1]$, we have

$$\begin{aligned}f(tx+(1-t)y)&\le\left[t(f(x))^r+(1-t)(f(y))^r\right]^{\frac{1}{r}},\ r\neq 0;\\ f(tx+(1-t)y)&\le(f(x))^t(f(y))^{1-t},\ r=0.\end{aligned}$$

Definition 3.2. (see [12]) The incomplete beta function is defined as follows:

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt,\ x \in [0,1],\ a,b > 0.$$

We also need the following elementary inequalities. One can find them in [13].

Lemma 3.1. *For $A \geqslant 0, B \geqslant 0$, it holds*

$$\begin{aligned} A^\theta + B^\theta \le (A+B)^\theta &\leqslant 2^{\theta-1}(A^\theta + B^\theta) \quad \text{when } \theta \geqslant 1, \\ 2^{\theta-1}(A^\theta + B^\theta) \le (A+B)^\theta &\leqslant A^\theta + B^\theta \qquad\qquad \text{when } 0 < \theta \leqslant 1. \end{aligned}$$

Lemma 3.2. *For $A > B > 0$, it holds*

$$\begin{aligned} (A-B)^\theta &\le A^\theta - B^\theta \quad \text{when } \theta \ge 1, \\ (A-B)^\theta &\ge A^\theta - B^\theta \quad \text{when } 0 < \theta \le 1. \end{aligned}$$

Lemma 3.3. *(see Lemma 2.1, [14]) For $\alpha > 0$ and $k > 0$,we have*

$$I(\alpha) = \int_0^1 t^{\alpha-1}k^t dt = k\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln k)^{i-1}}{(\alpha)_i} < +\infty,$$

where $(\alpha)_i = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1)$.

Lemma 3.4. *(see Lemma 2.2, [14]) For $\alpha > 0$ and $k > 0, z > 0$, we have*

$$\begin{aligned} J(\alpha,k) &= \int_0^1 (1-t)^{\alpha-1}k^t dt = \sum_{i=1}^{\infty}\frac{(\ln k)^{i-1}}{(\alpha)_i} < +\infty \\ H(\alpha,k,z) &= \int_0^z t^{\alpha-1}k^t dt = z^\alpha k^z \sum_{i=1}^{\infty}\frac{(-z\ln k)^{i-1}}{(\alpha)_i} < +\infty. \end{aligned}$$

Now we are ready to present our main results in this section.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping with $0 \le a < b$. If $|f''|$ is integrable and r-convex on $[a,b]$ for some fixed $0 \le r < \infty$, then the following inequality for fractional integrals holds*

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right) - (1+\lambda)\frac{f(a)+f(b)}{2} - (1-\lambda)f(\frac{a+b}{2})\right| \\ &\le K_r, \end{aligned} \tag{3.1}$$

where

$$K_r := 2^{\frac{1}{r}-1}(b-a)^2\left\{\frac{|f''(a)|+|f''(b)|}{\alpha+1}\left[\frac{r}{r+1} - \frac{r}{\alpha r+2r+1} - B(\frac{1}{r}+1,\alpha+2)\right]\right.$$

$$
\begin{aligned}
& +\frac{1-\lambda}{2}|f''(a)|\left(\frac{r}{r+1}-\frac{r}{2r+1}\right)\left[1-\left(\frac{1}{2}\right)^{\frac{1}{r}+1}\right] \\
& +\frac{1-\lambda}{2}|f''(b)|\left[B_{\frac{1}{2}}(2,\frac{1}{r}+1)+\frac{r}{2r+1}\left(\frac{1}{2}\right)^{\frac{1}{r}+2}\right]\Bigg\} \quad \text{for } 0<r\le 1, \\
K_r := \; & (b-a)^2\left\{\frac{|f''(a)|+|f''(b)|}{\alpha+1}\left[\frac{r}{r+1}-\frac{r}{\alpha r+2r+1}-B(\frac{1}{r}+1,\alpha+2)\right]\right. \\
& +\frac{1-\lambda}{2}|f''(a)|\left(\frac{r}{r+1}-\frac{r}{2r+1}\right)\left[1-\left(\frac{1}{2}\right)^{\frac{1}{r}+1}\right] \\
& +\frac{1-\lambda}{2}|f''(b)|\left[B_{\frac{1}{2}}(2,\frac{1}{r}+1)+\frac{r}{2r+1}\left(\frac{1}{2}\right)^{\frac{1}{r}+2}\right]\Bigg\} \quad \text{for } r>1, \\
K_r := \; & (b-a)^2|f''(b)|\left\{\frac{1}{\alpha+1}\left[\frac{|k|-1}{\ln|k|}-|k|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln|k|)^{i-1}}{(\alpha+2)_i}-\sum_{i=1}^{\infty}\frac{(\ln|k|)^{i-1}}{(\alpha+2)_i}\right]\right. \\
& \left.+\frac{1-\lambda}{4}|k|^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{(-\frac{1}{2}\ln|k|)^{i-1}}{(2)_i}+\frac{1-\lambda}{2}\left[\frac{|k|-|k|^{\frac{1}{2}}}{\ln|k|}-|k|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln|k|)^{i-1}}{(2)_i}\right]\right\} \\
& \text{for } r=0, \text{ and } k=\frac{f''(a)}{f''(b)}.
\end{aligned}
$$

Proof. We divide our proof into three cases.

