

## A CHARACTERIZATION OF $U_4(2)$ BY NSE\*

Farnoosh Hajati, Ali Iranmanesh and Abolfazl Tehranian

© 2019 by University of Niš, Serbia | Creative Commons Licence: CC BY-NC-ND

**Abstract.** Let  $G$  be a finite group and  $\omega(G)$  be the set of element orders of  $G$ . Let  $k \in \omega(G)$  and  $m_k$  be the number of elements of order  $k$  in  $G$ . Let  $nse(G) = \{m_k | k \in \omega(G)\}$ . The aim of this paper is to prove that, if  $G$  is a finite group such that  $nse(G) = nse(U_4(2))$ , then  $G \cong U_4(2)$ .

**Keywords.** element order; number of elements of the same order; projective special unitary group; simple  $K_n$  - group.

### 1. Introduction

This section contains the relevant definitions, some standard facts on nse, and a brief exposition of nse history. Throughout this paper,  $G$  is a finite group. We denote by  $\pi(G)$  the set of prime divisors of  $|G|$ , and by  $\omega(G)$ , we introduce the set of order of elements from  $G$ . Set  $m_k = m_k(G) = |\{g \in G | o(g) = k\}|$ , and  $nse(G) = \{m_k | k \in \omega(G)\}$ . In fact,  $m_k$  is the number of elements of order  $k$  in  $G$  and  $nse(G)$  is the set of sizes of elements with the same order in  $G$ .

To the world's mathematics and researchers, one of the important problems in group theory is characterization of a group by a given property, that is, to prove there exists only one group with a given property (up to isomorphism). Until now, different characterizations are investigated for finite simple groups. For instance, in [21, 22] motivated by one of the Thompson's problem, the authors introduced a new characterization for the finite simple group  $G$  by  $nse(G)$  and  $|G|$ . In fact, they proved that if  $G$  is a simple  $K_i$ -group ( $i = 3, 4$ ), then  $G$  is characterizable by  $nse(G)$  and  $|G|$  (The simple group  $G$  is called simple  $K_n$ -group if  $|\pi(G)| = n$ ). Following this result, several groups were characterized by nse and order. For example, in [5, 11], it is proved that Suzuki group, and sporadic groups are characterizable by nse and order.

---

Received February 02, 2019; accepted August 14, 2019  
2010 *Mathematics Subject Classification.* Primary 20D06; Secondary 20D20  
\*Corresponding author: Ali Iranmanesh

We remark here that not all groups can be characterized by their group orders and the set  $nse$ . As an illustration, let  $H_1 = C_4 \times C_4$  and  $H_2 = C_2 \times Q_8$ , where  $C_2$  and  $C_4$  are cyclic groups of order 2 and 4 respectively, and  $Q_8$  is a quaternion group of order 8. It is easy to see that  $nse(H_1) = nse(H_2) = \{1, 3, 12\}$  and  $|H_1| = |H_2| = 16$  but  $H_1 \not\cong H_2$ .

However, it is claimed that some simple groups could be characterized by exactly the set  $nse$  without considering the order of group. In fact, a finite non-abelian simple group  $H$  is called characterizable by  $nse$ , if every finite group  $G$  with  $nse(G) = nse(H)$  implies that  $G \cong H$ . In [7, 8, 9, 10, 12, 13, 24] it is proved that the alternating groups  $A_n$ , where  $n \in \{7, 8\}$ , the symmetric groups  $S_n$  where  $n \in \{3, 4, 5, 6, 7\}$ ,  $M_{12}$ ,  $L_2(27)$ ,  $L_2(q)$  where  $q \in \{16, 17, 19, 23\}$ ,  $L_2(q)$  where  $q \in \{7, 8, 11, 13\}$ ,  $L_2(q)$  where  $q \in \{17, 27, 29\}$ , are uniquely determined by  $nse(G)$ . Besides, in [1, 14, 15, 16] it is proved that  $U_3(4)$ ,  $L_3(4)$ ,  $U_3(5)$ ,  $L_3(5)$ , are uniquely determined by  $nse(G)$ . Recently, in [3, 6, 18, 19], it is proved that the simple groups  $U_3(3)$ ,  $L_3(3)$ ,  $G_2(4)$ ,  $L_2(3^n)$ , where  $|\pi(L_2(3^n))| = 4$ , and  $L_2(2^m)$ , where  $|\pi(L_2(2^m))| = 4$ , are uniquely determined by  $nse(G)$ . Therefore, it is natural to ask what happens with other kinds of simple groups.

