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ON KENMOTSU MANIFOLDS WITH A SEMI-SYMMERIC METRIC CONNECTION

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Abstract. The aim of the present paper is to study the properties of locally and globally ϕ-concircularly symmetric Kenmotsu manifolds endowed with a semi-symmetric metric connection. First, we will prove that the locally ϕ-symmetric and the globally ϕ-concircularly symmetric Kenmotsu manifolds are equivalent. Next, we will study three dimensional locally ϕ-symmetric, locally ϕ-concircularly symmetric and locally ϕ-concircularly recurrent Kenmotsu manifolds with respect to such connection and will obtain some geometrical results. In the end, we will construct a non-trivial example of Kenmotsu manifold admitting a semi-symmetric metric connection and validate our results.

Keywords: Kenmotsu manifolds, ϕ-symmetric manifolds, η-parallel Ricci tensor, semi-symmetric metric connection, concircular curvature tensor.

1. Introduction

The product of an almost contact manifold M and the real line $\mathbb{R}$ carries a natural almost complex structure. However, if one takes M to be an almost contact metric manifold and suppose that the product metric G on $M \times \mathbb{R}$ is Kähler, then the structure on M is cosymplectic [19] and not Sasakian. On the other hand, Oubina [25] pointed that if the conformally related metric $e^{2t}G$, t being the coordinates on $\mathbb{R}$ is Kähler, then M is Sasakian and vice versa.

In [34], Tanno classified almost contact metric manifolds whose automorphism group possesses the maximum dimension. For such manifold M, the sectional curvature of the plane section containing ξ is constant, say c. If $c >, =, and < 0$, then M is said be a homogeneous Sasakian manifold of constant sectional curvature, product of a line or a circle with Kähler manifold of constant holomorphic sectional curvature, and warped product space $\mathbb{R} \times_f C^n$, respectively. In 1972, Kenmotsu [23] characterized the geometrical properties of the manifold when $c < 0$, called Kenmotsu manifold. The geometrical properties of this manifold have been studied

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by many geometers, for instance (see, [3], [7]-[11], [15], [16], [22], [26], [33], [36], [40], [41]).

In general, a geodesic circle (a curve whose first curvature is constant and second curvature is identically zero) does not transform into a geodesic circle by the conformal transformation

$$\tilde{g}_{ij} = \psi^2 g_{ij} \tag{1.1}$$

of the fundamental tensor g_{ij}. A transformation which preserves the geodesic circle was first introduced by Yano [37]. The conformal transformation (1.1) satisfying the partial differential equation

$$\psi_{;i;j} = \phi g_{ij}$$

change a geodesic circle into a geodesic circle. Such transformation is known as the concircular transformation and the geometry which leads with such transformation is known as the concircular geometry [37].

A $(1, 3)$ type tensor C which remains invariant under the transformation (1.1), for an n-dimensional Riemannian manifold M, given by

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}\left[g(Y, Z)X - g(X, Z)Y\right] \tag{1.2}$$

for all vector fields X, Y and Z on M is known as a concircular curvature tensor [37], where R, r, and ∇ are the Riemannian curvature tensor, the scalar curvature, and the Levi-Civita connection, respectively. In view of (1.2), it is obvious that

$$(\nabla_W C)(X, Y)Z = (\nabla_W R)(X, Y)Z - \frac{dr(W)}{n(n-1)}\left[g(Y, Z)X - g(X, Z)Y\right]. \tag{1.3}$$

The importance of the concircular transformation and the concircular curvature tensor are well known in the differential geometry of F-structures such as complex, almost complex, Kähler, almost Kähler, contact and almost contact structures ([37], [6], [35]). In a recent paper, Ahsan and Siddiqui [1] have studied the application of concircular curvature in general relativity and cosmology.

Let (M, g) be a Riemannian manifold of dimension n. A linear connection $\tilde{\nabla}$ on (M, g), whose torsion tensor $\tilde{T}$ of type $(1, 2)$ is defined by

$$\tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]$$

For arbitrary vector, fields X and Y on M are said to be torsion free or symmetric if $\tilde{T}$ vanishes, otherwise it is non-symmetric. If the connection $\tilde{\nabla}$ satisfies $\tilde{\nabla} g = 0$ on (M, g), then it is called metric connection, otherwise it is non-metric. In [17], Friedmann and Schouten introduced the notion of semi-symmetric linear connection on a differentiable manifold. Hayden [18] introduced the idea of semi-symmetric linear connection with non-zero torsion on a Riemannian manifold. The systematic study of the semi-symmetric metric connection on the Riemannian manifold was introduced by Yano [38]. He proved that a Riemannian manifold endowed with a semi-symmetric metric connection has vanishing curvature tensor with respect to the semi-symmetric metric connection if and only if it is conformally flat. This result was generalized for vanishing Ricci tensor of the semi-symmetric metric connection by T. Imai ([20], [21]). Various geometrical and physical properties of this connection have been studied by many authors among whom are ([2]-[4], [12]-[14], [27]- [31], [39]). Motivated by the above studies, the authors will continue to study the properties of the Kenmotsu manifolds equipped with a semi-symmetric metric connection. The present paper is organized in the following manner:

After the introduction in Section 1, we will notify you on the basic results of the Kenmotsu manifolds and the semi-symmetric metric connection in Section 2 and Section 3, respectively. In section 4, we will start the study of globally ϕ-concircularly symmetric Kenmotsu manifold and prove that the manifold is η-Einstein as well as locally ϕ-symmetric.The following sections deal with the study of locally ϕ-symmetric, locally ϕ-concircularly symmetric, Ricci semisymmetric, η-parallel Ricci tensor and locally ϕ-concircularly recurrent Kenmotsu manifolds equipped with a semi-symmetric metric connection. In the last section, we will construct an example of three dimensional Kenmotsu manifold admitting a semi-symmetric metric connection to verify some results of our paper.