(i) Case 1: $0<r\le 1$. From the above definitions and lemmas we have

$$
\begin{aligned}
& \left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)-(1+\lambda)\frac{f(a)+f(b)}{2}-(1-\lambda)f(\frac{a+b}{2})\right| \\
\le \; & (b-a)^2\int_0^1|k(t,\lambda)|\left(t|f''(a)|^r+(1-t)|f''(b)|^r\right)^{\frac{1}{r}}dt \\
\le \; & 2^{\frac{1}{r}-1}(b-a)^2\int_0^1|k(t,\lambda)|\left(t^{\frac{1}{r}}|f''(a)|+(1-t)^{\frac{1}{r}}|f''(b)|\right)dt \\
\le \; & 2^{\frac{1}{r}-1}(b-a)^2\left[\int_0^{\frac{1}{2}}\left(\left|\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}\right|+\left|\frac{1-\lambda}{2}t\right|\right)\left(|f''(a)|t^{\frac{1}{r}}+|f''(b)|(1-t)^{\frac{1}{r}}\right)dt\right. \\
& \left.+\int_{\frac{1}{2}}^1\left(\left|\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}\right|+\left|\frac{1-\lambda}{2}(1-t)\right|\right)\left(|f''(a)|t^{\frac{1}{r}}+|f''(b)|(1-t)^{\frac{1}{r}}\right)dt\right] \\
= \; & 2^{\frac{1}{r}-1}(b-a)^2\left[\int_0^{\frac{1}{2}}\left(\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}|f''(a)|t^{\frac{1}{r}}+\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}|f''(b)|(1-t)^{\frac{1}{r}}\right.\right. \\
& \left.+\frac{1-\lambda}{2}|f''(a)|t^{\frac{1}{r}+1}+\frac{1-\lambda}{2}|f''(b)|t(1-t)^{\frac{1}{r}}\right)dt \\
& +\int_{\frac{1}{2}}^1\left(\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}|f''(a)|t^{\frac{1}{r}}+\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}|f''(b)|(1-t)^{\frac{1}{r}}\right. \\
& \left.\left.+\frac{1-\lambda}{2}|f''(a)|(1-t)t^{\frac{1}{r}}+\frac{1-\lambda}{2}|f''(b)|(1-t)^{\frac{1}{r}+1}\right)dt\right]
\end{aligned}
$$

$$= 2^{\frac{1}{r}-1}(b-a)^2\Big\{\frac{|f''(a)|+|f''(b)|}{\alpha+1}\Big[\frac{r}{r+1}-\frac{r}{\alpha r+2r+1}-B(\frac{1}{r}+1,\alpha+2)\Big]$$
$$+\frac{1-\lambda}{2}|f''(a)|\Big(\frac{r}{r+1}-\frac{r}{2r+1}\Big)\Big[1-\Big(\frac{1}{2}\Big)^{\frac{1}{r}+1}\Big]$$
$$+\frac{1-\lambda}{2}|f''(b)|\Big[B_{\frac{1}{2}}(2,\frac{1}{r}+1)+\frac{r}{2r+1}\Big(\frac{1}{2}\Big)^{\frac{1}{r}+2}\Big]\Big\}.$$

(ii) Case 2: $r > 1$. Like in Case 1, we have

$$\Big|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right)-(1+\lambda)\frac{f(a)+f(b)}{2}-(1-\lambda)f(\frac{a+b}{2})\Big|$$
$$\leq (b-a)^2\int_0^1|k(t,\lambda)|\left(t^{\frac{1}{r}}|f''(a)|+(1-t)^{\frac{1}{r}}|f''(b)|\right)dt$$
$$= (b-a)^2\Big[\int_0^{\frac{1}{2}}\Big|\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}+\frac{1-\lambda}{2}t\Big|\left(|f''(a)|t^{\frac{1}{r}}+|f''(b)|(1-t)^{\frac{1}{r}}\right)dt$$
$$+\int_{\frac{1}{2}}^1\Big|\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}+\frac{1-\lambda}{2}(1-t)\Big|\left(|f''(a)|t^{\frac{1}{r}}+|f''(b)|(1-t)^{\frac{1}{r}}\right)dt\Big]$$
$$\leq (b-a)^2\Big\{\frac{|f''(a)|+|f''(b)|}{\alpha+1}\Big[\frac{r}{r+1}-\frac{r}{\alpha r+2r+1}-B(\frac{1}{r}+1,\alpha+2)\Big]$$
$$+\frac{1-\lambda}{2}|f''(a)|\Big(\frac{r}{r+1}-\frac{r}{2r+1}\Big)\Big[1-\Big(\frac{1}{2}\Big)^{\frac{1}{r}+1}\Big]$$
$$+\frac{1-\lambda}{2}|f''(b)|\Big[B_{\frac{1}{2}}(2,\frac{1}{r}+1)+\frac{r}{2r+1}\Big(\frac{1}{2}\Big)^{\frac{1}{r}+2}\Big]\Big\}.$$