In an effort to fill some of the empty ground about the characterization of simple groups by  $nse$ , in this paper we will prove the following main theorem.

**Main Theorem.** Let  $G$  be a group such that  $nse(G) = nse(U_4(2))$ . Then  $G$  is isomorphic to  $U_4(2)$ .

## 2. Notation and Preliminaries

Before we get started, let us fix some notations that will be used throughout the paper. For a natural number  $n$  by  $\pi(n)$ , we mean the set of all prime divisors of  $n$ , so it is obvious that if  $G$  is a finite group, then  $\pi(G) = \pi(|G|)$ . A Sylow  $r$ -subgroup of  $G$  is denoted by  $P_r$  and by  $n_r(G)$ , we mean the number of Sylow  $r$ -subgroup of  $G$ . Also the largest element order of  $P_r$  is signified by  $exp(P_r)$ . Moreover, we denote by  $\phi$  the Euler function. In the following, we bring some useful lemmas which be used in the proof of the main theorem.

**Lemma 2.1.** [25]. *Let  $G$  be a group containing more than two elements. If the maximal number  $s$  of elements of the same order in  $G$  is finite, then  $G$  is finite and  $|G| \leq s(s^2 - 1)$ .*

**Lemma 2.2.** [24]. *Let  $G$  be a group. If  $1 \neq n \in nse(G)$  and  $2 \nmid n$ , then the following statements hold:*

- (1)  $2 \parallel |G|$ ;
- (2)  $m_2 = n$ ;
- (3) for any  $2 < t \in \omega(G)$ ,  $m_t \neq n$ .

**Lemma 2.3.** [2]. *Let  $G$  be a finite group and  $m$  be a positive integer dividing  $|G|$ . If  $L_m(G) = \{g \in G \mid g^m = 1\}$ , then  $m \parallel |L_m(G)|$ .*

**Lemma 2.4.** [23]. *Let  $G$  be a group and  $P$  be a cyclic Sylow  $p$ -group of  $G$  of order  $p^\alpha$ . If there is a prime  $r$  such that  $p^\alpha r \in \omega(G)$ , then  $m_{p^\alpha r} = m_r(C_G(P))m_{p^\alpha}$ . In particular  $\phi(r)m_{p^\alpha} | m_{p^\alpha r}$ , where  $\phi(r)$  is the Euler function of  $r$ .*

**Lemma 2.5.** [17]. *Let  $G$  be a finite group and  $p \in \pi(G)$  be odd. Suppose that  $P$  is a Sylow  $p$ -subgroup of  $G$  and  $n = p^s m$ , where  $(p, m) = 1$ . If  $P$  is not cyclic group and  $s > 1$ , then the number of elements of order  $n$  is always a multiple of  $p^s$ .*

We say that a group  $G$  acts semi regularly on set  $X$  if  $G$  acts on  $X$  in such a way that  $G_x = 1$  for all  $x \in X$ .

**Lemma 2.6.** [20]. *Let the finite group  $G$  acts on the finite set  $X$ . If the action is semi regular, then  $|G| \mid |X|$ .*

Let us mention the structure of simple  $K_3$ -groups, that will be needed in Section 3.

**Lemma 2.7.** [4]. *If  $G$  is a simple  $K_3$ -group, then  $G$  is isomorphic to one of the following groups:*

$$A_5, A_6, L_2(7), L_2(8), L_2(17), L_3(3), U_3(3), U_4(2).$$

**Lemma 2.8.** [22]. *Let  $G$  be a group and  $M$  a simple  $K_3$ -group. Then  $G \cong M$  if and only if the following hold: (1)  $|G| = |M|$ , (2)  $nse(G) = nse(M)$ .*

### 3. Main Theorem and its Proof

Suppose  $G$  is a group such that  $nse(G) = nse(U_4(2))$ . By Lemma 2.1, we can assume that  $G$  is finite. Let  $m_n$  be the number of elements of order  $n$ . We notice that  $m_n = k\phi(n)$ , where  $k$  is the number of cyclic subgroups of order  $n$  in  $G$ . In addition, we notice that if  $n > 2$ , then  $\phi(n)$  is even. If  $n \in \omega(G)$ , then by Lemma 2.3 and the above argument, we have

$$(3.1) \quad \begin{cases} \phi(n) | m_n \\ n | \sum_{d|n} m_d \end{cases}$$

In the proof of the main theorem, we often apply formula (3.1) and the above comments.

