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ON THE ROOTS OF TOTAL DOMINATION POLYNOMIAL OF GRAPHS, II

Saeid Alikhani,* Nasrin Jafari

Abstract. Let $G = (V, E)$ be a simple graph of order n. The total dominating set of G is a subset D of V that every vertex of V is adjacent to some vertices of D. The total domination number of G is equal to minimum cardinality of total dominating set in G and is denoted by $\gamma_t(G)$. The total domination polynomial of G is the polynomial $D_t(G, x) = \sum_{i=\gamma_t(G)}^{n} d_t(G, i)x^i$, where $d_t(G, i)$ is the number of total dominating sets of G of size i. A root of $D_t(G, x)$ is called a total domination root of G. The set of total domination roots of graph G is denoted by $Z(D_t(G, x))$. In this paper, we show that $D_t(G, x)$ has $\delta - 2$ non-real roots and if all roots of $D_t(G, x)$ are real, then $\delta \leq 2$, where δ is the minimum degree of vertices of G. Also we show that if $\delta \geq 3$ and $D_t(G, x)$ has exactly three distinct roots, then $Z(D_t(G, x)) \subseteq \{0, -2 \pm \sqrt{2}i, \frac{-3\pm\sqrt{3}i}{2}\}$. Finally we study the location roots of total domination polynomial of some families of graphs.
Keywords. graph; total domination number; total domination polynomial; root.

1. Introduction

Let $G = (V, E)$ be a simple graph. The order of G is the number of vertices of G. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subset V$, the open neighborhood of S is the set $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood of S is the set $N[S] = N(S) \cup S$. A leaf (end-vertex) of a graph is a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. The set $D \subset V$ is a total dominating set if every vertex of V is adjacent to some vertices of D, or equivalently, $N(D) = V$. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G. A total dominating set with cardinality $\gamma_t(G)$ is called a γ_t-set. An i-subset of V is a subset of V of cardinality i. Let

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*Corresponding author

$\mathcal{D}_t(G,i)$ be the family of total dominating sets of G which are i-subsets and let $d_t(G,i) = |\mathcal{D}_t(G,i)|$. The polynomial $D_t(G;x) = \sum_{i=1}^{n} d_t(G,i)x^i$ is defined as total domination polynomial of G. As an example, $D_t(K_n,x) = (x+1)^n - nx - 1$ and $D_t(K_{1,n},x) = x((x+1)^n - 1)$. A root of $D_t(G,x)$ is called a total domination root of G. The set of total domination roots of graph G is denoted by $Z(D_t(G,x))$. For many graph polynomials, their roots have attracted considerable attention. For example in [5] Brown, Hickman, and Nowakowski proved that the real roots of the independence polynomials are dense in the interval $(-\infty, 0]$, while the complex roots are dense in the complex plane. For matching polynomial, in [14] was proved that all roots of the matching polynomials are real. Also it was shown that if a graph has a Hamiltonian path, then all roots of its matching polynomial are simple (see Theorem 4.5 of [15]). For domination polynomial, Brown and Tufts in [4] studied the location of domination roots and they proved that the set of all domination roots is dense in the complex plane. For graphs with few domination roots see [1]. Related to the roots of total domination polynomials there are a few papers. See [2, 16] for more details. Recently authors in [16] shown that all roots of $D_t(G,x)$ lie in the circle with center $(-1,0)$ and radius $\sqrt[\delta]{2^n - 1}$, where δ is the minimum degree of G and n is the order of G. As a consequence, they proved that if $\delta \geq \frac{2n}{3}$, then every integer root of $D_t(G,x)$ lies in the set $\{-3,-2,-1,0\}$.

In this paper we show that $D_t(G,x)$ has $\delta - 2$ non-real roots and if all roots of $D_t(G,x)$ are real, then $\delta \leq 2$. Also we show that if $\delta \geq 3$ and $D_t(G,x)$ has exactly three distinct roots, then $Z(D_t(G,x)) \subseteq \{0, -2 \pm \sqrt{2}i, \frac{-3\pm\sqrt{3}i}{2}\}$. Finally we study the location roots of total domination polynomial of some families of graphs.

