

CERTAIN GEOMETRIC PROPERTIES OF A NORMALIZED HYPER-BESSEL FUNCTION

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Abstract. In the present paper, by making use of previous results on analytic functions, certain geometric properties including starlikeness of order α and convexity of order α of a normalized hyper-Bessel function have been investigated. In addition, some conditions of the mentioned function which belongs to the Hardy space have been given. Moreover, specific examples of the results which refer to special values of the parameters have also been presented.

Keywords: analytic function, hyper-Bessel function, starlikeness of order α , convexity of order α , Hardy space.

1. Introduction

Special functions have a great importance in applied sciences. Due to their remarkable properties, a great number of mathematicians have investigated the geometric properties of special functions like Bessel, Struve and Lommel functions of the first kind. Especially, these geometric properties include univalence, starlikeness and convexity of the above mentioned functions. Actually, the first results on the univalence of Bessel function can be found in the papers [9] and [13] written by Brown and, Kreyszig and Todd in 1960, respectively. In recent years, the geometric properties of some special functions have become very interesting to mathematicians. More precisely, the authors in [3, 4, 5, 6] began to study the geometric properties of the above mentioned special functions and their generalizations. Also, the authors in [7, 14] studied Hardy space of generalized Bessel and hypergeometric functions, respectively. Especially, Bessel function has some extensions like q -Bessel and hyper-Bessel functions. Some geometric properties of these extensions may be found in [1, 2, 3] and the references therein.

Now, we would like to remind you about the definitions of the Bessel and hyper-Bessel functions, respectively. Bessel function is defined by the following infinite

series (see [16]):

$$(1.1) \quad J_\nu(z) = \sum_{n \geq 0} \frac{(-1)^n}{n! \Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{2n+\nu}, z \in \mathbb{C}$$

where $\Gamma(z)$ denotes the familiar gamma function. Similarly, the hyper-Bessel function is defined by (see [10]):

$$(1.2) \quad J_{\gamma_d}(z) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{z}{d+1}\right)^{n(d+1) + \sum_{i=1}^d \gamma_i}}{n! \prod_{i=1}^d \Gamma(\gamma_i + n + 1)}.$$

It is clear that, the hyper-Bessel function is reduced to the classical Bessel function given by (1.1) for $d = 1$ and $\gamma_1 = \nu$. Due to this relationship, the earlier studies on the classical Bessel function can be extended to the hyper-Bessel function. Motivated by some earlier works, our main aim in this study is to obtain new conditions on the starlikeness and convexity of the hyper-Bessel functions. Also, we will deal with the Hardy space of the hyper-Bessel function.

2. Starlikeness and convexity of hyper-Bessel function

In the beginning of this section, we are going to mention some basic notions in geometric function theory. Let \mathbb{D}_r be the open disk $\{z \in \mathbb{C} : |z| < r\}$ with radius $r > 0$ and $\mathbb{D}_1 = \mathbb{D}$. Let \mathcal{A} denote the class of analytic functions $f : \mathbb{D}_r \rightarrow \mathbb{C}$,

$$(2.1) \quad f(z) = z + \sum_{n \geq 2} a_n z^n,$$

which satisfies the normalization conditions $f(0) = f'(0) - 1 = 0$. By \mathcal{S} we mean the class of functions belonging to \mathcal{A} which are univalent in \mathbb{D}_r . Also, for $0 \leq \alpha < 1$, by $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ we will denote the subclasses of \mathcal{A} consisting of functions which are starlike and convex of order α , respectively. The analytic characterizations of these subclasses are

$$\mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > \alpha \text{ for } z \in \mathbb{D} \right\}$$

and

$$\mathcal{C}(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} \right) > \alpha \text{ for } z \in \mathbb{D} \right\},$$

respectively.

The following results on analytic functions given by Silverman in [15] will be used in order to prove our main results.

Lemma 2.1. *Let f is of the form (2.1). If*

$$(2.2) \quad \sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha,$$

then $f \in \mathcal{S}^(\alpha)$.*

Lemma 2.2. *Let f is of the form (2.1). If*

$$(2.3) \quad \sum_{n=2}^{\infty} n(n - \alpha) |a_n| \leq 1 - \alpha,$$

then $f \in \mathcal{C}(\alpha)$.

Before revealing our main results concerning starlikeness and convexity of the hyper-Bessel function, we will apply the following natural normalization for the function $z \mapsto J_{\gamma_d}(z)$ given by (1.2):

$$(2.4) \quad \begin{aligned} F_{\gamma_d}(z) &= z \frac{\prod_{i=1}^d \Gamma(\gamma_i + 1)}{\left(\frac{d+\sqrt{z}}{d+1}\right)^{\sum_{i=1}^d \gamma_i}} J_{\gamma_d}(\sqrt{d+1}z) \\ &= z + \sum_{n \geq 2} \frac{(-1)^n}{(n-1)! \rho^{n-1} \prod_{i=1}^d (\gamma_i + 1)_{n-1}} z^n, \end{aligned}$$

where $\rho = (d + 1)^{d+1}$ and $(\beta)_n$ is the Pochhammer symbol which is defined by $(\beta)_0 = 1$ and $(\beta)_n = \beta(\beta + 1) \dots (\beta + n - 1)$ for $n \geq 1$. As a consequence, the function $z \mapsto F_{\gamma_d} \in \mathcal{A}$.