**Proof of the Main Theorem.** Let  $G$  be a group with

$$nse(G) = nse(U_4(2)) = \{1, 315, 800, 3780, 4320, 5184, 5760\}$$

where  $U_4(2)$  is the projective special unitary group of degree 4 over field of order 2. We have divided the proof into a sequence of lemmas.

**Remark 3.1.** Let  $2 \neq p \in \pi(G)$ , by formula (3.1),  $p|(1 + m_p)$  and  $(p - 1)|m_p$ , which implies that  $p \in \{3, 5, 7, 17, 19\}$ .

In the following lemma, we prove some basic properties of group  $G$ :

**Lemma 3.1.** *If  $p \in \pi(G)$  and  $p \in \{2, 3, 5\}$ , then*

- (1)  $2 \in \pi(G)$  and  $m_2 = 315$ ;
- (2)  $m_3 = 800$ ,  $m_5 = 5184$ ;
- (3)  $\{5^2, 3^6, 2^9\} \cap \omega(G) = \emptyset$ ;
- (4)  $|P_2||2^9$ .

*Proof.* The proof is straightforward according to Lemma 2.2, Lemma 2.3, and formula (3.1).  $\square$

**Lemma 3.2.**  $\{17, 19\} \cap \pi(G) = \emptyset$ .

*Proof.* We prove that  $17 \notin \pi(G)$ . Conversely, suppose that  $17 \in \pi(G)$ . Then formula (3.1) implies  $m_{17} = 5184$ . On the other hand, by formula (3.1), we conclude that if  $2.17 \in \omega(G)$ , then  $m_{2.17} \in \{800, 4320, 5184, 5760\}$  and  $2.17|1 + m_2 + m_{17} + m_{2.17} (= 6300, 9820, 10684, 11260)$ , which is a contradiction, and hence  $2.17 \notin \omega(G)$ . Since  $2.17 \notin \omega(G)$ , the group  $P_{17}$  acts fixed point freely on the set of elements of order 2 of  $G$  and by Lemma 2.6,  $|P_{17}||m_2$ , which is a contradiction. Hence,  $17 \notin \pi(G)$ . Similarly, we can prove that  $19 \notin \pi(G)$ .  $\square$

To remove the prime 7, let us first show that  $5 \in \pi(G)$ .

**Lemma 3.3.**  $\{5\} \cap \pi(G) = \{5\}$ .

*Proof.* Assume that  $5 \notin \pi(G)$ .

• If  $3, 7 \notin \pi(G)$ , then  $G$  is a 2-group. Since  $2^9 \notin \omega(G)$ , we have  $\omega(G) \subseteq \{1, 2, 2^2, \dots, 2^8\}$ . Hence  $|G| = 2^m = 20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$ , where  $k_1, k_2, k_3, k_4, k_5$  and  $m$  are non-negative integers, and  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 2$ . It is obvious that  $20160 \leq |G| \leq 20160 + (k_1 + k_2 + k_3 + k_4 + k_5)5760$  and so  $20160 \leq |G| \leq 20160 + 2.5760$ . Now, it is easily seen that the equation has no solution.

Hence 3 or 7 belongs to  $\pi(G)$ , and the following cases are considered.

• If  $7 \in \pi(G)$ , by formula (3.1)  $m_7 = 5760$ , then as  $\exp(P_7) = 7$ ,  $|P_7||1 + m_7$  and so  $|P_7| = 7$ . Since  $n_7 = \frac{m_7}{\phi(7)} = 2^6.3.5|G|$ , it follows that  $5 \in \pi(G)$ , which is a contradiction.

• If  $3 \in \pi(G)$ , then  $\exp(P_3) = 3, 3^2, 3^3, 3^4, 3^5$ .