2. Main results

In this section we obtain some results on total domination roots. Oboudi in [20] has studied graphs whose domination polynomials have only real roots. More precisely he obtained the number of non-real roots of domination polynomial of graphs. Similarly, we do it for total domination roots, in the next theorem.

Theorem 2.1. *Let G be a connected graph of order $n \geq 2$.*

i) If all roots of G are real, then $\delta = 1$ or 2.

ii) The polynomial $D_t(G,x)$ has at least $\delta - 2$ non-real roots.

Proof. Let $g(x) = D_t(G,x)$ and $g^{(m)}(x)$ be the m-th derivative of $g(x)$ with respect to x. It is easy to see that if $i \geq n - \delta + 1$, then $d_t(G,i) = \binom{n}{i}$ and if $i \leq n - \delta$, then $d_t(G,i) < \binom{n}{i}$, where $d_t(G,i)$ is the number of total dominating sets of G with cardinality i, for every natural number i. Thus there exists a polynomial $f(x)$ with positive coefficients and with degree $n - \delta$ such that $D_t(G,x) = (x+1)^n - f(x)$. Since all roots of $g(x)$ are real, by Rolle's theorem we conclude that all roots of $g^{(n-\delta)}(x)$ are real as well. On the other hand $g^{(n-\delta)}(x) = \frac{n!}{\delta!}(x+1)^{\delta} - a(n-\delta)!$, where a is the coefficient of $x^{n-\delta}$ in $f(x)$. Since all roots of $g^{(n-\delta)}(x)$ are real, this shows that $\delta \leq 2$. Since G is connected, so $\delta = 1$ or 2.

Now suppose that $g(x)$ has exactly r real roots. Using Rolle's theorem one can see that $g^{(n-\delta)}(x)$ has at least $r-(n-\delta)$ real roots. On the other hand $g^{(n-\delta)}(x) = \frac{n!}{\delta!}(x+1)^{\delta} - a(n-\delta)!$. Thus $r-(n-\delta) \leq 2$. Therefore $g(x)$ has at least $\delta - 2$ non-real roots. $\square$

`Theorem 2.2.` [2] *If $G = (V, E)$ is a graph of order n with r support vertices, then $d_t(G, n-1) = n - r$.*

`Theorem 2.3.` [15] *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq \frac{n}{2}$.*

The study of graphs which their polynomials have few roots can give sometimes a surprising information about the structure of the graph. If A is the adjacency matrix of G, then the eigenvalues of A, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are said to be the eigenvalues of the graph G. These are the roots of the characteristic polynomial $\phi(G, \lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$. For more details on the characteristic polynomials. The characterization of graphs with few distinct roots of characteristic polynomials (i.e. graphs with few distinct eigenvalues) have been the subject of many researches. Graphs with three adjacency eigenvalues have been studied by Bridges and Mena [3] and Klin and Muzychuk [17]. Also van Dam studied graphs with three and four distinct eigenvalues [6, 7, 8, 9]. Graphs with three distinct eigenvalues and index less than 8 were studied by Chuang and Omidi in [18]. Graphs with few domination roots were studied by Akbari, Alikhani and Peng in [1]. In [2], authors studied graphs with exactly two total domination roots $\{-3, 0\}$, $\{-2, 0\}$ and $\{-1, 0\}$. Here we study graphs with three distinct total domination roots.

`Theorem 2.4.` *Let G be a graph with $\delta \geq 3$. If $D_t(G, x)$ has exactly three distinct roots, then*

$$Z(D_t(G, x)) \subseteq \{0, -2 \pm \sqrt{2}i, \frac{-3 \pm \sqrt{3}i}{2}\}.$$

Proof. Let G be a connected graph of order n and $Z(D_t(G, x)) = \{0, a, b\}$ that $a \neq b$. Therefore $D_t(G, x) = x^i(x-a)^j(x-b)^k$, for some i, j, k. So by Theorem 2.2, we have

$$-(ja + kb) = n. \tag{2.1}$$

Also because $d_t(G, i) = \binom{n}{i}$ for $i \geq n - \delta + 1$, we have

$$\binom{j}{2}a^2 + \binom{k}{2}b^2 + jkab = d_t(G, n-2) = \binom{n}{2}. \tag{2.2}$$

Let $P(x)$ be the minimal polynomial of a over $\mathbb{Q}$. Clearly, all roots of $P(x)$ are simple. This implies that $deg(P(x)) = 1$ *or* 2. We consider two cases.