(iii) Case 3: $r = 0$. We have

$$\Big|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right)-(1+\lambda)\frac{f(a)+f(b)}{2}-(1-\lambda)f(\frac{a+b}{2})\Big|$$
$$\leq (b-a)^2\int_0^1|k(t,\lambda)|\left(|f''(a)|^t|f''(b)|^{1-t}\right)dt$$
$$\leq (b-a)^2\Big[\int_0^{\frac{1}{2}}\Big(\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}+\frac{1-\lambda}{2}t\Big)\left(|f''(a)|^t|f''(b)|^{1-t}\right)dt$$
$$+\int_{\frac{1}{2}}^1\Big(\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}+\frac{1-\lambda}{2}(1-t)\Big)\left(|f''(a)|^t|f''(b)|^{1-t}\right)dt\Big]$$
$$= (b-a)^2|f''(b)|\Big[\frac{1}{\alpha+1}\int_0^1[1-t^{\alpha+1}-(1-t)^{\alpha+1}]\Big|\frac{f''(a)}{f''(b)}\Big|^t dt$$
$$+\frac{1-\lambda}{2}\int_0^{\frac{1}{2}}t\Big|\frac{f''(a)}{f''(b)}\Big|^t dt+\frac{1-\lambda}{2}\int_{\frac{1}{2}}^1(1-t)\Big|\frac{f''(a)}{f''(b)}\Big|^t dt\Big]$$

$$= (b-a)^2|f''(b)|\Bigg\{\frac{1}{\alpha+1}\Bigg[\frac{\left|\frac{f''(a)}{f''(b)}\right|-1}{\ln\left|\frac{f''(a)}{f''(b)}\right|}-\left|\frac{f''(a)}{f''(b)}\right|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left(\ln\left|\frac{f''(a)}{f''(b)}\right|\right)^{i-1}}{(\alpha+2)_i}$$

$$-\sum_{i=1}^{\infty}\frac{\left(\ln\left|\frac{f''(a)}{f''(b)}\right|\right)^{i-1}}{(\alpha+2)_i}\Bigg]+\frac{1-\lambda}{4}\left|\frac{f''(a)}{f''(b)}\right|^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{\left(-\frac{1}{2}\ln\left|\frac{f''(a)}{f''(b)}\right|\right)^{i-1}}{(2)_i}$$

$$+\frac{1-\lambda}{2}\Bigg[\frac{\left|\frac{f''(a)}{f''(b)}\right|-\left|\frac{f''(a)}{f''(b)}\right|^{\frac{1}{2}}}{\ln\left|\frac{f''(a)}{f''(b)}\right|}-\left|\frac{f''(a)}{f''(b)}\right|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\left(\ln\left|\frac{f''(a)}{f''(b)}\right|\right)^{i-1}}{(2)_i}\Bigg]\Bigg\}.$$

The proof is done. □

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping with $0\le a<b$. If $|f''|^q$, $q>1$ is integrable and r-convex on $[a,b]$ for some fixed $0\le r<\infty$, then the following inequality for fractional integrals holds*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right)-(1+\lambda)\frac{f(a)+f(b)}{2}-(1-\lambda)f(\frac{a+b}{2})\right|$$

$$\le K_r, \tag{3.2}$$

where

$$K_r := 2^{\frac{1}{qr}}(b-a)^2\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1-\lambda}{4}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}$$

$$\times\left[(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}} \quad \text{for } 0<r\le 1,$$

$$K_r := 2^{\frac{1}{q}}(b-a)^2\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1-\lambda}{4}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}$$

$$\times\left[(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}} \quad \text{for } r>1,$$

$$K_r := 2^{\frac{1}{q}}(b-a)^2|f''(b)|\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1-\lambda}{4}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}$$

$$\times\left(\frac{|k|^q-1}{q\ln|k|}\right)^{\frac{1}{q}} \quad \text{for } r=0, \text{ and } k=\frac{f''(a)}{f''(b)}.$$