2. Preliminaries

Let M be an $n(=2m+1)$-dimensional connected almost contact metric manifold with an almost contact structure (ϕ, ξ, η, g), that is, M admits a $(1,1)$-type tensor field ϕ, a $(1,0)$-type vector field ξ, a 1-form η, and a compatible Riemannian metric g satisfies

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X) \tag{2.2}$$

for all X, $Y \in T(M)$, where $T(M)$ denotes the tangent space of M [5]. If an almost contact metric manifold M satisfies

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{2.3}$$

for all X, $Y \in T(M)$, then M is called a Kenmotsu manifold [23]. From (2.1)-(2.3), it can be easily prove that

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.4}$$

and

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.5}$$

for all X, $Y \in T(M)$. Let S denote the Ricci tensor of M. It is noticed that M satisfies the following relations.

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi \tag{2.7}$$

and

$$S(X,\xi) = -(n-1)\eta(X) \tag{2.8}$$

for all $X, Y \in T(M)$. The curvature tensor R in a 3-dimensional Kenmotsu manifold M assumes the form

$$R(X,Y)Z = \left(\frac{r+4}{2}\right)[g(Y,Z)X - g(X,Z)Y] - \left(\frac{r+6}{2}\right)[g(Y,Z)\eta(X)\xi$$
$$-g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \tag{2.9}$$

After contracting X it becomes

$$S(Y,Z) = \frac{1}{2}\left[(r+2)g(Y,Z) - (r+6)\eta(X)\eta(Y)\right] \tag{2.10}$$

for all $X, Y \in T(M)$.

An n-dimensional Kenmotsu manifold (M,g) is said to be an η-Einstein manifold if its non-vanishing Ricci-tensor S takes the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.11}$$

for all $X, Y \in T(M)$, where a and b are smooth functions on (M,g). If $b = 0$ and a is constant, then η-Einstein manifold becomes Einstein manifold. Kenmotsu [23] proved that if (M,g) is an n-dimensional η-Einstein manifold, then $a+b = -(n-1)$.

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3. Semi-symmetric metric connection on Kenmotsu manifold

Let M be an n-dimensional Kenmotsu manifold endowed with a Riemannian metric g. A linear connection $\tilde{\nabla}$ on (M,g) is said to be a semi-symmetric metric connection [38] if the torsion tensor $\tilde{T}$ of the connection $\tilde{\nabla}$ and the Riemannian metric g satisfies

(3.1)

$$ll.,l,kmlkvmmmmmmmmmmmmmmmmmmmm\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

and

$$\tilde{\nabla}g = 0 \tag{3.2}$$

for all $X, Y \in T(M)$. The Levi-Civita connection ∇ and the semi-symmetric metric connection $\tilde{\nabla}$ on (M,g) are connected by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi \tag{3.3}$$

for all $X, Y \in T(M)$ [38]. From (2.1), (2.2) and (3.3), it follows that

$$(\tilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y) - \eta(X)\eta(Y) + g(X,Y). \tag{3.4}$$

The curvature tensors R and $\tilde{R}$ with respect to ∇ and $\tilde{\nabla}$, respectively, are connected by

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X + g(X,Z)AY - g(Y,Z)AX, \tag{3.5}$$

where α is a tensor field of type $(0,2)$ and A, a tensor field of type $(1,1)$, are related by

$$\alpha(Y,Z) = g(AY,Z) = (\nabla_Y\eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}g(Y,Z), \tag{3.6}$$

for all X, Y, $Z \in T(M)$ [38]. From (2.1), (2.5), (3.5) and (3.6), it follows that

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z &- 3g(Y,Z)X + 3g(X,Z)Y + 2\eta(Y)\eta(Z)X \\ &-2\eta(X)\eta(Z)Y + 2\eta(X)g(Y,Z)\xi - 2\eta(Y)g(X,Z)\xi.\end{aligned} \tag{3.7}$$

Contracting (3.7) along X, we get

$$\tilde{S}(Y,Z) = S(Y,Z) - (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z), \tag{3.8}$$

which becomes

$$\tilde{r} = r - n(3n-7) - 4. \tag{3.9}$$

Here $\tilde{S}$ and $\tilde{r}$ denote the Ricci tensor and the scalar curvature with respect to the connection $\tilde{\nabla}$. Replacing Z by ξ in (3.8) and using (2.8), we have

$$\tilde{S}(Y,\xi) = -2(n-1)g(Y,\xi). \tag{3.10}$$

Thus we can state:

Proposition 3.1. *Let M be an n-dimensional, $n \geqslant 3$, Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. Then ξ is an eigen vector of $\tilde{S}$ corresponding to the eigenvalue $-2(n-1)$.*

4. Globally ϕ-concircularly symmetric Kenmotsu manifold with a semi-symmetric metric connection

In this section, we will study the properties of the globally ϕ-concircularly symmetric Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$ and prove our result in the form of theorems.