Case 1. $deg(P(x)) = 1$. So $D_t(G, x) = x^i(x-a)^j(x-b)^k$, where $-a, -b \in \mathbb{N}$. By Theorem 2.1, we have $\delta = 1$ or 2, a contradiction.

Case 2. $deg(P(x)) = 2$. In this case since $D_t(G,x)$ has three distinct roots, the minimal polynomial of b over $\mathbb{Q}$ is also $P(x)$, Thus we have $D_t(G,x) = x^i(x^2+rx+s)^j$, where $P(x) = x^2+rx+s$. We have $i+2j = n$, and also by (2.1), $-(a+b)j = n$. By Theorem 2.3, $i \leq \frac{n}{2}$. Therefore $j \geq \frac{n}{4}$. Since $-(a+b)j = n$ and $a+b$ is an integer, we have $-(a+b) \in \{1,2,3,4\}$. We consider four cases:

Subcase 2.1. If $a+b=-1$, then $j=n$, a contradiction.

Subcase 2.2. If $a+b=-2$, then $j=\frac{n}{2}$, a contradiction.

Subcase 2.3. If $a+b=-3$, then $i=j=\frac{n}{3}$, so we have $D_t(G,x) = x^{\frac{n}{3}}(x^2+rx+s)^{\frac{n}{3}}$. Now, by (2.2) we have

$$\binom{\frac{n}{3}}{2}(a^2+b^2) + \frac{n^2ab}{9} = \binom{n}{2}.$$

In the other hand, since $a+b=-3$, we conclude that $a^2+b^2 = 9-2ab$. Thus by simple calculation we obtain $nab = 3n$. Therefore $ab = 3$. By using $a+b=-3$, we have

$$a \in \{\frac{-3 \pm \sqrt{3}i}{2}\}$$

Subcase 2.4. Now, suppose that $a+b=-4$. Then $i=\frac{n}{2}$ and $j=\frac{n}{4}$. With the same calculations, we have $ab = 6$. Using the fact that $a+b=-4$, we have $a \in \{-2 \pm \sqrt{2}i\}$. □

□

As noted before, in [2], authors studied graphs with exactly two total domination roots $\{-3,0\}$, $\{-2,0\}$ and $\{-1,0\}$. Here we present a family of graphs whose total domination roots are -1 and 0.

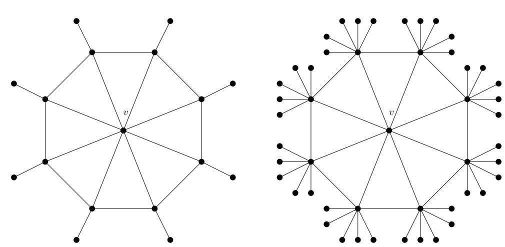

FIG. 2.1: Helm graph H_8 and generalized helm graph $H_{8,5}$, respectively.

The helm graph H_n is obtained from the wheel graph W_n by attaching a pendent edge at each vertex of the n-cycle of the wheel. We define generalized helm graph $H_{n,m}$, the graph is obtained from the wheel graph W_n by attaching m pendent edges at each vertex of the n-cycle of the wheel (Figure 2.1). We recall that corona product of two graphs G and H is denoted by $G \circ H$ and was introduced by Harary [12, 13]. This graph formed from one copy of G and $|V(G)|$ copies of H, where the i-th vertex of G is adjacent to every vertex in the i-th copy of H. We need the following theorems:

`Theorem 2.5.` [10] *Let $G = (V, E)$ be a graph and $u, v \in V$ two non-adjacent vertices of the graph with $N(u) \subseteq N(v)$. Then*

$$D_t(G, x) = D_t(G \setminus v, x) + xD_t(G/v, x) + x^2 \sum_{w \in N(v) \cap N(u)} D_t(G \setminus N[\{v, w\}], x).$$

`Theorem 2.6.` [16] *For any graph G of order $n \geq 2$, $D_t(G \circ \overline{K_m}, x) = x^n(1+x)^{mn}$.*

`Theorem 2.7.` *For every natural number n, m, we have*

i) $D_t(H_n, x) = x^n(x+1)^{n+1}$,

ii) $D_t(H_{n,m}, x) = x^n(1+x)^{mn+1}$.

