ON THE EIGENVALUES OF N-CAYLEY GRAPHS: A SURVEY

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Abstract. A graph Γ is called an *n-Cayley graph over a group G* if Aut(Γ) contains a semi-regular subgroup isomorphic to G with n orbits. In this paper, we review some recent results and future directions around the problem of computing the eigenvalues on n-Cayley graphs.
Keywords: n-Cayley graph; eigenvalues; semi-regular subgroup.

1. Introduction

The spectrum of a graph is one of the most important algebraic invariants as it is known that numerous proofs in graph theory depend on the spectrum of graphs. In particular, eigenvalues of Cayley graphs have attracted increasing attention due to their prominent roles in algebraic graph theory and applications in many areas such as expanders, chemical graph theory, quantum computing, etc [21]. This paper is a survey of the literature on the eigenvalues of graphs having a semi-regular of subgroup of their automorphism groups.

A digraph Γ is a pair (V, E) of vertices V and edges E where $E \subseteq V \times V$. A graph is a digraph with no edges of the form (α, α) and with the property that $(\alpha, \beta) \in E$ implies $(\beta, \alpha) \in E$. The set of all permutations of V which preserve the adjacency structure of Γ is called the automorphism group of Γ; it is denoted by Aut(Γ). In this paper all digraphs have no loops. For the most part our notation and terminology are standard and mainly taken from [9] (for graph theory) and [16] (for representation theory of finite groups). For the graph-theoretic and group-theoretic terminology not defined here we refer the reader to [9, 16].

Let Γ be a (di)graph with n vertices. The adjacency matrix of A of Γ is an $n \times n$ matrix with ij-entry equal to 1 if ith and jth vertices are adjacent and 0 otherwise. The *spectrum* of a graph is the multi-set of eigenvalues of its adjacency matrix. It

is known that numerous proofs in graph theory depend on the spectrum of graphs and the spectrum of a graph is one of the most important algebraic invariants [9].

Let G be a group and S be a subset of G not containing the identity element of G. The *Cayley (di)graph of G with respect to S* is a graph with a vertex set G where (g, h) is an arc whenever $hg^{-1} \in S$. A large number of results on spectra of Cayley graphs have been produced over the last more than four decades. For a survey of the literature on eigenvalues of Cayley graphs and their applications see [21].

By a theorem of Sabidussi [26], a (di)graph Γ is a Cayley graph over a group G if $\mathrm{Aut}(\Gamma)$ contains a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G. As a generalization, a (di)graph Γ is called an *n-Cayley (di)graph over a group G* if there exists a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G with n orbits (of equal size). Since every regular subgroup is a transitive semi-regular subgroup, every Cayley (di)graphs is a 1-Cayley (di)graph. Also a Cayley graph over a finite group G having a subgroup H of index n is an n-Cayley graph over H [1, Lemma 8]. n-Cayley graphs over cyclic groups are called n-circulant. In particular 2-Cayley and 3-Cayley graphs over cyclic groups are called bicirculant and tricirculant graphs [24], respectively. Unlike Cayley graphs, in general n-Cayley graphs are not vertex-transitive for $n \geq 2$. Furthermore, there are vertex-transitive n-Cayley graphs which are not Cayley graphs such as generalized Petersen graphs. Undirected and loop-free 2-Cayley graphs are called, by some authors, semi-Cayley graphs [25, 3] and also bi-Cayley graphs [17]. In this paper, we follow [25] to use the term semi-Cayley.

n-Cayley graphs, in particular when $n = 2$ or $n = 3$, have played an important role in many classical fields of graph theory, such as strongly regular graphs [19, 22, 23, 24, 25], automorphisms [2, 15, 28], isomorphisms [3, 5], symmetry properties of graphs [10, 11, 20] and the spectrum of graphs [1, 4, 8, 12, 13]. In this paper, we review recent results and future directions of some problems related to the spectrum of n-Cayley graphs.