Proof. (i) Case 1: $0<r\le 1$. Like Theorem 3.1, we have

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right)-(1+\lambda)\frac{f(a)+f(b)}{2}-(1-\lambda)f(\frac{a+b}{2})\right|$$

$$\begin{aligned}
&\leq (b-a)^2\left(\int_0^1 |k(t,\lambda)|^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\
&\leq (b-a)^2\left[\int_0^{\frac{1}{2}}\left(\left|\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}\right|+\left|\frac{1-\lambda}{2}t\right|\right)^{\frac{q}{q-1}}dt\right.\\
&\left.+\int_{\frac{1}{2}}^1\left(\left|\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}\right|+\left|\frac{1-\lambda}{2}(1-t)\right|\right)^{\frac{q}{q-1}}dt\right]^{1-\frac{1}{q}}\\
&\times\left(2^{\frac{1}{r}-1}\int_0^1\left[|f''(a)|^q t^{\frac{1}{r}}+|f''(b)|^q(1-t)^{\frac{1}{r}}\right]dt\right)^{\frac{1}{q}}\\
&\leq (b-a)^2\left\{2^{\frac{1}{q-1}}\int_0^{\frac{1}{2}}\left[\left(\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\right)^{\frac{q}{q-1}}+\left(\frac{1-\lambda}{2}t\right)^{\frac{q}{q-1}}\right]dt\right.\\
&+2^{\frac{1}{q-1}}\int_{\frac{1}{2}}^1\left[\left(\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\right)^{\frac{q}{q-1}}\right.\\
&\left.\left.+\left(\frac{1-\lambda}{2}(1-t)\right)^{\frac{q}{q-1}}\right]dt\right\}^{1-\frac{1}{q}}\times\left[2^{\frac{1}{r}-1}(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}}\\
&\leq 2^{\frac{1}{q}}(b-a)^2\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\int_0^1\left(1-[t^{\alpha+1}+(1-t)^{\alpha+1}]^{\frac{q}{q-1}}\right)dt\right.\\
&\left.+\left(\frac{1-\lambda}{2}\right)^{\frac{q}{q-1}}\left(\frac{1}{2}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}\\
&\times\left[2^{\frac{1}{r}-1}(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}}\\
&\leq 2^{\frac{1}{q}}(b-a)^2\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\int_0^1\left(1-[t^{\frac{q(\alpha+1)}{q-1}}\right.\right.\\
&\left.\left.+(1-t)^{\frac{q(\alpha+1)}{q-1}}]\right)dt+\left(\frac{1-\lambda}{2}\right)^{\frac{q}{q-1}}\left(\frac{1}{2}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}\\
&\times\left[2^{\frac{1}{r}-1}(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}}\\
&= 2^{\frac{1}{qr}}(b-a)^2\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1-\lambda}{4}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}\\
&\times\left[(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}}.
\end{aligned}$$

(ii) Case 2: $r>1$. Like in Case 1, we have

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right)-(1+\lambda)\frac{f(a)+f(b)}{2}-(1-\lambda)f(\frac{a+b}{2})\right|$$

$$
\begin{aligned}
&\le (b-a)^2\left(\int_0^1 |k(t,\lambda)|^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\
&\le (b-a)^2\Bigg[\int_0^{\frac{1}{2}}\left(\left|\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}\right|+\left|\frac{1-\lambda}{2}t\right|\right)^{\frac{q}{q-1}}dt\\
&\quad+\int_{\frac{1}{2}}^1\left(\left|\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}\right|+\left|\frac{1-\lambda}{2}(1-t)\right|\right)^{\frac{q}{q-1}}dt\Bigg]^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1\left[|f''(a)|^q t^{\frac{1}{r}}+|f''(b)|^q(1-t)^{\frac{1}{r}}\right]dt\right)^{\frac{1}{q}}\\
&\le 2^{\frac{1}{q}}(b-a)^2\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1-\lambda}{4}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}\\
&\quad\times\left[(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}}.
\end{aligned}
$$

(iii) Case 3: $r = 0$. We have

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right)-(1+\lambda)\frac{f(a)+f(b)}{2}-(1-\lambda)f(\frac{a+b}{2})\right|\\
&\le (b-a)^2\left(\int_0^1 |k(m,\lambda)|^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\
&\le (b-a)^2\Bigg[\int_0^{\frac{1}{2}}\left(\left|\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}\right|+\left|\frac{1-\lambda}{2}t\right|\right)^{\frac{q}{q-1}}dt\\
&\quad+\int_{\frac{1}{2}}^1\left(\left|\frac{t^{\alpha+1}+(1-t)^{\alpha+1}-1}{\alpha+1}\right|+\left|\frac{1-\lambda}{2}(1-t)\right|\right)^{\frac{q}{q-1}}dt\Bigg]^{1-\frac{1}{q}}\\
&\quad\times\left(|f''(b)|^q\frac{\left|\frac{f''(a)}{f''(b)}\right|^q-1}{q\ln\left|\frac{f''(a)}{f''(b)}\right|}\right)^{\frac{1}{q}}\\
&\le 2^{\frac{1}{q}}(b-a)^2|f''(b)|\Bigg[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)\\
&\quad+\left(\frac{1-\lambda}{4}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\Bigg]^{1-\frac{1}{q}}\left(\frac{\left|\frac{f''(a)}{f''(b)}\right|^q-1}{q\ln\left|\frac{f''(a)}{f''(b)}\right|}\right)^{\frac{1}{q}}.
\end{aligned}
$$