★ If  $\exp(P_3) = 3$ , then by Lemma 2.3,  $|P_3||(1 + m_3)$  and so  $|P_3||3^2$ . We will consider two cases for  $|P_3|$ .

*Case 1* If  $|P_3| = 3$ , then since  $n_3 = \frac{m_3}{\phi(3)} = 2^3.5^3|G|$ ,  $5 \in \pi(G)$  which is a contradiction.

*Case 2* If  $|P_3| = 3^2$ , then since  $5, 7 \notin \pi(G)$  and  $\pi(G) \subseteq \{2, 3, 5, 7\}$ , we can assume that  $\{2\} \subseteq \pi(G) \subseteq \{2, 3\}$ , and so we have

$$\omega(G) \subseteq \{1, 2, \dots, 2^8\} \cup \{3, 3.2, 3.2^2, 3.2^3, \dots, 3.2^7\}$$

$(2^8.3 \notin \omega(G)$  by formula (3.1)) and  $|\omega(G)| \leq 17$ . Therefore  $20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5 = |G| = 2^a.9$  where  $k_1, k_2, k_3, k_4, k_5$ , and  $a$  are non-negative integers and  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 10$ . Since  $20160 \leq 2^a.9 \leq 20160 + 10.5760$ , we have  $a = 12$ , or  $a = 13$ . If  $a = 12$ , then since  $|P_2||2^9$ , we have a contradiction. Similarly, we can rule out  $a = 13$ .

★ If  $\exp(P_3) = 3^2$ , then, by Lemma 2.3,  $|P_3|(1 + m_3 + m_{3^2})$  and so  $|P_3||3^8$ . (for example, when  $m_9 = 5760$ ). We will consider seven cases for  $|P_3|$ .

*Case 1.* If  $|P_3| = 3^2$ , then  $n_3 = \frac{m_9}{\phi(9)}$ , since  $m_9 \in \{3780, 4320, 5184, 5760\}$ ,  $n_3 = 3^2.2.5.7$ ,  $n_3 = 2^4.3^2.5$ , or  $n_3 = 2^6.3.5$ , and so  $5 \in \pi(G)$ , which is a contradiction, and if  $n_3 = 2^5.3^3$ , since  $n_3 \not\equiv 1 \pmod{3}$ , we have a contradiction.

*Case 2.* If  $|P_3| = 3^3$ , then since  $5, 7 \notin \pi(G)$ , we can assume that  $\{2\} \subseteq \pi(G) \subseteq \{2, 3\}$  and so we have  $\omega(G) \subseteq \{1, 2, \dots, 2^8\} \cup \{3, 3.2, 3.2^2, \dots, 3.2^7\} \cup \{3^2, 3^2.2, 3^2.2^2, \dots, 3^2.2^7\}$  ( $2^8.3 \notin \omega(G)$ ,  $2^8.3^2 \notin \omega(G)$  by formula (3.1)) and  $|\omega(G)| \leq 25$ . Therefore  $20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5 = |G| = 2^a.27$ , where  $k_1, k_2, k_3, k_4, k_5$ , and  $a$  are non-negative integers and  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 18$ . Since  $20160 \leq 2^a.27 \leq 20160 + 18.5760$ , we have  $a = 10$ ,  $a = 11$ , or  $a = 12$ .

If  $a = 10$ , then since  $|P_2||2^9$ , we have a contradiction. Similarly, we can rule out  $a = 11$  and  $a = 12$ .

*Case 3.* If  $|P_3| = 3^4$ , then since  $\exp(P_3) = 3^2$  and  $2^8.3, 2^8.9 \notin \omega(G)$ ,  $\omega(G) \subseteq \{1, \dots, 2^8\} \cup \{3, \dots, 3.2^7\} \cup \{3^2, \dots, 3^2.2^7\}$ . On the other hand, if  $2^8 \in \omega(G)$  since  $2^8.3 \notin \omega(G)$ , the group  $P_3$  acts fixed point freely on the set of elements of order  $2^8$ . Hence  $|P_3||m_{2^8} = 5760$ , which is a contradiction. Hence  $2^8 \notin \omega(G)$  and  $|\omega(G)| \leq 24$ . Therefore  $20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5 = |G| = 2^a.81$ , where  $k_1, k_2, k_3, k_4, k_5$ , and  $a$  are non-negative integers and  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 17$ . Since  $20160 \leq 2^a.81 \leq 20160 + 17.5760$ , we have  $a = 8$ ,  $a = 9$ , or  $a = 10$ .