Definition 4.1. A Kenmotsu manifold M of dimension n is said to be locally ϕ-symmetric with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if the non-vanishing curvature tensor $\tilde{R}$ satisfies the relation

$$\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) = 0$$

for all vector fields X, Y, Z and W orthogonal to ξ.

This notion was introduced by Takahashi [32] for Sasakian manifold.

Definition 4.2. An n-dimensional Kenmotsu manifold M is said to be a globally ϕ-concircularly symmetric manifold with respect to ∇ if the non-zero concircular curvature tensor C satisfies

$$\phi^2((\nabla_W C)(X,Y)Z) = 0 \tag{4.1}$$

for all vector fields X, Y, Z, $W \in T(M)$.

Definition 4.3. An n-dimensional Kenmotsu manifold M equipped with the semi-symmetric metric connection $\tilde{\nabla}$ is said to be a globally ϕ-concircularly symmetric Kenmotsu manifold with respect to $\tilde{\nabla}$ if the non-vanishing concircular curvature tensor $\tilde{C}$ with respect to $\tilde{\nabla}$ satisfies

$$\phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = 0 \tag{4.2}$$

for arbitrary vector fields X, Y, Z and W. Here $\tilde{C}$ is a concircular curvature tensor [37] with respect to $\tilde{\nabla}$ and is defined by

$$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right]. \tag{4.3}$$

Theorem 4.1. *An n-dimensional, $n \geq 3$, globally ϕ-concircularly symmetric Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is an η-Einstein manifold.*

Proof. We suppose that M is a globally ϕ-concircularly symmetric Kenmotsu manifold with respect to a semi-symmetric metric connection $\tilde{\nabla}$. Then we have

$$\phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = 0.$$

In view of (2.1), the above equation becomes

$$-(\tilde{\nabla}_W \tilde{C})(X,Y)Z + \eta((\tilde{\nabla}_W \tilde{C})(X,Y)Z)\xi = 0.$$

Equation (1.3) along with above equation give

$$\begin{aligned}
&-g((\tilde{\nabla}_W \tilde{R})(X,Y)Z,U) + \frac{d\tilde{r}(W)}{n(n-1)}\left[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\right] \\
&+\eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\eta(U) - \frac{d\tilde{r}(W)}{n(n-1)}\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right]\eta(U) = 0.
\end{aligned}$$

Replacing $X = U = e_i$, where $\{e_i, i = 1,2,3,...,n\}$, be an orthonormal basis of the tangent space at each point of the manifold M and then summing over i, $1 \leq i \leq n$, we get

$$\begin{aligned}
-(\tilde{\nabla}_W \tilde{S})(Y,Z) + \frac{d\tilde{r}(W)}{n}g(Y,Z) + \eta((\tilde{\nabla}_W \tilde{R})(\xi,Y)Z) \\
-\frac{d\tilde{r}(W)}{n(n-1)}\left[g(Y,Z) - \eta(Y)\eta(Z)\right] = 0.
\end{aligned}$$

Putting $Z = \xi$ in the above equation and using (2.1), we get

$$-(\tilde{\nabla}_W \tilde{S})(Y,\xi) + \frac{d\tilde{r}(W)}{n}\eta(Y) + \eta((\tilde{\nabla}_W \tilde{R})(\xi, Y)\xi) = 0. \tag{4.4}$$

In view of (2.1), (2.2), (2.4), (2.6), (2.7), (3.3) and (3.7), we conclude that

$$\eta((\tilde{\nabla}_W \tilde{R})(\xi, Y)\xi) = 0$$

and hence the equation (4.4) becomes

$$(\tilde{\nabla}_W \tilde{S})(Y,\xi) = \frac{d\tilde{r}(W)}{n}\eta(Y). \tag{4.5}$$

Substituting $Y = \xi$ in (4.5) and using (2.1) and (2.8), we get $d\tilde{r}(W) = 0$. This implies that $\tilde{r}$ is a constant. So from (4.5), we obtain

$$(\tilde{\nabla}_W \tilde{S})(Y,\xi) = 0. \tag{4.6}$$

It is well known that

$$(\tilde{\nabla}_W \tilde{S})(Y,\xi) = \tilde{\nabla}_W \tilde{S}(Y,\xi) - \tilde{S}(\tilde{\nabla}_W Y,\xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi).$$

In view of (2.1), (2.2), (2.4), (2.5), (2.8), (3.3), (3.4), (3.10) and (4.6), above equation takes the form

$$S(Y,W) = (n-3)g(Y,W) - 2(n-2)\eta(Y)\eta(W).$$

Hence the statement of the Theorem 4.1 is proved. □

From the above equation, it is clear that $r = (n-1)(n-4)$. Hence the scalar curvature under consideration is constant. Thus we have

Corollary 4.1. *An n-dimensional, $n > 3$, globally ϕ-concircularly symmetric Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ possesses a constant scalar curvature.*

Theorem 4.2. *Let M be an n-dimensional, $n \geqslant 3$, Kenmotsu manifold admits a semi-symmetric metric connection $\tilde{\nabla}$. Then the globally ϕ-concircularly symmetric manifold and the locally ϕ-symmetric manifold with respect to $\tilde{\nabla}$ coincide.*