Proof. Let v be the center vertex of wheel in helm graph H_n and $H_{n,m}$. By Theorems 2.5 and 2.6 we have

i) $D_t(H_n, x) = D_t(C_n \circ K_1, x) + xD_t(K_n \circ K_1, x) = (1+x)(x(1+x))^n$,

ii) $D_t(H_{n,m}, x) = D_t(C_n \circ \overline{K_m}, x) + xD_t(K_n \circ \overline{K_m}, x) = (1+x)(x(1+x)^m)^n$.

So we have the result. □

The lexicographic product is also known as graph substitution, a name that bears witness to the fact that $G[H]$ can be obtained from G by substituting a copy H_u of H for every vertex u of G and then joining all vertices of H_u with all vertices of H_v if $\{u, v\} \in E(G)$.

`Theorem 2.8.` *Let K_m, K_n be complete graphs of order m and n. The total domination polynomial of lexicographic product of K_m and K_n is*

$$D_t(K_m[K_n], x) = D_t(K_m, D(K_n, x)) + mD_t(K_n, x).$$

Proof. Note that $K_m[K_n] \cong K_{mn}$, So the result is obtained. □

The generalized friendship graph $F_{n,q}$ is a collection of n cycles (all of order q), meeting at a common vertex (see Figure 2.4). The generalized friendship graph may also be referred to as a flower [19]. For $q = 3$ the graph $F_{n,q}$ is denoted simply by F_n and is friendship graph. The total domination polynomial of F_n and its roots studied in [16]. Here, we compute the total domination number of $F_{n,4}$. To study the total domination roots of $F_{n,4}$ we first obtain a formula for the total domination polynomial of graph G_n depicted in Figure 2.2. We need the following theorem:

`Theorem 2.9.` [10]

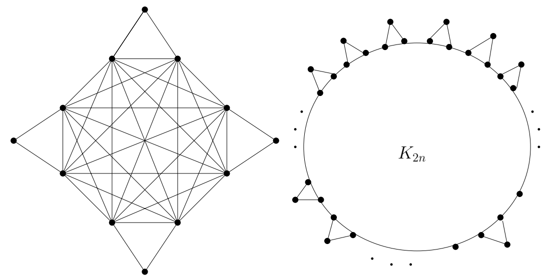

FIG. 2.2: Graphs G_4 and G_n in proof of Theorem 2.9, respectively.

(i) For any vertex u in the graph G we have

$$D_t(G,x) = D_t(G\setminus u,x) + xD_t(G/u,x) + x^2 \sum_{v\in N(u)} D_t(G\setminus N[\{u,v\}],x) - (1+x)p_u(G),$$

where $p_u(G,x)$ is the polynomial counting the total dominating sets of $G\setminus u$ which do not contain any vertex of $N(u)$ in G.

(ii) Let $u,v\in V(G)$ be two non-adjacent vertices of G with $N(v)\subseteq N(u)$. Then $D_t(G,x)$

$$= D_t(G\setminus u,x) + xD_t(G/u,x) + x^2 \sum_{w\in N(u)\cap N(v)} D_t(G\setminus N[\{u,w\}],x).$$

`Theorem 2.10.` *For any $n\in\mathbb{N}$, $D_t(G_n,x) = (x(x+1)(x+2))^n$.*

Proof. Consider the graph G_n shown in Figure 2.2 and v be a vertex of degree two of this graph. By Theorem 2.9(i) and the fact that $p_v(G_n,x) = D_t(G_{n-1},x)$ and $G_n - v \cong G_n/v$, we have

$$D_t(G_n,x) = (x+1)D_t(G_n - v,x) - (x+1)D_t(G_{n-1},x).$$

Now by Theorem 2.9(ii) for graph $G_n - v$ and the vertex u of this graph (see figure 2.3):

$$D_t(G_n,x) = (x+1)^2 D_t(G_n - v/u,x) - (x+1)D_t(G_{n-1},x).$$

Again by Theorem 2.9(ii) for the vertex w of the graph $G_n - v/u$ shown in figure