If  $a = 8$ , then  $576 = 800k_1 + 3760k_2 + 4320k_3 + 5184k_4 + 5760k_5$  where  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 17$ . By a computer calculation, it is easy to see this equation has no solution.

If  $a = 9$ , then  $21312 = 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$  where  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 17$ . The only solution of this equation is  $(0, 0, 0, 3, 1)$ . We show this is impossible. Since  $|\omega(G)| = 11$  and  $2^8 \notin \omega(G)$ ,  $\exp(P_2) = 2^i$ , where  $3 \leq i \leq 7$ . Hence, if  $\exp(P_2) = 2^i$  where  $3 \leq i \leq 7$ , then  $|P_2|(1 + m_2 + m_4 + \dots + m_{2^i})$  by Lemma 2.3. In fact  $|P_2|(1 + 315 + 800t_1 + 3780t_2 + 4320t_3 + 5184t_4 + 5760t_5)$  where  $t_1, t_2, t_3, t_4, t_5$ , are non-negative integers and  $0 \leq t_1 + t_2 + t_3 + t_4 + t_5 \leq 6$ . Because  $k_1 = 0$  and  $m_3 = 800$ ,  $m_{2^i} \neq 800$  for  $1 \leq i \leq 7$ ,  $t_1 = 0$ . Since  $k_2 = 0$ ,  $0 \leq t_2 \leq 1$ . We claim  $t_2 = 0$ . Suppose, contrary to our claim,  $t_2 = 1$ . If  $m_4 = 3780$ , then since  $m_9 \in \{3780, 4320, 5184, 5760\}$ , we have a contradiction and so  $t_2 = 0$ . If  $m_4 \neq 3780$ , then by a computer calculation  $m_8 = 3780$ , since  $m_9 \in \{3780, 4320, 5184, 5760\}$ , we have a contradiction and so  $t_2 = 0$ . Also  $k_3 = 0, k_4 = 3$ , and  $k_5 = 1$ , thus  $0 \leq t_3 \leq 1, 0 \leq t_4 \leq 4$ , and  $0 \leq t_5 \leq 2$ . By an easy computer calculation, this is impossible.

If  $a = 10$ , then since  $|P_2||2^9$ , we have a contradiction.

Similarly, we can rule out the other cases.

★ If  $\exp(P_3) = 3^3$ , then by Lemma 2.3,  $|P_3|(1 + m_3 + m_{3^2} + m_{3^3})$  and so  $|P_3||3^4$  (for example when  $m_9 = 5184$  and  $m_{27} = 5760$ ). We will consider two cases for  $|P_3|$ .

*Case 1.* If  $|P_3| = 3^3$ , then  $n_3 = \frac{m_{27}}{\phi(27)}$ , since  $m_{27} \in \{3780, 4320, 5184, 5760\}$ ,  $n_3 = 2.3.5.7$ ,  $n_3 = 2^4.3.5$ , or  $n_3 = 2^6.5$ , and so  $5 \in \pi(G)$ , which is a contradiction, and if  $n_3 = 2^5.3^2$ , since  $n_3 \not\equiv 1 \pmod{3}$ , we have a contradiction.

*Case 2.* If  $|P_3| = 3^4$ , then by Lemma 2.5,  $27|m_{27}$ . Since  $(27 \nmid 5760)$ , it is understood that  $m_{27} \in \{3780, 4320, 5184\}$ . Since  $2^8.3 \notin \omega(G)$ ,  $2^8.3^2 \notin \omega(G)$ ,  $2^8.3^3 \notin \omega(G)$ , and  $2^8 \notin \omega(G)$ ,  $|\omega(G)| \leq 32$ . Therefore  $20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5 = |G| = 2^a.81$ , where  $k_1, k_2, k_3, k_4, k_5$ , and  $a$  are non-negative integers and  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 25$ . Since  $20160 \leq 2^a.81 \leq 20160 + 25.5760$ , we have  $a = 8$ ,  $a = 9$ , or  $a = 10$ .