Proof. We suppose that the manifold M is globally ϕ-concircularly symmetric with respect to a semi-symmetric metric connection $\tilde{\nabla}$. Since r is constant on M and therefore $\tilde{r}$ is also constant. The covariant derivative of (4.3) gives

$$(\tilde{\nabla}_W \tilde{C})(X,Y)Z = (\tilde{\nabla}_W \tilde{R})(X,Y)Z. \tag{4.7}$$

In view of (3.3), (3.4) and (3.7), we get

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z &= (\tilde{\nabla}_W R)(X,Y)Z + 2\{(\nabla_W\eta)(Y) - \eta(Y)\eta(W)\\ &+g(Y,W)\}\eta(Z)X + 2\{(\nabla_W\eta)(Z) - \eta(Z)\eta(W) + g(Z,W)\}\eta(Y)X\\ &-2\{(\nabla_W\eta)(X) - \eta(X)\eta(W) + g(X,W)\}\eta(Z)Y\\ &-2\{(\nabla_W\eta)(Z) - \eta(Z)\eta(W) + g(Z,W)\}\eta(X)Y\\ &+2g(Y,Z)\{(\nabla_W\eta)(X) - \eta(X)\eta(W) + g(X,W)\}\xi\\ &+2\{\nabla_W\xi + W - \eta(W)\xi\}\{\eta(X)g(Y,Z) + \eta(Y)g(X,Z)\}\\ &-2g(X,Z)\{(\nabla_W\eta)(Y) + \eta(Y)\eta(W) - g(Y,W)\}\xi. \quad (4.8)\end{aligned}$$

Using (2.4) and (2.5) in (4.8), we obtain

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z &= (\tilde{\nabla}_W R)(X,Y)Z + 4\{-\eta(Y)\eta(W) + g(Y,W)\}\eta(Z)X\\ &+4\{-\eta(Z)\eta(W) + g(Z,W)\}\eta(Y)X\\ &-4\{-\eta(X)\eta(W) + g(X,W)\}\eta(Z)Y\\ &-4\{-\eta(Z)\eta(W) + g(Z,W)\}\eta(X)Y\\ &+4g(Y,Z)\{-\eta(X)\eta(W) + g(X,W)\}\xi\\ &+4\{\eta(X)g(Y,Z) + \eta(Y)g(X,Z)\}\{W - \eta(W)\xi\}. \quad (4.9)\end{aligned}$$

If X, Y, Z and W are orthogonal to ξ then from above equation, we get

$$(\tilde{\nabla}_W\tilde{R})(X,Y)Z = (\tilde{\nabla}_W R)(X,Y)Z + 4g(Y,Z)g(X,W)\xi. \quad (4.10)$$

In view of (4.7) and (4.10), we have

$$(\tilde{\nabla}_W\tilde{C})(X,Y)Z = (\tilde{\nabla}_W R)(X,Y)Z + 4g(Y,Z)g(X,W)\xi.$$

Operating ϕ^2 on either sides of the above equation and then using (2.1) we get

$$\phi^2(\tilde{\nabla}_W\tilde{C})(X,Y)Z = \phi^2(\tilde{\nabla}_W R)(X,Y)Z \quad (4.11)$$

for all vector fields X, Y, Z and W orthogonal to ξ. From the equations (4.7) and (4.9), it is clear that the equation (4.11) satisfies for all vector fields X, Y, Z and W on M. Hence the statement of the Theorem 4.2 is proved. □

Remark 4.1. *The last equation shows that a locally ϕ-symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is always globally ϕ-concircularly symmetric manifold. Thus we conclude that on a Kenmotsu manifold locally ϕ-symmetric and globally ϕ-symmetric manifolds are equivalent corresponding to the connection $\tilde{\nabla}$.*

5. Three dimensional locally ϕ-symmetric Kenmotsu manifolds with respect to the semi-symmetric metric connection

This section deals with the study of the locally ϕ-symmetric Kenmotsu manifold M with respect to a semi-symmetric metric connection $\tilde{\nabla}$. Now, we will consider a 3-dimensional locally ϕ-symmetric Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$ and prove the following:

Theorem 5.1. *A 3-dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is locally ϕ-symmetric with respect to the connection $\tilde{\nabla}$ if and only if $dr(W) = 0$, W is an orthonormal vector field to ξ.*

Proof. From (2.9) and (3.7), we get

$$\begin{aligned}\tilde{R}(X,Y)Z = &\left(\frac{r-2}{2}\right)\{g(Y,Z)X - g(X,Z)Y\} \\ &+\left(\frac{r+2}{2}\right)[\eta(Y)g(X,Z)\xi + \eta(X)\eta(Z)Y \\ &-\eta(X)g(Y,Z)\xi - \eta(Y)\eta(Z)X].\end{aligned} \tag{5.1}$$

Taking covariant differentiation of (5.1) with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along W, we have

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z = &\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y - \eta(X)\eta(Z)Y \\ &+\{-g(X,Z)\eta(Y) + g(Y,Z)\eta(X)\}\xi + \eta(Y)\eta(Z)X] \\ &+\left(\frac{r+2}{2}\right)[g(X,Z)(\tilde{\nabla}_W\eta)(Y)\xi + g(X,Z)\eta(Y)\tilde{\nabla}_W\xi \\ &-g(Y,Z)(\tilde{\nabla}_W\eta)(X)\xi - g(Y,Z)\eta(X)\tilde{\nabla}_W\xi \\ &+\eta(Z)(\tilde{\nabla}_W\eta)(X)Y + \eta(X)(\tilde{\nabla}_W\eta)(Z)Y \\ &-\eta(Z)(\tilde{\nabla}_W\eta)(Y)X - \eta(Y)(\tilde{\nabla}_W\eta)(Z)X].\end{aligned} \tag{5.2}$$