The proof is done. □

4. An example

In this section we point out some particular inequalities.

Proposition 4.1. *Under the assumptions Theorem 3.1 with* $\lambda = 0$ *in Theorem 3.1, then we get the following inequality,*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right) - \frac{f(a)+f(b)}{2} - f(\frac{a+b}{2})\right| \quad \leq \quad K_r,$$

where

$$\begin{aligned} K_r \quad &:= \quad 2^{\frac{1}{r}-1}(b-a)^2\left\{\frac{|f''(a)|+|f''(b)|}{\alpha+1}\left[\frac{r}{r+1} - \frac{r}{\alpha r+2r+1} - B(\frac{1}{r}+1,\alpha+2)\right]\right. \\ &+\frac{|f''(a)|}{2}\left(\frac{r}{r+1} - \frac{r}{2r+1}\right)\left[1-\left(\frac{1}{2}\right)^{\frac{1}{r}+1}\right] \\ &\left.+\frac{|f''(b)|}{2}\left[B_{\frac{1}{2}}(2,\frac{1}{r}+1) + \frac{r}{2r+1}\left(\frac{1}{2}\right)^{\frac{1}{r}+2}\right]\right\} \quad \text{for } 0 < r \leq 1, \\ K_r \quad &:= \quad (b-a)^2\left\{\frac{|f''(a)|+|f''(b)|}{\alpha+1}\left[\frac{r}{r+1} - \frac{r}{\alpha r+2r+1} - B(\frac{1}{r}+1,\alpha+2)\right]\right. \\ &+\frac{|f''(a)|}{2}\left(\frac{r}{r+1} - \frac{r}{2r+1}\right)\left[1-\left(\frac{1}{2}\right)^{\frac{1}{r}+1}\right] \\ &\left.+\frac{|f''(b)|}{2}\left[B_{\frac{1}{2}}(2,\frac{1}{r}+1) + \frac{r}{2r+1}\left(\frac{1}{2}\right)^{\frac{1}{r}+2}\right]\right\} \quad \text{for } r > 1, \\ K_r &:= (b-a)^2|f''(b)|\left\{\frac{1}{\alpha+1}\left[\frac{|k|-1}{\ln|k|} - |k|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln|k|)^{i-1}}{(\alpha+2)_i} - \sum_{i=1}^{\infty}\frac{(\ln|k|)^{i-1}}{(\alpha+2)_i}\right]\right. \\ &\left.+\frac{|k|^{\frac{1}{2}}}{4}\sum_{i=1}^{\infty}\frac{(-\frac{1}{2}\ln|k|)^{i-1}}{(2)_i} + \frac{1}{2}\left[\frac{|k|-|k|^{\frac{1}{2}}}{\ln|k|} - |k|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln|k|)^{i-1}}{(2)_i}\right]\right\} \\ &\text{for } r = 0, \text{ and } k = \frac{f''(a)}{f''(b)}. \end{aligned}$$

Proposition 4.2. *(Trapezoid Inequality). Under the assumptions Theorem 3.1 with* $\lambda = 1$ *in Theorem 3.1, then we get the following inequality,*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right) - \frac{f(a)+f(b)}{2}\right| \quad \leq \quad K_r,$$

where

$$\begin{aligned} K_r \quad &:= \quad \frac{2^{\frac{1}{r}-2}(b-a)^2(|f''(a)|+|f''(b)|)}{\alpha+1}\left[\frac{r}{r+1} - \frac{r}{\alpha r+2r+1} - B(\frac{1}{r}+1,\alpha+2)\right] \\ &\text{for } 0 < r \leq 1, \end{aligned}$$

$$K_r := \frac{(b-a)^2(|f''(a)|+|f''(b)|)}{2(\alpha+1)}\left[\frac{r}{r+1}-\frac{r}{\alpha r+2r+1}-B(\frac{1}{r}+1,\alpha+2)\right]$$
for $r>1$,
$$K_r := \frac{(b-a)^2|f''(b)|}{2(\alpha+1)}\left[\frac{|k|-1}{\ln|k|}-|k|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln|k|)^{i-1}}{(\alpha+2)_i}-\sum_{i=1}^{\infty}\frac{(\ln|k|)^{i-1}}{(\alpha+2)_i}\right]$$
for $r=0$, *and* $k=\dfrac{f''(a)}{f''(b)}$.