If  $a = 8$ , then  $576 = 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$  where  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 25$ . By a computer calculation, it is easily seen that the equation has no solution.

If  $a = 9$ , then  $21312 = 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$  where  $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 25$ . By a computer calculation, the only solution of this equation is  $(0, 0, 0, 3, 1)$ . We show this is impossible. Since  $|\omega(G)| = 11$  and  $2^8 \notin \omega(G)$ ,  $\exp(P_2) = 2^i$ , where  $3 \leq i \leq 7$ . Hence, if  $\exp(P_2) = 2^i$ , where  $3 \leq i \leq 7$  then  $|P_2|(1 + m_2 + m_4 + \dots + m_{2^i})$  by Lemma 2.3. In fact  $|P_2|(1 + 315 + 800t_1 + 3780t_2 + 4320t_3 + 5184t_4 + 5760t_5)$  where  $t_1, t_2, t_3, t_4, t_5$ , are non-negative integers and  $0 \leq t_1 + t_2 + t_3 + t_4 + t_5 \leq 6$ . Because  $k_1 = 0$  and  $m_3 = 800$ ,  $m_{2^i} \neq 800$  for  $1 \leq i \leq 7$ ,  $t_1 = 0$ . Since  $k_2 = 0$ ,  $0 \leq t_2 \leq 1$ . We claim  $t_2 = 0$ . Suppose, contrary to our claim,  $t_2 = 1$ . If  $m_4 = 3780$ , then since  $m_{27} \in \{3780, 4320, 5184\}$ , we have a contradiction and so  $t_2 = 0$ . If  $m_4 \neq 3780$ , then by computer calculation  $m_8 = 3780$ , since  $m_{27} \in \{3780, 4320, 5184\}$ , we have a contradiction and so  $t_2 = 0$ . Also  $k_3 = 0, k_4 = 3$ , and  $k_5 = 1$ , thus  $0 \leq t_3 \leq 1$ ,  $0 \leq t_4 \leq 4$ , and  $0 \leq t_5 \leq 2$ . By an easy computer calculation, this is impossible.

If  $a = 10$ , then since  $|P_2||2^9$ , we have a contradiction.

★ If  $\exp(P_3) = 3^4$ , then by Lemma 2.3,  $|P_3|(1 + m_3 + m_{3^2} + m_{3^3} + m_{3^4})$  and so  $|P_3||3^4$  (for example when  $m_9 = 5760$ ,  $m_{27} = 3780$ , and  $m_{81} = 4320$ ).

If  $|P_3| = 3^4$ , then  $n_3 = \frac{m_{81}}{\phi(81)}$ , since  $m_{81} \in \{3780, 4320, 5184\}$ ,  $n_3 = 2^4.5$ , or  $n_3 = 2.5.7$ , and so  $5 \in \pi(G)$ , which is a contradiction, and if  $n_3 = 2^5.3$ , since a cyclic group of order 81 has two elements of order 3,  $m_3 \leq 2^5.3.2 = 192$ , which is a contradiction.

★ If  $\exp(P_3) = 3^5$ , then by Lemma 2.3,  $|P_3|(1 + m_3 + m_{3^2} + m_{3^3} + m_{3^4} + m_{3^5})$  and so  $|P_3||3^5$  (for example when  $m_9 = 5184$ ,  $m_{27} = 5760$ , and  $m_{81} = m_{243} = 5184$ ). In a similar way we have a contradiction. Therefore,  $5 \in \pi(G)$ .  $\square$

**Lemma 3.4.**  $\{7\} \cap \pi(G) = \emptyset$ .

*Proof.* By Lemma 2.3  $|P_5||1 + m_5$  and so  $|P_5| = 5$ . In the following, that the prime 7 do not belong to  $\pi(G)$  is proved. Let  $7 \in \pi(G)$ . Then formula (3.1) implies

$m_{7.5} \in \{4320, 5184, 5760\}$  and  $7.5|1+m_5+m_7+m_{5.7}(= 15256, 16129, 16705)$ , which is a contradiction, and hence  $5.7 \notin \omega(G)$ . It follows that the Sylow 7-subgroup of  $G$  acts fixed point freely on the set of elements of order 5 and so  $|P_7||m_5$ , which is a contradiction. Hence  $7 \notin \pi(G)$ .  $\square$

From what has already been proved, we conclude  $2, 5 \in \pi(G)$ , so the following cases will be considered  $\{2, 5\}, \{2, 3, 5\}$ .