In consequence of (3.3) and (3.4), (5.2) becomes

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z = &\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y - g(X,Z)\eta(Y)\xi \\ &-\eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi + \eta(Y)\eta(Z)X] \\ &+\left(\frac{r+2}{2}\right)[-\eta(X)g(Y,Z)\{\nabla_W\xi + W - \eta(W)\xi\} \\ &-g(Y,Z)\{(\nabla_W\eta)(X) - \eta(X)\eta(W) + g(X,W)\}\xi \\ &+\eta(Z)\{(\nabla_W\eta)(X) - \eta(X)\eta(W) + g(X,W)\}Y \\ &+\eta(X)\{(\nabla_W\eta)(Z) - \eta(W)\eta(Z) + g(Z,W)\}Y \\ &-\eta(Z)\{(\nabla_W\eta)(Y) - \eta(Y)\eta(W) + g(Y,W)\}X \\ &-\eta(Y)\{(\nabla_W\eta)(Z) - \eta(Z)\eta(W) + g(Z,W)\}X \\ &+g(X,Z)\{(\nabla_W\eta)(Y) - \eta(Y)\eta(W) + g(Y,W)\}\xi \\ &+g(X,Z)\eta(Y)\{\nabla_W\xi + W - \eta(W)\xi\}].\end{aligned} \tag{5.3}$$

Let us suppose that the vector fields X, Y, Z and W are orthogonal to ξ, therefore

(5.3) becomes

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z &= \frac{dr(W)}{2}\{g(Y,Z)X - g(X,Z)Y\}\\ &+\left(\frac{r+2}{2}\right)[g(X,Z)\{(\nabla_W\eta)(Y)+g(Y,W)\}\\ &-g(Y,Z)\{(\nabla_W\eta)(X)+g(X,W)\}]\xi.\end{aligned} \tag{5.4}$$

Operating ϕ^2 on both sides of (5.4) and then using (2.1) and (2.2), we obtain

$$\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) = -\frac{dr(W)}{2}\{g(Y,Z)X - g(X,Z)Y\}. \tag{5.5}$$

From the equation (5.5), it is obvious that the manifold M is locally ϕ-symmetric Kenmotsu manifold with respect to $\tilde{\nabla}$ if and only if $dr(W) = 0$. Hence the statement of the Theorem 5.1 is proved. □

6. Three dimensional Locally ϕ-concircularly symmetric Kenmotsu manifold with a semi-symmetric metric connection

Definition 6.1. A Kenmotsu manifold M is said to be locally ϕ-concircularly symmetric with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if its concircular curvature tensor $\tilde{C}$ satisfies

$$\phi^2((\tilde{\nabla}_W\tilde{C})(X,Y)Z) = 0$$

for all vector fields W, X, Y and Z orthogonal to ξ.

Theorem 6.1. *A 3-dimensional Kenmotsu manifold M with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is locally ϕ-concircularly symmetric manifold with respect to the connection $\tilde{\nabla}$ if and only if the scalar curvature r is constant.*

Proof. From (2.9) and (3.7), it follows that

$$\begin{aligned}\tilde{R}(X,Y)Z &= \left(\frac{r-2}{2}\right)\{g(Y,Z)X - g(X,Z)Y\}\\ &+\left(\frac{r+2}{2}\right)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\\ &+\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi].\end{aligned} \tag{6.1}$$

In view of (1.2) and (6.1), we get

$$\begin{aligned}\tilde{C}(X,Y)Z &= \left(\frac{r-2}{2}\right)\{g(Y,Z)X - g(X,Z)Y\}\\ &+\left(\frac{r+2}{2}\right)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\\ &+\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi]\\ &+\frac{r}{6}\{g(Y,Z)X - g(X,Z)Y\}.\end{aligned} \tag{6.2}$$

Taking covariant derivative of (6.2) with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along W, we have

$$\begin{aligned}(\tilde{\nabla}_W\tilde{C})(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y \\ &+ \{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\xi - \eta(Y)\eta(Z)X] \\ &+ \left(\frac{r+2}{2}\right)[g(X,Z)(\tilde{\nabla}_W\eta)(Y)\xi + g(X,Z)\eta(Y)\tilde{\nabla}_W\xi \\ &-g(Y,Z)(\tilde{\nabla}_W\eta)(X)\xi - g(Y,Z)\eta(X)\tilde{\nabla}_W\xi + \eta(X)(\tilde{\nabla}_W\eta)(Z)Y \\ &-\eta(Z)(\tilde{\nabla}_W\eta)(Y)X - \eta(Y)(\tilde{\nabla}_W\eta)(Z)X + \eta(Z)(\tilde{\nabla}_W\eta)(X)Y] \\ &+\frac{dr(W)}{6}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{6.3}$$