Our first main result regarding starlikeness of order α of the function $F_{\gamma_d}(z)$ is the following:

Theorem 2.1. *Let $\alpha \in [0, 1)$, $d=1, 2, \dots$, $\rho = (d + 1)^{d+1}$ and $\mu = \prod_{i=1}^d (\gamma_i + 1)$. If*

$$\alpha \leq \frac{4\rho^2\mu^2 - 12\rho\mu + 3}{4\rho^2\mu^2 - 8\rho\mu + 3},$$

then the normalized hyper-Bessel function $F_{\gamma_d}(z)$ is starlike of order α in \mathbb{D} .

Proof. It is known from Lemma 2.1 that, if the function $f \in \mathcal{A}$ satisfies the inequality (2.2), then the function f is starlike of order α . Because of this, in order to prove starlikeness of order α of the function $F_{\gamma_d}(z)$, it is enough to show that

$$\sum_{n \geq 2} (n - \alpha) \left| \frac{(-1)^{n-1}}{(n-1)! \rho^{n-1} \prod_{i=1}^d (\gamma_i + 1)_{n-1}} \right| \leq 1 - \alpha$$

for $\alpha \in [0, 1)$. Now using the well-known inequalities

$$(2.5) \quad 2^{n-2} \leq (n-1)!$$

and

$$(2.6) \quad (\beta)_n \leq (\beta)^n$$

for $n \geq 2$, we can write that

$$\begin{aligned} \sum_{n \geq 2} (n - \alpha) \left| \frac{(-1)^{n-1}}{(n-1)! \rho^{n-1} \prod_{i=1}^d (\gamma_i + 1)_{n-1}} \right| &= \sum_{n \geq 2} \frac{(n - \alpha)}{(n-1)! \rho^{n-1} \prod_{i=1}^d (\gamma_i + 1)_{n-1}} \\ &\leq 2 \sum_{n \geq 2} \frac{n - \alpha}{(2\rho\mu)^{n-1}} \\ &= 2 \sum_{n \geq 2} n \left(\frac{1}{2\rho\mu} \right)^{n-1} - 2\alpha \sum_{n \geq 2} \left(\frac{1}{2\rho\mu} \right)^{n-1}. \end{aligned}$$

Considering the known series sums

$$\sum_{n \geq 2} t^{n-1} = \frac{t}{1-t} \quad \text{and} \quad \sum_{n \geq 2} nt^{n-1} = \frac{t(2-t)}{(1-t)^2}$$

for $|t| < 1$ in the last step, we have

$$\sum_{n \geq 2} (n - \alpha) \left| \frac{(-1)^{n-1}}{(n-1)! \rho^{n-1} \prod_{i=1}^d (\gamma_i + 1)_{n-1}} \right| \leq \frac{4\rho\mu(2-\alpha) + 2(\alpha-1)}{(2\rho\mu-1)^2} \leq 1 - \alpha$$

for $\left| \frac{1}{2\rho\mu} \right| < 1$ and

$$\alpha \leq \frac{4\rho^2\mu^2 - 12\rho\mu + 3}{4\rho^2\mu^2 - 8\rho\mu + 3}.$$

So, the proof is completed. \square

By setting $d = 1$ and $\gamma_1 = \nu$ in the Theorem 2.1, we have the following:

Corollary 2.1. *Let $\alpha \in [0, 1)$. If*

$$\alpha \leq \frac{64\nu^2 + 80\nu + 19}{64\nu^2 + 96\nu + 35},$$

then the function $z \mapsto F_\nu(z) = 2^\nu z^{1-\frac{\alpha}{2}} \Gamma(\nu+1) J_\nu(\sqrt{z})$ is starlike of order α .

It is well-known from [8, p.13–14] that the basic trigonometric functions can be represented by the classical Bessel function J_ν for the appropriate values of the parameter ν . Clearly, for $\nu = -\frac{1}{2}$, $\nu = \frac{1}{2}$ and $\nu = \frac{3}{2}$, respectively, we have the following:

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z, \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z \quad \text{and} \quad J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right).$$

Now, taking $\nu = \frac{1}{2}$ and $\nu = \frac{3}{2}$ in Corollary 2.1 we have the following examples on the elementary trigonometric functions.