**Lemma 3.5.**  $\pi(G) = \{2, 3, 5\}$ .

*Proof.* If  $\pi(G) = \{2, 5\}$ , since  $\exp(P_5) = 5$ , then by Lemma 2.3,  $|P_5||1+m_5$ , and so  $|P_5| = 5$ . Since  $n_5 = \frac{m_5}{\phi(5)} = 2^4.3^4$ , it follows that 3 belongs to  $\pi(G)$ , which is a contradiction. Hence  $\pi(G) = \{2, 3, 5\}$ . The proof is completed by showing that  $|G| = |U_4(2)|$ .  $\square$

**Lemma 3.6.**  $G \cong U_4(2)$ .

*Proof.* First, we show that  $|G| = |U_4(2)|$ . From the above arguments, we have  $|P_5| = 5$ . Now, we prove  $10 \notin \omega(G)$ . Conversely, suppose that  $10 \in \omega(G)$ . Then formula (3.1) implies  $m_{10} \in \{800, 3780, 4320, 5760\}$ . On the other hand, if  $2.5 \in \omega(G)$ , then by Lemma 2.4,  $m_{2.5} = m_5.\phi(2).t$  for some integer  $t$ , which is a contradiction and hence  $2.5 \notin \omega(G)$ . Since  $2.5 \notin \omega(G)$ , the group  $P_2$  acts fixed point freely on the set of elements of order 5, and so  $|P_2||m_5$ , hence  $|P_2||3^4.2^6$ . In fact  $|P_2||2^6$ . In the same way, since  $15 \notin \omega(G)$ ,  $|P_3||m_5$  and hence  $|P_3||3^4.2^6$ . In fact  $|P_3||3^4$ . Therefore we have  $|G| = 2^m.3^n.5$ . Since  $20160 = 2^6.3^2.5.7 \leq |G| = 2^m.3^n.5$ ,  $|G| = 2^6.3^4.5$ . Hence  $|G| = 2^6.3^4.5 = |U_4(2)|$  and by assumption  $nse(G) = nse(U_4(2))$ , so by Lemma 2.8,  $G \cong U_4(2)$  and the proof is completed.  $\square$

**Acknowledgments:** The authors would like to express their deep gratitude to the referees for their helpful comments and valuable suggestion for improvement of this paper. Part of this research work was done while the second author was spending his sabbatical leave at the Department of Mathematics of University of California, Berkeley. This author expresses his thanks for the hospitality and facilities provided by Department of Mathematics of UCB.

## REFERENCES

1. D. CHEN : *A characterization of  $PSU(3,4)$  by nse*. International Journal of Algebra and Statistics **2** (2013), 51-56.
2. G. FROBENIUS: *Verallgemeinerung des Sylowschen Satze*. Berl. Ber. (1895), 981-993.
3. F. HAJATI, A. IRANMANESH, and A. TEHRANIAN: *A characterization of projective special unitary group  $PSU(3,3)$  and projective special linear group  $PSL(3,3)$  by nse*. Mathematics **6**, no.7 (2018).
4. M. HERZOG : *On finite simple groups of order divisible by three primes only*. J. Algebra **10**, no.3, (1968), 383-388.