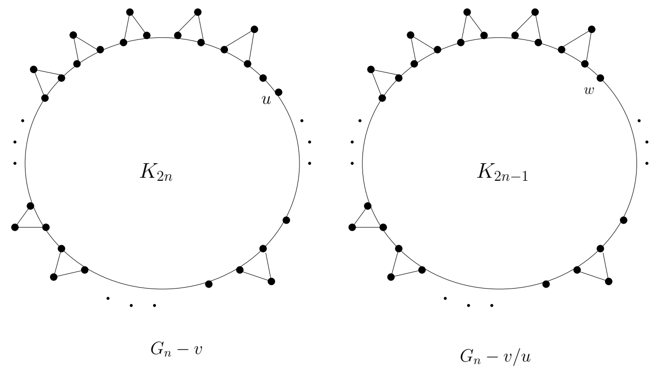

FIG. 2.3: Graphs in proof of Theorem 2..

2.3, we have the following equations.

$$\begin{aligned} D_t(G_n, x) &= (x+1)^2 D_t(G_n - v/u, x) - (x+1)D_t(G_{n-1}, x) \\ &= (x+1)^3 D_t(G_{n-1}, x) - (x+1)D_t(G_{n-1}, x) \\ &= x(x+1)(x+2)D_t(G_{n-1}, x) \\ &= (x(x+1)(x+2))^n \end{aligned}$$

So we have result. □

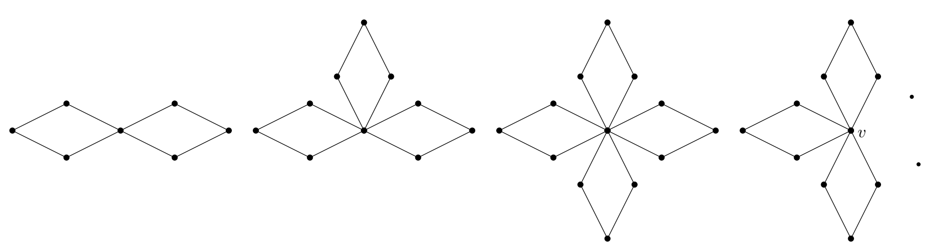

FIG. 2.4: Friendship graphs $F_{2,4}, F_{3,4}, F_{4,4}$ and $F_{n,4}$, respectively.

Theorem 2.11. *For every natural number n, total domination polynomial of generalize friendship graph $F_{n,4}$ is*

$$D_t(F_{n,4}, x) = x^{n+1}(x+2)^n((x+1)^n + x^{n-1}).$$

Proof. Let v be center vertex of $F_{n,4}$. By theorem 2.5 we have

$$D_t(F_{n,4}, x) = (D_t(P_3, x))^n + xD_t(G_n, x)$$

where G_n is graph in Figure 2.2 and so by Theorem 2.10 we have the result. □

We need the following lemma to obtain more results:

`Lemma 2.12.` [4] $lim_{n\to\infty} ln(n)\Big(\frac{ln(n)-1}{ln(n)}\Big)^n = 0$.

The basic idea of the following result follows from the proof of Theorem 8 in [4].

`Theorem 2.13.` *For natural number $n \geq 2$,*

i) The total domination polynomial of the generalized friendship graph, $D_t(F_{n,4}, x)$, has a real root in the interval $(-1, 0)$

ii) The total domination polynomial of the generalized friendship graph, $D_t(F_{n,4}, x)$, has a real root in the interval $(-n, -ln(n))$, for n sufficiently large.

Proof. i) Let $f(x) = (x + 1)^n + x^{n-1}$. So $f(0) = 1$ and $f(-1) = (-1)^{n-1} = -1$. By the intermediate value theorem, we have result.

ii) Suppose that

$$f_{2n}(x) = x^{n+1}((x + 1)^n + x^{n-1}).$$

Observe that

$$f_{2n}(x) = x^{2n+1}+(n+1)x^{2n}+\binom{n}{n-2}x^{2n-1}+\binom{n}{n-3}x^{2n-2}+...+nx^{n+2}+x^{n+1}.$$