2. Presentation of n-Cayley graphs

Recall that a (di)graph Γ is called an n-Cayley graph over a group G if $\mathrm{Aut}(\Gamma)$ contains a semi-regular subgroup isomorphic to G with n orbits (of equal size). It is proved that every n-Cayley graph over a group G can be presented by suitable n^2 subsets of G:

Lemma 2.1. *([1, Lemma 2]) A digraph Γ is n-Cayley digraph over G if and only if there exist subsets T_{ij} of G, where $1 \leq i, j \leq n$, such that Γ is isomorphic to a digraph X with vertex set $G \times \{1, 2, \ldots, n\}$ and edge set*

$$E(X) = \bigcup_{1 \leq i,j \leq n} \left\{ ((g, i), (tg, j)) \mid g \in G \ \text{ and } \ t \in T_{ij} \right\}.$$

By Lemma 2.1, an n-Cayley (di)graph is characterized by a group G and n^2 subsets T_{ij} of G (some subsets may be empty). So we denote an n-Cayley (di)graph

with respect to n^2 subsets T_{ij} by $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$. It is easy to see that $\mathrm{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$ is undirected if and only if $T_{ij}^{-1} = T_{ji}$ for all $1 \leq i, j \leq n$. Also it is loop-free if $1 \notin T_{ii}$ for all $1 \leq i \leq n$. Let Γ be a 2-Cayley graph which is undirected and loop-free. Then there exist three subsets $R = T_{11}$, $L = T_{22}$, $S = T_{12}$ and $T_{21} = S^{-1}$ of G such that $R = R^{-1}$, $L = L^{-1}$ and $1 \notin R \cup L$ and $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i, j \leq 2)$. We denote this graph with $\mathrm{SC}(G; R, L, S)$ and call it semi-Cayley graph. In the case $R = L = \emptyset$, we denote it by $\mathrm{BCay}(G; S)$ and call it bi-Cayley graph.

There are a lot of examples of n-Cayley graphs, $n \geq 2$. Here we provide some.

Example 2.1. Let P be the Petersen graph. Then $P = \mathrm{SC}(G; R, L, S)$, where $G = \langle a \rangle \cong \mathbb{Z}_5$, $R = \{a, a^4\}$, $L = \{a^2, a^3\}$ and $S = \{1\}$.

Example 2.2. ([1, Lemma 8]) Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley (di)graph. Suppose that there exists a subgroup H of G with index n. If $\{t_1, \ldots, t_n\}$ is a left transversal to H in G, then $\Gamma \cong \mathrm{Cay}(H, T_{ij} \mid 1 \leq i, j \leq n)$, where $T_{ij} = \{h \in H \mid t_j^{-1} h t_i \in S\} = H \cap t_j S t_i^{-1}$.

Example 2.3. The I-graph $I(n, j, k)$ is a cubic graph of order $2n$ with vertex set $\{u_i, v_i \mid 0 \leq i \leq n - 1\}$ and edge set $\{u_i u_{i+j}, u_i v_i, v_i v_{i+k}\}$. Graphs $I(n, 1, k)$ are called generalized Petersen graphs. It is easy to see that $I(n, j, k) = \mathrm{SC}(G; R, L, S)$, where $G = \langle a \rangle \cong \mathbb{Z}_n$, $R = \{a^j, a^{-j}\}$, $L = \{a^k, a^{-k}\}$ and $S = \{1\}$.

Example 2.4. Let $RW(n, j, k)$ be a Rose Window graph, for the definition of graph see [18]. $RW(n, j, k)$ is a 4-valent bicirculant graph isomorphic to $\mathrm{SC}(G; R, L, S)$, where $G = \langle a \rangle \cong \mathbb{Z}_n$, $R = \{a, a^{-1}\}$, $L = \{a^j, a^{-j}\}$ and $S = \{1, a^k\}$.

Example 2.5. For given natural numbers $n \geq 3$ and $1 \leq r, j, k \leq n - 1$, with $j \neq n/2$ and $r \neq k$, the *Tabačjn graph* $T(n, r, k, j)$ is a pentavalent graph with vertex set $\{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$ and edge set

$$\{x_i x_{i+1} \mid i \in \mathbb{Z}_n\} \cup \{y_i y_{i+j} \mid i \in \mathbb{Z}_n\} \cup \{x_i y_{i+r} \mid i \in \mathbb{Z}_n\} \cup \{x_i y_{i+k} \mid i \in \mathbb{Z}_n\}.$$

It is easy to see that $T(n, r, k, j) = \Gamma \cong \mathrm{SC}(G; R, L, S)$, where $G = \langle a \rangle \cong \mathbb{Z}_n$, $R = \{a, a^{-1}\}$, $L = \{a^j, a^{-j}\}$ and $S = \{1, a^r, a^k\}$.