5. A. IRANMANESH, H. PARVIZI MOSAED and A. TEHRANIAN,: *Characterization of Suzuki group by nse and order of group*. Bull.Korean Math.Soc. **53**, no. 3, (2016), 651-656, .
6. M. JAHANDIDEH KHANGHESHLAGHI and M. R. DARAFSHEH: *Nse characterization of the Chevalley group  $G_2(4)$* . Arabian Journal of Mathematics **7** (2018), 21-26.
7. A. KHALILI ASBOEI, S. S. SALEHI AMIRI, A. IRANMANESH, and A. TEHRANIAN: *A new characterization of  $A_7, A_8$* .. Anale Stintifice ale Universitatii Ovidius Constanta **21** (2013), 43-50 .
8. A. KHALILI ASBOEI, S. S. SALEHI AMIRI and A. IRANMANESH: *A new characterization of Symmetric groups for some n*. Hacettepe Journal of Mathematics and Statistics **43**, (2013), 715-723.
9. A. KHALILI ASBOEI, S. S. SALEHI AMIRI and A. IRANMANESH: *A new note on characterization of a Mathieu group of degree 12*. Southeast Asian Bulletin of Mathematics **38** (2014), 383-388.
10. A. KHALILI ASBOEI : *A new characterization of  $PSL(2,27)$* . Bol. Soc. Paran. Mat. **32** (2014), no.1, 43-50.
11. A. KHALILI ASBOEI, S. S. SALEHI AMIRI, A. IRANMANESH, and A. TEHRANIAN: *A characterization of sporadic simple groups by nse and order*. J.Algebra and its Applications **12** (2013), no.2.
12. A. KHALILI ASBOEI, S. S. SALEHI AMIRI, and A. IRANMANESH: *A new characterization of  $PSL(2,q)$  for some q*.. Ukrainian Mathematical Journal **67** (2016), no.9, 1297-1305.
13. M. KHATAMI, B. KHOSRAVI and Z. AKHLAGHI: *A new characterization for some linear groups*. Monatsh. Math. **163** (2011), no.1, 39-50.
14. S. LIU : *A characterization of  $L_3(4)$* . Sci. Asia **39** (2013), 436-439.
15. S. LIU : *A characterization of projective special unitary group  $U_3(5)$  by nse*. Arab Journal of Mathematical Sciences **20** (2014), no.1, 133-140.
16. S. LIU : *A characterization of projective special linear group  $L_3(5)$  by nse* . Italian Journal of Pure and Applied Mathematics (2014), no.32, 203-212.
17. G. A. MILLER : *Addition to a theorem due to Frobenius*.. Bull. Amer. Math. Soc. **11** (1904), no.1, 6-7.
18. H. PARVIZI MOSAED, A. IRANMANESH and A. TEHRANIAN: *Nse characterization of simple group  $L_2(3^n)$* . Publications De L' institut Mathematique Nouvelle Serie **99** (2016), no.113, 193-201.
19. H. PARVIZI MOSAED, A. IRANMANESH, M. FOROUDI GHASEMABADI and A. TEHRANIAN: *A new characterization of simple group  $L_2(2^m)$* .. Hacettepe Journal of Mathematics and Statistics **44** (2016), no.4, 875-886 .
20. D. PASSMAN: *Permutation Groups*. W.A. Benjamin, New York, 1968.
21. C. SHOA, W. SHI and Q. JIANG: *Characterization of simple  $K_4$ -groups*.. Front. Math.China **3** (2008), no.3, 355-370.
22. C. SHOA, W. SHI, and Q. JIANG: *A characterization of simple  $K_3$ -groups*. Advances in Mathematics **38** (2009), no.3, 327-330.
23. C. SHOA and Q. JIANG: *A new characterization of some linear groups by nse*. Journal of Algebra and its Applications **13** (2014), no. 2.
24. C. SHOA and Q. JIANG,: *Characterization of groups  $L_2(q)$  by nse where  $q \in \{17, 27, 29\}$* . Chin. Ann. Math. **37B** (2016), no.1, 103-110.



25. R. SHEN, C. SHOA, W. SHI, Q. JIANG, and V. MAZUROV: *A new characterization of  $A_5$* , *Monatsh. Math.* **160** (2010), 337-341.
26. W. SHI: *A new characterization of sporadic simple groups*. In: *Group Theory*, (Proceeding of the 1987 Singapore Conference on Group Theory) Walter de Gruyter, Berlin 1989, pp. 531-540.

Farnoosh Hajati  
Department of Mathematics, Science and Research Branch  
Islamic Azad University  
P. O. Box 14515-775  
Tehran, Iran  
`F_hajati@azad.ac.ir`

Ali. Iranmanesh  
Faculty of Mathematical Sciences  
Department of Mathematics  
Tarbiat Modares University  
P. O. Box 14115-137  
Tehran, Iran  
`iranmanesh@modares.ac.ir`

Abolfazl. Tehranian  
Department of Mathematics, Science and Research Branch  
Islamic Azad University  
P. O. Box 14515-775  
Tehran, Iran  
`tehranian@srbiau.ac.ir`