Proposition 4.3. *(Midpoint Inequality). Under the assumptions Theorem 3.1 with* $\lambda = -1$ *in Theorem 3.1, then we get the following inequality,*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left(J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right)-f(\frac{a+b}{2})\right| \leq K_r, \tag{4.1}$$

where

$$K_r := 2^{\frac{1}{r}-2}(b-a)^2\left\{\frac{|f''(a)|+|f''(b)|}{\alpha+1}\left[\frac{r}{r+1}-\frac{r}{\alpha r+2r+1}-B(\frac{1}{r}+1,\alpha+2)\right]\right.$$
$$+|f''(a)|\left(\frac{r}{r+1}-\frac{r}{2r+1}\right)\left[1-\left(\frac{1}{2}\right)^{\frac{1}{r}+1}\right]$$
$$\left.+|f''(b)|\left[B_{\frac{1}{2}}(2,\frac{1}{r}+1)+\frac{r}{2r+1}\left(\frac{1}{2}\right)^{\frac{1}{r}+2}\right]\right\} \quad \textit{for } 0<r\leq 1,$$
$$K_r := \frac{(b-a)^2}{2}\left\{\frac{|f''(a)|+|f''(b)|}{\alpha+1}\left[\frac{r}{r+1}-\frac{r}{\alpha r+2r+1}-B(\frac{1}{r}+1,\alpha+2)\right]\right.$$
$$+|f''(a)|\left(\frac{r}{r+1}-\frac{r}{2r+1}\right)\left[1-\left(\frac{1}{2}\right)^{\frac{1}{r}+1}\right]$$
$$\left.+|f''(b)|\left[B_{\frac{1}{2}}(2,\frac{1}{r}+1)+\frac{r}{2r+1}\left(\frac{1}{2}\right)^{\frac{1}{r}+2}\right]\right\} \quad \textit{for } r>1,$$
$$K_r := \frac{(b-a)^2}{2}|f''(b)|\left\{\frac{1}{\alpha+1}\left[\frac{|k|-1}{\ln|k|}-|k|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln|k|)^{i-1}}{(\alpha+2)_i}-\sum_{i=1}^{\infty}\frac{(\ln|k|)^{i-1}}{(\alpha+2)_i}\right]\right.$$
$$\left.+\frac{1}{2}|k|^{\frac{1}{2}}\sum_{i=1}^{\infty}\frac{(-\frac{1}{2}\ln|k|)^{i-1}}{(2)_i}+\frac{1-\lambda}{2}\left[\frac{|k|-|k|^{\frac{1}{2}}}{\ln|k|}-|k|\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln|k|)^{i-1}}{(2)_i}\right]\right\}$$
for $r=0$, *and* $k=\dfrac{f''(a)}{f''(b)}$.

Proposition 4.4. *Under the assumptions Theorem 3.2 with* $\lambda = 0$ *in Theorem 3.2, then we get the following inequality,*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right)-\frac{f(a)+f(b)}{2}-f(\frac{a+b}{2})\right| \leq K_r,$$

where

$$K_r := 2^{\frac{1}{qr}}(b-a)^2\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1}{4}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}$$
$$\times\left[(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}} \quad \text{for } 0<r\le 1,$$
$$K_r := 2^{\frac{1}{q}}(b-a)^2\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1}{4}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}$$
$$\times\left[(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}} \quad \text{for } r>1,$$
$$K_r := 2^{\frac{1}{q}}(b-a)^2|f''(b)|\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1}{4}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}$$
$$\times\left(\frac{|k|^q-1}{q\ln|k|}\right)^{\frac{1}{q}} \quad \text{for } r=0, \text{ and } k=\frac{f''(a)}{f''(b)}.$$

Proposition 4.5. *Under the assumptions Theorem 3.2 with* $\lambda=1$ *in Theorem 3.2, then we get the following trapezoid inequality,*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left(J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right)-\frac{f(a)+f(b)}{2}\right| \le K_r,$$

where

$$K_r := \frac{2^{\frac{1}{qr}-1}(b-a)^2}{\alpha+1}\left[1-\frac{2(q-1)}{q\alpha+2q-1}\right]^{1-\frac{1}{q}}\left[(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}}$$
$$\text{for } 0<r\le 1,$$
$$K_r := \frac{2^{\frac{1}{q}-1}(b-a)^2}{\alpha+1}\left[1-\frac{2(q-1)}{q\alpha+2q-1}\right]^{1-\frac{1}{q}}\left[(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}}$$
$$\text{for } r>1,$$
$$K_r := \frac{2^{\frac{1}{q}-1}(b-a)^2|f''(b)|}{\alpha+1}\left[1-\frac{2(q-1)}{q\alpha+2q-1}\right]^{1-\frac{1}{q}}\left(\frac{|k|^q-1}{q\ln|k|}\right)^{\frac{1}{q}}$$
$$\text{for } r=0, \text{ and } k=\frac{f''(a)}{f''(b)}.$$