Example 2.6. Let $K_{r,r,\ldots,r}$ be the n-partite complete graph. Then $K_{r,r,\ldots,r} = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$, where G is a finite group of order r, and for all $1 \leq i, j \leq n$ where $j \neq i$, $T_{ii} = \emptyset$ and $T_{ij} = G$.

3. Eigenvalues of n-Cayley (di)graphs

In 2007, the spectrum of bi-Cayley graphs over finite abelian groups computed in [29]:

Theorem 3.1. *Let* $\Gamma = \mathrm{BCay}(G, S)$ *be a bi-Cayley graph over finite abelian group* $G = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_t}$ *with respect to* S. *Then eigenvalues of* Γ *are*

$$\pm \Big| \sum_{(i_1,\ldots,i_t) \in S} \omega_{n_1}^{r_1 i_1} \ldots \omega_{n_t}^{r_t i_t} \Big|, \quad r_j = 0, \ldots, n_j - 1, \quad j = 1, \ldots, t.$$

In 2010, Gao and Luo improved Theorem 3.1. They studied the spectrum of semi-Cayley graphs over finite abelian groups. Using matrix theory, they derived a formula of the spectrum of semi-Cayley graphs over finite abelian groups:

Theorem 3.2. *([12, Theorem 3.2]) Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be a semi-Cayley graph over a finite abelian group $G = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_t}$. Then Γ has eigenvalues*

$$\frac{\lambda^R_{r_1 \ldots r_t} + \lambda^R_{r_1 \ldots r_t} \pm \sqrt{(\lambda^R_{r_1 \ldots r_t} - \lambda^L_{r_1 \ldots r_t})^2 + 4|\lambda^S_{r_1 \ldots r_t}|^2}}{2},$$

$r_j = 0, \ldots, n_j - 1$, $j = 1, \ldots, t$, where $\lambda^X_{r_1 \ldots r_t} = \sum_{(i_1, \ldots, i_t) \in X} \omega^{r_1 i_1}_{n_1} \ldots \omega^{r_t i_t}_{n_t}$ and ω_n is the primitive nth root of unity.

Also the spectrum of a bi-Cayley graph of an arbitrary group with respects to a normal subset computed in [6, Theorem 2.1], a generalization of Theorem 3.1. In 2013, Theorem 3.2 extended to n-Cayley graphs, $n \geq 2$, over arbitrary groups by Arezoomand and Taeri in [1] using representation theory of finite groups. Let us recall some definitions of the latter paper. Let G be a finite group and $\mathbb{C}[G]$ be the complex vector space of dimension $|G|$ with basis $\{e_g \mid g \in G\}$. We identify $\mathbb{C}[G]$ with the vector space of all complex-valued functions on G. Thus a function $\varphi : G \to \mathbb{C}$ corresponds to the vector $\varphi = \sum_{g \in G} \varphi(g) e_g$ and vice versa. In particular, the vector e_g, where $g \in G$, of the standard basis corresponds to the function e_g, where

$$e_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g. \end{cases}$$

The (left) regular representation ρ_{reg} of G on $\mathbb{C}[G]$ is defined by its action on the basis $\{e_h \mid h \in G\}$; that is for all $g, h \in G$, $\rho_{\mathrm{reg}}(g)e_h = e_{gh}$. Let $\mathrm{Irr}(G) = \{\rho_1, \ldots, \rho_m\}$ be the set of all irreducible inequivalent $\mathbb{C}$-representations of G and d_k be the degree of ρ_k, $k = 1, \ldots, m$. Let e^i_g be the $1 \times n|G|$ vector with n blocks, where ith block is e_g, as defined, and other blocks are $0_{1 \times |G|}$ vectors. Let V be the vector space with basis $\{e^i_g \mid g \in G, 1 \leq i \leq n\}$. Clearly $V \cong \underbrace{\mathbb{C}[G] \oplus \mathbb{C}[G] \oplus \cdots \oplus \mathbb{C}[G]}_{n-\text{times}}$, as

$\mathbb{C}[G] = \langle e_g \mid g \in G \rangle$. So $\dim_{\mathbb{C}} V = n \dim_{\mathbb{C}} \mathbb{C}[G] = n|G|$. Let $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$ and $A = [a_{(g,i)(h,j)}]_{g,h \in G, 1 \leq i,j \leq n}$ be the adjacency matrix of Γ. Viewing A as the linear map

$$A : V \to V$$
$$e^i_g \mapsto \sum_{j=1}^n \sum_{h \in G} a_{(h,j)(g,i)} e^j_h, \qquad 1 \leq i \leq n, \ g \in G,$$

it is proved that:

Theorem 3.3. *([1, Theorem 6]) Let $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$ be an n-Cayley digraph over a finite group G and $\mathrm{Irr}(G) = \{\rho_1, \ldots, \rho_m\}$. For each $l \in \{1, \ldots, m\}$,*

we define $nd_l \times nd_l$ block matrix $A_l := \left[A_{ij}^{(l)} \right]$, where $A_{ij}^{(l)} = \sum_{t \in T_{ji}} \rho^{(l)}(t)$. Let $\chi_{A_l}(\lambda)$ and $\chi_A(\lambda)$ be the characteristic polynomial of A_l and A, respectively. Then $\chi_A(\lambda) = \Pi_{l=1}^m \chi_{A_l}(\lambda)^{d_l}$.

Example 3.1. ([7, Corollary 2.3]) The eigenvalues of $I(n, j, k)$ are

$$\cos(2lj\pi/n) + \cos(2lk\pi/n) \pm \sqrt{(\cos(2lj\pi/n) - \cos(2lk\pi/n))^2 + 1}, \quad l = 0, \dots, n-1.$$

Example 3.2. ([7, Corollary 2.4]) The eigenvalues of $RW(n, j, k)$ are

$$\cos(2l\pi/n) + \cos(2lj\pi/n) \pm \sqrt{(\cos(2l\pi/n) - \cos(2lj\pi/n))^2 + 2 + 2\cos(2lk\pi/n)}, \quad l = 0, \dots, n-1.$$

Example 3.3. ([7, Corollary 2.5]) The eigenvalues of $T(n, r, k, j)$ are

$$\cos(2l\pi/n) + \cos(2lj\pi/n) \pm \sqrt{(\cos(2l\pi/n) - \cos(2lj\pi/n))^2 + \alpha_l}, \quad l = 0, \dots, n-1,$$

where $\alpha_l = 3 + 2 \Big(\cos(2\pi lr/n) + \cos(2\pi lk/n) + \cos(2\pi l(r-k)/n) \Big)$.

Since any Cayley graph over a group G is a 1-Cayley graph over G, as a direct consequence of Theorem 3.3, we can reprove the following result which is proved in [27]:

Corollary 3.1. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley digraph over a finite group G with irreducible matrix representations $\varrho^{(1)}, \dots, \varrho^{(m)}$. Let d_l be the degree of $\varrho^{(l)}$. For each $l \in \{1, \dots, m\}$, define a $d_l \times d_l$ block matrix $A_l := \left[A_S^{(l)} \right]$, where $A_S^{(l)} = \sum_{s \in S} \varrho^{(l)}(s)$. Let $\chi_{A_l}(\lambda)$ and $\chi_A(\lambda)$ be the characteristic polynomial of A_l and A, the adjacency matrix of Γ, respectively. Then $\chi_A(\lambda) = \Pi_{l=1}^m \chi_{A_l}(\lambda)^{d_l}$.*

Let G be a finite abelian group. Then by [16, Theorem 9.8], putting $n = 2$ in Theorem 3.3, Theorem 3.2 directly follows. Also Theorems 4.6 and 4.3 of [12] improved in [1]:

Corollary 3.2. *([1, Corollary 9]) Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley digraph, $H = \langle a \rangle$ a cyclic subgroup of G of order n and of index 2 with left transversal $\{t_1, t_2\}$. Then the characteristic polynomial of the adjacency matrix of Γ is $\chi_A(\lambda) = \Pi_{k=0}^{n-1} (\lambda - \lambda_k^+)(\lambda - \lambda_k^-)$, where*

$$\lambda_k^+ = \frac{\lambda_k^{11} + \lambda_k^{22} + \sqrt{(\lambda_k^{11} - \lambda_k^{22})^2 + 4\lambda_k^{12}\lambda_k^{21}}}{2},$$

$$\lambda_k^- = \frac{\lambda_k^{11} + \lambda_k^{22} - \sqrt{(\lambda_k^{11} - \lambda_k^{22})^2 + 4\lambda_k^{12}\lambda_k^{21}}}{2},$$

$\lambda_k^{ij} = \sum_{t \in T_{ji}} \omega_n^{kt}$ and $T_{ij} = \{t \mid 0 \le t \le n-1, a^t \in t_j S t_i^{-1}\}$.