Let us consider that the vector fields X, Y and Z are orthonormal to ξ and therefore (6.3) converts into the form

$$\begin{aligned}(\tilde{\nabla}_W\tilde{C})(X,Y)Z &= \frac{dr(W)}{2}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \left(\frac{r+2}{2}\right)\Big\{g(X,Z)(\tilde{\nabla}_W\eta)(Y) - g(Y,Z)(\tilde{\nabla}_W\eta)(X)\Big\}\xi \\ &+\frac{dr(W)}{6}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{6.4}$$

Using (3.4) in (6.4), we obtain

$$\begin{aligned}(\tilde{\nabla}_W\tilde{C})(X,Y)Z &= \frac{2dr(W)}{3}\{g(Y,Z)X - g(X,Z)Y\} + \left(\frac{r+2}{2}\right)[g(X,Z)(\nabla_W\eta)(Y) \\ &-g(X,Z)\eta(Y)\eta(W) - g(Y,Z)(\nabla_W\eta)(X) + g(Y,W)g(X,Z) \\ &-g(Y,Z)g(X,W) + g(Y,Z)\eta(X)\eta(W)]\xi. \end{aligned} \tag{6.5}$$

Applying ϕ^2 on both sides of (6.5) and using (2.1), we get

$$\phi^2\Big((\tilde{\nabla}_W\tilde{C})(X,Y)Z\Big) = \frac{2dr(W)}{3}\{g(Y,Z)X - g(X,Z)Y\}.$$

This proved the statement of the Theorem 6.1. □

From the Theorem 5.1 and the Theorem 6.1, we can state the following:

Corollary 6.1. *A 3-dimensional Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is locally ϕ-concircularly symmetric with respect to the connection $\tilde{\nabla}$ if and only if it is locally ϕ-symmetric with respect to $\tilde{\nabla}$.*

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7. Three dimensional Ricci semisymmetric Kenmotsu manifold with respect to the semi-symmetric metric connection

The following section delas with the study of a 3-dimensional Ricci semisymmetric Kenmotsu manifold with respect to the semi-symmetric metric connection with tha aim to prove some geometrical results.

Theorem 7.1. *A 3-dimensional Ricci semisymmetric Kenmotsu manifold with respect to a semi-symmetric metric connection $\tilde{\nabla}$ possesses a constant scalar curvature.*

Proof. Let us consider a 3-dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ which satisfies $\tilde{R}(X,Y)\cdot\tilde{S}=0$, that is, M is Ricci semisymmetric with respect to $\tilde{\nabla}$ and then we have

$$\tilde{S}(\tilde{R}(X,Y)Z,W)+\tilde{S}(Z,\tilde{R}(X,Y)W)=0. \tag{7.1}$$

Replacing X by ξ in (7.1), we get

$$\tilde{S}(\tilde{R}(\xi,Y)Z,W)+\tilde{S}(Z,\tilde{R}(\xi,Y)W)=0. \tag{7.2}$$

From (5.1), it is obvious that

$$\tilde{R}(\xi,Y)Z=-2\left\{g(Y,Z)\xi-\eta(Z)Y\right\}. \tag{7.3}$$

By virtue of (3.10), (7.2) and (7.3), we obtain

$$\eta(Z)\tilde{S}(Y,W)+4\eta(W)g(Y,Z)+\eta(W)\tilde{S}(Z,Y)+4\eta(Z)g(Y,W)=0. \tag{7.4}$$

Let $\{e_i\}$, $i=1,2,3$, is an orthonormal basis of the tangent space at each point of the manifold M. Putting $Y=Z=e_i$ in (7.4) and taking summation over i, $1\le i\le 3$, we get

$$(\tilde{r}+12)\eta(W)=0.$$

Since $\eta(W)\neq 0$, in general, therefore $\tilde{r}=-12\,(constant)$. This proved the statement of the Theorem 7.1. □

In consequence of the Theorem 6.1 and Theorem 7.1, we state:

Corollary 7.1. *If a 3-dimensional Kenmotsu manifold M with respect to a semi-symmetric metric connection $\tilde{\nabla}$ satisfies the condition $\tilde{R}(X,Y)\cdot\tilde{S}=0$, then M is locally ϕ-symmetric as well as locally ϕ-concircularly symmetric with respect to $\tilde{\nabla}$, respectively.*

8. η-parallel Ricci tensor with respect to the semi-symmetric metric connection

Definition 8.1. A Ricci tensor $\tilde{S}$ of a Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is called η-parallel with respect to $\tilde{\nabla}$ if it $\tilde{S}$ is non-zero and satisfies

$$(\tilde{\nabla}_X \tilde{S})(\phi Y, \phi Z) = 0 \tag{8.1}$$

for all vector fields X, Y and Z on M.

The notion of η-parallel Ricci tensor on a Sasakian manifold was introduced by M. Kon [24]. Since then, many authors studied the geometrical and physical properties of this tensor.