Proposition 4.6. *Under the assumptions Theorem 3.2 with* $\lambda=-1$ *in Theorem 3.2, then we get the following midpoint inequality,*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left(J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right)-f\left(\frac{a+b}{2}\right)\right| \le K_r, \tag{4.2}$$

where

$$K_r := 2^{\frac{1}{qr}-1}(b-a)^2\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1}{2}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}$$
$$\times\left[(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}} \quad \textit{for } 0<r\le 1,$$
$$K_r := 2^{\frac{1}{q}-1}(b-a)^2\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1}{2}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}$$
$$\times\left[(|f''(a)|^q+|f''(b)|^q)\frac{r}{r+1}\right]^{\frac{1}{q}} \quad \textit{for } r>1,$$
$$K_r := 2^{\frac{1}{q}-1}(b-a)^2|f''(b)|\left[\left(\frac{1}{\alpha+1}\right)^{\frac{q}{q-1}}\left(1-\frac{2(q-1)}{q\alpha+2q-1}\right)+\left(\frac{1}{2}\right)^{\frac{q}{q-1}}\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}}$$
$$\times\left(\frac{|k|^q-1}{q\ln|k|}\right)^{\frac{1}{q}} \quad \textit{for } r=0,\ \textit{and } k=\frac{f''(a)}{f''(b)}.$$

5. Applications to special means

We shall consider the following special means:

(a) The arithmetic mean: $A=A(a,b):=\frac{a+b}{2}, a,b\ge 0$,

(b) The geometric mean: $G=G(a,b):=\sqrt{ab}, a,b\ge 0$,

(c) The harmonic mean: $H=H(a,b):=\frac{2ab}{a+b}, a,b>0$,

(d) The logarithmic mean: $L=L(a,b):=\begin{cases} a & if\ a=b \\ \frac{b-a}{\ln b-\ln a} & if\ a\neq b\end{cases} \quad a,b>0$,

(e) The identric mean: $I=I(a,b):=\begin{cases} a & if\ a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & if\ a\neq b\end{cases} \quad a,b>0$,

(f) The p-logarithmic mean: $L_p=L_p(a,b):=\begin{cases} a & if\ a=b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & if\ a\neq b\end{cases}$,
$p\in\mathbb{R}\backslash\{-1,0\}; a,b>0$.

Clearly, L_p is monotonic nondecreasing over $p\in\mathbb{R}$ with $L_{-1}:=L$ and $L_0=I$. In particular, $H\le G\le L\le I\le A$. Now, using the results of Section 4, some new inequalities are derived for the above means.

Proposition 5.1. *Let $a,b\in\mathbb{R}, 0<a<b$ and $n\in\mathbb{N}, n>2$. Then, we have*

$$\left|2L_n^n(a,b)-A(a^n,b^n)-A^n(a,b)\right| \quad\le\quad K_r,$$

where

$$K_r := 2^{\frac{1}{r}-2}n(n-1)(b-a)^2\left[\left(a^{n-2}+b^{n-2}\right)\left(\frac{r}{r+1}-\frac{r}{3r+1}-B(\frac{1}{r}+1,3)\right)\right.$$

$$+a^{n-2}\left(\frac{r}{r+1}-\frac{r}{2r+1}\right)\left(1-\left(\frac{1}{2}\right)^{\frac{1}{r}+1}\right)$$
$$+b^{n-2}\left(B_{\frac{1}{2}}(2,\frac{1}{r}+1)+\frac{r}{2r+1}\left(\frac{1}{2}\right)^{\frac{1}{r}+2}\right)\Bigg]$$
for $0 < r \leq 1$,

$$K_r := \frac{n(n-1)(b-a)^2}{2}\Bigg[\left(a^{n-2}+b^{n-2}\right)\left(\frac{r}{r+1}-\frac{r}{3r+1}-B(\frac{1}{r}+1,3)\right)$$
$$+a^{n-2}\left(\frac{r}{r+1}-\frac{r}{2r+1}\right)\left(1-\left(\frac{1}{2}\right)^{\frac{1}{r}+1}\right)$$
$$+b^{n-2}\left(B_{\frac{1}{2}}(2,\frac{1}{r}+1)+\frac{r}{2r+1}\left(\frac{1}{2}\right)^{\frac{1}{r}+2}\right)\Bigg]$$
for $r > 1$,