Consider

$$f_{2n}(-n) = (-1)^{2n+1}n^{2n+1}\Big(1 - \frac{n+1}{n} + \frac{\binom{n}{2}}{(n)^2} - ... + \frac{(-1)^n}{(n)^n}\Big).$$

So $f_{2n}(-n) < 0$ for n sufficiently large, because the following inequality is true for n sufficiently large,

$$\frac{n+1}{n} - \frac{\binom{n}{2}}{(n)^2} + ... - \frac{(-1)^n}{(n)^n} < 1.$$

Now consider

$$\begin{aligned} f_{2n}(-ln(n)) &= (-ln(n))^{n+1}(1 - ln(n))^n + (-ln(n))^{2n} \\ &= (ln(n))^{2n}\Big(1 - ln(n)\Big(\frac{ln(n)-1}{ln(n)}\Big)^n\Big). \end{aligned}$$

From Lemma 2.12, we have $ln(n)\Big(\frac{ln(n)-1}{ln(n)}\Big)^n \to 0$, as $n \to \infty$ which implies that $f_{2n}(-ln(n)) > 0$. By the Intermediate Value Theorem, for sufficiently large n, $f_{2n}(x) = D_t(F_n, x)$ has a real root in the interval $(-n, -ln(n))$.

□

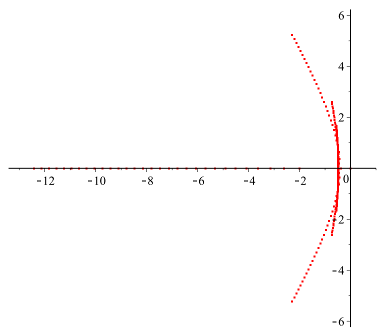

FIG. 2.5: Total domination roots of $F_{n,4}$, for $2 \leq n \leq 30$.

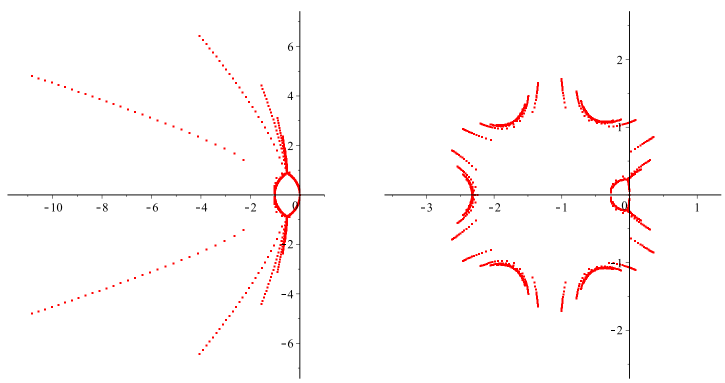

FIG. 2.6: Total domination roots of $K_{1,n}[K_2]$ and $K_{1,n}[K_7]$, for $2 \leq n \leq 30$, respectively.

Figure 2.5 shows the total domination roots of $F_{n,4}$ for $2 \leq n \leq 30$.

Theorem 2.14. *Let G and H be two graphs of order m and n, respectively. The total domination polynomial of join of these two graphs is*

$$D_t(G \vee H) = ((1+x)^m - 1)((1+x)^n - 1) + D_t(G,x) + D_t(H,x).$$

Theorem 2.15. *For every natural numbers m, n,*

$$D_t(K_{1,n}[K_m], x) = (1+x)^{mn}((1+x)^m - 1) + ((1+x)^m - mx - 1)^n - mx.$$

Proof. For two natural numbers m, n, $K_{1,n}[K_m] \cong k_m \vee nK_m$. So by Theorem 2.14, it is easy to see the equation is true. □

Using Maple we think that for two natural numbers m,n, if m and n are even or n is odd, then the total domination polynomial of $K_{1,n}[K_m]$ has no real roots.

However, until now all attempts to prove this failed. See the total domination roots of $K_{1,n}[K_2]$ and $K_{1,n}[K_7]$ for $2 \leq n \leq 30$ in Figure 2.6.

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Saeid Alikhani
Department of Mathematics
Yazd University, 89195-741
Yazd, Iran
alikhani@yazd.ac.ir

Nasrin Jafari
Department of Mathematics
Yazd University, 89195-741
Yazd, Iran
nasrin7190@yahoo.com