Theorem 8.1. *If a 3-dimensional Kenmotsu manifold M with respect to the semi-symmetric metric connection $\tilde{\nabla}$ possesses an η-parallel Ricci tensor, then the scalar curvature of M is constant.*

Proof. In view of (2.2), (2.9) and (3.8), we have

$$\tilde{S}(\phi X, \phi Y) = \left(\frac{\tilde{r}+4}{2}\right)\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{8.2}$$

Differentiating (8.2) covariantly with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along W, we get

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(\phi X, \phi Y) = {} & \frac{d\tilde{r}(W)}{2}\{g(X,Y) - \eta(X)\eta(Y)\} \\ & - \left(\frac{\tilde{r}+4}{2}\right)\Big\{(\tilde{\nabla}_W \eta)(X)\eta(Y) + (\tilde{\nabla}_W \eta)(Y)\eta(X)\Big\} \\ & - \tilde{S}((\tilde{\nabla}_W \phi)(X), \phi Y) - \tilde{S}(\phi X, (\tilde{\nabla}_W \phi)(Y)).\end{aligned} \tag{8.3}$$

In view of (2.1), (2.3), (2.5), (3.3), (3.4), (8.1) and (8.3), it can be easily found that

$$\begin{aligned}& \frac{d\tilde{r}(W)}{2}\{g(X,Y) - \eta(X)\eta(Y)\} + 2\eta(X)\tilde{S}(\phi W, \phi Y) + 2\eta(Y)\tilde{S}(\phi W, \phi X) \\ & -(\tilde{r}+4)\{\eta(Y)g(X,W) + \eta(X)g(Y,W) - 2\eta(X)\eta(Y)\eta(W)\} = 0.\end{aligned} \tag{8.4}$$

In consequence of (8.2), (8.4) becomes

$$d\tilde{r}(W)\{g(X,Y) - \eta(X)\eta(Y)\} = 0,$$

which gives

$$d\tilde{r}(W) = 0 \Longleftrightarrow \tilde{r} \textit{ is constant}.$$

Hence the statement of the Theorem 8.1 is proved. □

In the light of the Theorem 6.1 and Theorem 8.1, we state the following corollary.

Corollary 8.1. *If a 3-dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ has η-parallel Ricci tensor, then the manifold is locally ϕ-symmetric as well as locally ϕ-concircularly symmetric with respect to $\tilde{\nabla}$, respectively.*

9. Three dimensional locally ϕ-concircularly recurrent Kenmotsu manifold with respect to the semi-symmetric metric connection

Definition 9.1. A Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is said to be ϕ-concircularly recurrent with respect to $\tilde{\nabla}$ if there exists a non-zero 1-form A on M such that

$$\phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = A(W)\tilde{C}(X,Y)Z \tag{9.1}$$

for arbitrary vector fields X, Y, Z and W, where $\tilde{C}$ is the concircular curvature tensor with respect to the semi-symmetric metric connection $\tilde{\nabla}$. If the 1-form A vanishes identically on M, then the manifold M with $\tilde{\nabla}$ is reduced to a locally ϕ-concircularly symmetric manifold with respect to $\tilde{\nabla}$.

Theorem 9.1. *If a 3-dimensional locally ϕ-concircularly recurrent Kenmotsu manifold admits a semi-symmetric metric connection $\tilde{\nabla}$, then the curvature tensor with respect to $\tilde{\nabla}$ assumes the form (9.7).*

Proof. From (3.9) and (5.5), we have

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = -\frac{d\tilde{r}(W)}{2}\{g(Y,Z)X - g(X,Z)Y\}. \tag{9.2}$$

On the other hand, from (1.3), it is seen that (for $n = 3$)

$$(\tilde{\nabla}_W \tilde{C})(X,Y)Z = (\tilde{\nabla}_W \tilde{R})(X,Y)Z - \frac{d\tilde{r}(W)}{6}\{g(Y,Z)X - g(X,Z)Y\}. \tag{9.3}$$

Applying ϕ^2 on both sides of (9.3), we get

$$\phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = \phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) - \frac{d\tilde{r}(W)}{6}\left\{g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y\right\}. \tag{9.4}$$

In consequence of (2.1), (9.1) and (9.2), it is obvious that

$$\begin{aligned} A(W)\tilde{C}(X,Y)Z = &-\frac{d\tilde{r}(W)}{3}\{g(Y,Z)X - g(X,Z)Y\} \\ &-\frac{d\tilde{r}(W)}{6}\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi. \end{aligned} \tag{9.5}$$

Replacing W with ξ in (9.5), we get

$$\begin{aligned} \tilde{C}(X,Y)Z = &-\frac{d\tilde{r}(\xi)}{3A(\xi)}\{g(Y,Z)X - g(X,Z)Y\} \\ &-\frac{d\tilde{r}(\xi)}{6A(\xi)}\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi, \end{aligned} \tag{9.6}$$

provided $A(\xi) \neq 0$. In view of (1.2) and (9.6), we have

$$\tilde{R}(X,Y)Z = a\{g(Y,Z)X - g(X,Z)Y\} - b\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi, \tag{9.7}$$

where $a = \left\{\frac{\tilde{r}}{6} - \frac{d\tilde{r}(\xi)}{3A(\xi)}\right\}$, $b = \frac{d\tilde{r}(\xi)}{6A(\xi)}$ and A is a non-zero 1-form. □

10. Example of a Kenmotsu manifold admitting a semi-symmetric metric connection

In this section, we will construct a non-trivial example of a Kenmotsu manifold admitting the semi-symmetric metric connection and after that we will validate our results.