$$K_r := \frac{n(n-1)b^{n-2}(b-a)^2}{2}\Bigg[\frac{k-1}{\ln k}-k\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln k)^{i-1}}{(3)_i}-\sum_{i=1}^{\infty}\frac{(\ln k)^{i-1}}{(3)_i}$$
$$+\frac{k^{\frac{1}{2}}}{2}\sum_{i=1}^{\infty}\frac{(-\frac{1}{2}\ln k)^{i-1}}{(2)_i}+\frac{k-k^{\frac{1}{2}}}{\ln k}-k\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln k)^{i-1}}{(2)_i}\Bigg]$$
for $r = 0$, *and* $k = \left(\frac{a}{b}\right)^{n-2}$.

Proof. Let $\alpha = 1$ in Proposition 4.1 and applied to r-convex mapping $f(x) = x^n, n > 2, x \in \mathbb{R}$, here $|f'''(x)| = n(n-1)(n-2)x^{n-3} > 0$, so $|f''(x)|$ is an ordinary convex function, then it is a r-convex mapping (ordinary convexity implies r-convexity, see [11]). □

Proposition 5.2. *Let* $a, b \in \mathbb{R}, 0 < a < b$. *Then, for all* $q > 1$, *we have*

$$\left|L^{-1}(a,b)-H^{-1}(a,b)\right| \leq K_r,$$

where

$$K_r := 2^{\frac{1}{r}-2}(b-a)^2\left(\frac{1}{a^3}+\frac{1}{b^3}\right)\left[\frac{r}{r+1}-\frac{r}{3r+1}-B(\frac{1}{r}+1,3)\right]$$
for $0 < r \leq 1$,

$$K_r := \frac{(b-a)^2(\frac{1}{a^3}+\frac{1}{b^3})}{2}\left[\frac{r}{r+1}-\frac{r}{3r+1}-B(\frac{1}{r}+1,3)\right]$$
for $r > 1$,

$$K_r := \frac{(b-a)^2}{2b^3}\left[\frac{k-1}{\ln k}-k\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln k)^{i-1}}{(3)_i}-\sum_{i=1}^{\infty}\frac{(\ln k)^{i-1}}{(3)_i}\right]$$

$$\text{for } r = 0, \text{ and } k = \left(\frac{b}{a}\right)^3.$$

Proof. Let $\alpha = 1$ in Proposition 4.2 and applied to r-convex mapping $f(x) = \frac{1}{x}, x \in [a, b]$, here $|f'''(x)| = \frac{6}{x^4} > 0$, so $|f''(x)|$ is an ordinary convex function, then it is a r-convex mapping (ordinary convexity implies r-convexity, see [11]). □

Proposition 5.3. *Let $a, b \in \mathbb{R}, 0 < a < b$. Then, for all $q > 1$, we have*

$$\left|2L^{-1}(a,b) - H^{-1}(a,b) - A^{-1}(a,b)\right| \leq K_r,$$

where

$$\begin{aligned} K_r &:= 2^{\frac{1}{qr}-1}(b-a)^2\left[1 - \frac{2(q-1)}{3q-1} + \left(\frac{1}{2}\right)^{\frac{q}{q-1}} \frac{q-1}{2q-1}\right]^{1-\frac{1}{q}} \\ &\quad \times\left[\left(\left(\frac{2}{a^3}\right)^q + \left(\frac{2}{b^3}\right)^q\right)\frac{r}{r+1}\right]^{\frac{1}{q}} \quad \text{for } 0 < r \leq 1, \\ K_r &:= 2^{\frac{1}{q}-1}(b-a)^2\left[1 - \frac{2(q-1)}{3q-1} + \left(\frac{1}{2}\right)^{\frac{q}{q-1}} \frac{q-1}{2q-1}\right]^{1-\frac{1}{q}} \\ &\quad \times\left[\left(\left(\frac{2}{a^3}\right)^q + \left(\frac{2}{b^3}\right)^q\right)\frac{r}{r+1}\right]^{\frac{1}{q}} \quad \text{for } r > 1, \\ K_r &:= \frac{2^{\frac{1}{q}}(b-a)^2}{b^3}\left[1 - \frac{2(q-1)}{3q-1} + \left(\frac{1}{2}\right)^{\frac{q}{q-1}} \frac{q-1}{2q-1}\right]^{1-\frac{1}{q}} \\ &\quad \times\left(\frac{k^q - 1}{q \ln k}\right)^{\frac{1}{q}} \quad \text{for } r = 0, \text{ and } k = \left(\frac{b}{a}\right)^3. \end{aligned}$$

Proof. Let $\alpha = 1$ in Proposition 4.4 and applied to r-convex mapping $f(x) = \frac{1}{x}, x \in [a, b]$. □

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