Let Γ be a k-regular graph with n vertices and adjacency matrix A and A^c be the adjacency matrix of the complement of Γ. Then $(\lambda + k + 1)\chi_{A^c}(\lambda) = (-1)^n(\lambda - n + k + 1)\chi_A(-\lambda - 1)$, see [9, p. 20]. Despite of Cayley graphs, n-Cayley graphs $n \geq 2$, are not necessarily regular, but we have a similar relation between the characteristic polynomials of any n-Cayley graph and its complement which is given in the next theorem:

Theorem 3.4. *([1, Theorem 10]) Let $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1 \leq i, j \leq n)$ be an n-Cayley graph over a finite group G, $n \geq 1$. Let Γ^c be the complement of Γ with adjacency matrix A^c. Then the characteristic polynomials of Γ and Γ^c are related with the following equation:*

$$\chi_{B_1}(\lambda)\chi_A(-\lambda - 1) = (-1)^{|G|-1}\chi_{A_1}(-\lambda - 1)\chi_{A^c}(\lambda),$$

where $B_1 = |G|J - I_n - A_1$, J is the all ones matrix of degree n, and $A_1 = [|T_{ji}|]_{1 \leq i, j \leq n}$.

An eigenvector of the adjacency matrix of a graph Γ is said to be *main eigenvector* if it is not orthogonal to the all ones vector $\mathbf{j}$. An eigenvalue of a graph Γ is said to be a *main eigenvalue* if it has a main eigenvector. By Perron-Frobenius Theorem, the largest eigenvalue of a graph is a main eigenvalue. It is also well known that a graph is regular if and only if it has exactly one main eigenvalue. So for every Cayley graph $\Gamma = \mathrm{Cay}(G, S)$, $|S|$ is the only main eigenvalue of Γ. Since n-Cayley graphs, for $n \geq 2$ are not necessarily regular, determining the main eigenvalues of these graphs seems to be important. This problems reduced to determining main eigenvalues of the matrix A_1:

Theorem 3.5. *([1, Corollary 12]) Let $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1 \leq i, j \leq n)$ be an n-Cayley graph over a finite group G and $n \geq 2$. The main eigenvalues of Γ is equal to main eigenvalues of the matrix $A_1 = [|T_{ji}|]_{1 \leq i, j \leq n}$.*

4. Integrality of n-Cayley graphs

A graph Γ is called *integral* if all eigenvalues of the adjacency matrix of Γ are integers. The concept of integral graphs was first defined by Harary and Schwenk [14]. During the last forty years many mathematicians have tried to construct and classify some special classes of integral graphs including Cayley graphs(for a survey see [21]). It seems that integral graphs are very rare and determining all the integral n-Cayley graphs, even for $n = 2$, is difficult. It is easy to construct integral semi-Cayley graphs over finite abelian groups, as the following corollary shows:

Corollary 4.1. *([12, Corollary 3.5]) Let $\Gamma = \mathrm{SC}(G; R, R, S)$ be a semi-Cayley graph over a finite abelian group G. If $\mathrm{Cay}(G, R)$ and $\mathrm{Cay}(G, S)$ are integral then Γ is integral.*

The study of integrality of bi-Cayley graphs started by Arezoomand and Taeri in 2015:

Theorem 4.1. *([4, Corollary 3.10]) Every bi-Cayley graph of a finite group G is integral if and only if G is isomorphic to one of the groups $\mathbb{Z}_2^k$, $k \geq 1$, $\mathbb{Z}_3$ or S_3.*

Also finite groups admitting a connected cubic integral bi-Cayley graph determined in the following theorem:

Theorem 4.2. *([8, Theorem A]) A finite group G admits a connected cubic integral bi-Cayley graph if and only if G is isomorphic to one of the groups*

$$\mathbb{Z}_2^2, \ \mathbb{Z}_3, \ \mathbb{Z}_4, \ \mathbb{Z}_6, \ \mathbb{Z}_2 \times \mathbb{Z}_6, \ S_3, \ A_4, \ Dic_{12}.$$

The following questions naturally arise:

Problem 4.1. Determine finite groups admitting a connected k-regular, $k \geq 4$, bi-Cayley graphs.