Example 10.1. Let

$$M = \{(x, y, z) \in \mathbb{R}^3 : x, y, z(\neq 0) \in \mathbb{R}\},$$

be a three dimensional Riemannian manifold, where (x, y, z) denotes the standard coordinates of a point in $\mathbb{R}^3$. Let us suppose that

$$e_1 = z\frac{\partial}{\partial x}, \;\; e_2 = z\frac{\partial}{\partial y}, \;\; e_3 = -z\frac{\partial}{\partial z}$$

be a set of linearly independent vector fields at each point of the manifold M and therefore it forms a basis for the tangent space $T(M)$. We also define the Riemannian metric g of the manifold by $g(e_i, e_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta and $i, j = 1, 2, 3$. Let us consider a 1-form η defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in T(M)$ and a tensor field ϕ of type $(1, 1)$ defined by

$$\phi(e_1) = -e_2, \;\; \phi(e_2) = e_1, \;\; \phi(e_3) = 0.$$

By the linearity properties of ϕ and g, we can easily verify the following relations

$$\phi^2 X = -X + \eta(X)e_3, \;\; \eta(e_3) = 1, \;\; g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for arbitrary vector fields $X, Y \in T(M)$. This shows that for $\xi = e_3$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

If ∇ represents the Levi-Civita connection with respect to the Riemannian metric g, then with the help of above relations, we can easily calculate the non-vanishing components of Lie bracket as:

$$[e_1, e_2] = 0, \;\; [e_1, e_3] = e_1, \;\; [e_2, e_3] = e_2.$$

We recall the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \end{aligned}$$

for all vector fields $X, Y, Z \in T(M)$. It is obvious from Koszul's formula that

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_3 = e_1, \\ &\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = -e_3, && \nabla_{e_2} e_3 = e_2, \\ &\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0. \end{aligned}$$

From the above calculations, we can observe that $\nabla_X \xi = X - \eta(X)\xi$ for $\xi = e_3$. Thus the manifold (M, g) is a Kenmotsu manifold of dimension 3 and the structure (ϕ, η, ξ, g) denotes the Kenmotsu structure on the manifold M [16].

In consequence of (3.3) and the above results, we can find that

$$\begin{aligned}
&\tilde\nabla_{e_1}e_1 = -2e_3, && \tilde\nabla_{e_1}e_2 = 0, && \tilde\nabla_{e_1}e_3 = 2e_1,\\
&\tilde\nabla_{e_2}e_1 = 0, && \tilde\nabla_{e_2}e_2 = -2e_3, && \tilde\nabla_{e_2}e_3 = 2e_2,\\
&\tilde\nabla_{e_3}e_1 = 0, && \tilde\nabla_{e_3}e_2 = 0, && \tilde\nabla_{e_3}e_3 = 0
\end{aligned}$$

and also the components of torsion tensor $\tilde T$ are

$$\begin{aligned}
&\tilde T(e_i, e_i) = \tilde\nabla_{e_i}e_i - \tilde\nabla_{e_i}e_i - [e_i, e_i] = 0, \;\; for \;\; i = 1, 2, 3\\
&\tilde T(e_1, e_2) = 0, \quad \tilde T(e_1, e_3) = e_1, \quad \tilde T(e_2, e_3) = e_2.
\end{aligned}$$

This shows that $\tilde T \neq 0$ and, therefore, by the equation (3.1), we can say that the linear connection defined in (3.3) is a semi-symmetric connection on (M, g). By straightforward calculation, we can also find

$$(\tilde\nabla_{e_1}g)(e_2, e_3) = 0, \quad (\tilde\nabla_{e_2}g)(e_3, e_1) = 0, \quad (\tilde\nabla_{e_3}g)(e_1, e_2) = 0$$

and other components by symmetric properties. This demonstrates that the equation (3.2) is satisfied and hence the linear connection defined by (3.3) is a semi-symmetric metric connection on M. Thus, we can say that the manifold (M, g) is a 3-dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection defined by (3.3).

With the help of the above discussions, we can calculate the curvature and Ricci tensors of M with respect to the semi-symmetric metric connection $\tilde\nabla$ as

$$\begin{aligned}
&\tilde R(e_1, e_2)e_3 = 0, \quad \tilde R(e_1, e_3)e_3 = -2e_1, \quad \tilde R(e_3, e_2)e_2 = -2e_3,\\
&\tilde R(e_3, e_1)e_1 = -2e_3, \quad \tilde R(e_2, e_1)e_1 = -4e_2, \quad \tilde R(e_2, e_3)e_3 = -2e_2,\\
&\tilde R(e_1, e_2)e_2 = 0, \quad \tilde S(e_1, e_1) = -6, \quad \tilde S(e_2, e_2) = -2, \quad \tilde S(e_3, e_3) = -4
\end{aligned}$$

and other components can be calculated by skew-symmetric properties. We can easily observe that the equation (3.10) is verified.

Next, we have to prove that the manifold (M, g) is a Ricci semisymmetric with respect to the connection $\tilde\nabla$, *i.e.*, $\tilde R \cdot \tilde S = 0$. For instance,

$$(\tilde R(e_3, e_1) \cdot \tilde S)(e_1, e_1) = 0, \quad (\tilde R(e_3, e_2) \cdot \tilde S)(e_1, e_1) = 0, \quad (\tilde R(e_3, e_3) \cdot \tilde S)(e_1, e_1) = 0.$$

In a similar way, we can verify other components. Also, we can prove that $\tilde r = -12$ (constant) and hence the Theorem 7.1 is verified. Moreover, it can be easily seen that the Theorem 5.1, Theorem 6.1 and the Theorem 8.1have been verified.

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