Problem 4.2. Let $\Gamma = \mathrm{BCay}(G, S)$. In what conditions on S, is Γ an integral?

Problem 4.3. Determine finite groups in which all bi-Cayley graphs over them of the valency at most $k \geq 2$ are integral.

Problem 4.4. Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be a semi-Cayley graph over a group G. In what conditions on R, L and S is Γ an integral?

5. Laplacian and signless Laplacian eigenvalues of n-Cayley graphs

Let Γ be a graph with vertex set $\{v_1, \ldots, v_n\}$. Recall that the adjacency matrix of Γ is an $n \times n$ matrix $A = [a_{ij}]$, where $a_{ij} = 1$ whenever v_i and v_j are adjacent and $a_{ij} = 0$, otherwise. The degree matrix of Γ is a diagonal $n \times n$ matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$, where d_i is the number of vertices adjacent to v_i. The matrices $L = D - A$ and $Q = D + A$ are called *Laplacian* and *signless Laplacian* matrices of Γ, respectively. The characteristic polynomial of an $n \times n$ matrix X is $\det(\lambda I_n - X)$, where I_n is the $n \times n$ identity matrix and the roots of this polynomial are called eigenvalues of X. In this paper, the Laplacian eigenvalues and signless Laplacian eigenvalues of a graph Γ are eigenvalues of Laplacian and signless Laplacian matrices of Γ, respectively.

In 2015, the Laplacian and signless Laplacian spectrum of semi-Cayley graphs over abelian groups computed:

Theorem 5.1. *([13, Theorem 1]) Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be a semi-Cayley graph over a finite abelian group $G = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_t}$. Then Γ has Laplacian eigenvalues (resp. signless Laplacian eigenvalues)*

$$\frac{\mu_{r_1 \ldots r_t}^R + \mu_{r_1 \ldots r_t}^L + 2|S| \pm \sqrt{(\mu_{r_1 \ldots r_t}^R - \mu_{r_1 \ldots r_t}^L)^2 + 4|\lambda_{r_1 \ldots r_t}^S|^2}}{2}$$

$r_j = 0, \ldots, n_j - 1$, $j = 1, \ldots, t$, where $\lambda^S_{r_1 \ldots r_t}$ are eigenvalues of $\mathrm{Cay}(G, S)$, and $\mu^R_{r_1 \ldots r_t}, \mu^L_{r_1 \ldots r_t}$ are the Laplacian (resp. signless Laplacian) eigenvalues of $\mathrm{Cay}(G, R)$ and $\mathrm{Cay}(G, L)$, respectively.

The *n-sunlet* graph on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n. It is easy to see that $\Gamma = \mathrm{SC}(G, R, S, T)$, where $G = \langle a \rangle \cong \mathbb{Z}_n$, $R = \{a, a^{-1}\}$, $S = \varnothing$ and $T = \{1\}$.

Example 5.1. Let Γ be an n-sunlet graph. Then

(1) Lpalcian eigenvalues of Γ are

$$2 - \cos \frac{2\pi l}{n} \pm \sqrt{(1 - \cos \frac{2\pi l}{n})^2 + 1},$$

where $l = 0, \ldots, n - 1$.

(2) signless Laplacian eigenvalues of Γ are

$$2 + \cos \frac{2\pi l}{n} \pm \sqrt{(1 + \cos \frac{2\pi l}{n})^2 + 1},$$

where $l = 0, \ldots, n - 1$.

As a corollary, one can construct semi-Cayley graphs with an integral Laplacian and signless Laplacian spectrum:

Corollary 5.1. *([13, Corollary 4.6]) Let* $\Gamma = \mathrm{SC}(G; R, R, S)$ *be a semi-Cayley graph over a finite abelian group G. If $\mathrm{Cay}(G, R)$ and $\mathrm{Cay}(G, S)$ are integral graphs then Γ is a Laplacian and signless Laplacian integral graph.*

We end the paper with some open problems:

Problem 5.1. Determine the Laplacian and signless Laplacian eigenvalues of semi-Cayley graphs over non-abelian groups. Also do this for n-Cayley graphs when $n \geq 3$.

Problem 5.2. In what conditions on R, L and S, $\mathrm{SC}(G; R, L, S)$ is Laplacian (and signless Laplacian) an integral?

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