Example 2.1. The following assertions are true for $z \in \mathbb{D}$:

- a. If $\alpha \leq \alpha_1 \cong 0.75$, then the function $F_{\frac{1}{2}}(z) = \sqrt{z} \sin \sqrt{z}$ is in $\mathcal{S}^*(\alpha)$.
- b. If $\alpha \leq \alpha_2 \cong 0.88$, then the function $F_{\frac{3}{2}}(z) = 3 \left(\frac{\sin \sqrt{z}}{\sqrt{z}} - \cos \sqrt{z} \right)$ is in $\mathcal{S}^*(\alpha)$.

Our second main result concerning convexity of order α of the function $F_{\gamma_d}(z)$ is the following:

Theorem 2.2. Let $\alpha \in [0, 1)$, $d=1,2,\dots$, $\rho = (d + 1)^{d+1}$ and $\mu = \prod_{i=1}^d (\gamma_i + 1)$. If

$$\alpha \leq \frac{(2\rho\mu - 1)^3 - 8\rho^2\mu^2 + 12\rho\mu - 2}{(2\rho\mu - 1)^3 - 16\rho^2\mu^2 + 12\rho\mu - 2},$$

then the normalized hyper-Bessel function $F_{\gamma_d}(z)$ is convex of order α in \mathbb{D} .

Proof. By using the Lemma 2.2, it is possible to show that the function $F_{\gamma_d}(z)$ is convex of order α . For this reason, we need to show that the function $F_{\gamma_d}(z)$ satisfies the inequality (2.3). That is, it is sufficient to prove that

$$\sum_{n \geq 2} n(n - \alpha) \left| \frac{(-1)^{n-1}}{(n - 1)! \rho^{n-1} \prod_{i=1}^d (\gamma_i + 1)_{n-1}} \right| \leq 1 - \alpha.$$

By making use of the known inequalities (2.5) and (2.6), we can write that

$$\begin{aligned} \sum_{n \geq 2} n(n - \alpha) \left| \frac{(-1)^{n-1}}{(n - 1)! \rho^{n-1} \prod_{i=1}^d (\gamma_i + 1)_{n-1}} \right| &= \sum_{n \geq 2} \frac{n(n - \alpha)}{(n - 1)! \rho^{n-1} \prod_{i=1}^d (\gamma_i + 1)_{n-1}} \\ &\leq 2 \sum_{n \geq 2} \frac{n(n - \alpha)}{(2\rho\mu)^{n-1}} \\ &= 2 \sum_{n \geq 2} n^2 \left(\frac{1}{2\rho\mu} \right)^{n-1} - 2\alpha \sum_{n \geq 2} n \left(\frac{1}{2\rho\mu} \right)^{n-1}. \end{aligned}$$

If we consider the known series sums

$$\sum_{n \geq 2} nt^{n-1} = \frac{t(2 - t)}{(1 - t)^2} \text{ and } \sum_{n \geq 2} n^2t^{n-1} = \frac{t(t^2 - 3t + 4)}{(1 - t)^3}$$

for $|t| < 1$ in the above last inequality, we get

$$\sum_{n \geq 2} n(n - \alpha) \left| \frac{(-1)^{n-1}}{(n - 1)! \rho^{n-1} \prod_{i=1}^d (\gamma_i + 1)_{n-1}} \right| \leq \frac{8\rho^2\mu^2(1 - 2\alpha) + 2(6\rho\mu - 1)(\alpha - 1)}{(2\rho\mu - 1)^3}$$

for $\left| \frac{1}{2\rho\mu} \right| < 1$. Since

$$\frac{8\rho^2\mu^2(1 - 2\alpha) + 2(6\rho\mu - 1)(\alpha - 1)}{(2\rho\mu - 1)^3} \leq 1 - \alpha$$

under assumption, the proof is thus completed. \square

By setting $d = 1$ and $\gamma_1 = \nu$ in the Theorem 2.2, we have the following:

Corollary 2.2. *Let $\alpha \in [0, 1)$. If*

$$\alpha \leq \frac{512\nu^3 + 1216\nu^2 + 968\nu + 261}{512\nu^3 + 1088\nu^2 + 712\nu + 133},$$

then the function $z \mapsto F_\nu(z) = 2^\nu z^{1-\frac{\nu}{2}} \Gamma(\nu + 1) J_\nu(\sqrt{z})$ is convex of order α .

3. Hardy space of hyper-Bessel function

Let \mathcal{H} denote the set of all analytic (holomorphic, regular) functions on the open unit disk \mathbb{D} . For any $\eta \in (0, \infty]$, any function $f \in \mathcal{H}$ and any $r \in [0, 1)$ set

$$M_\eta(r, f) = \begin{cases} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta \right)^{\frac{1}{\eta}}, & \text{if } 0 < \eta < \infty \\ \max_{|z| \leq r} |f(z)|, & \text{if } \eta = \infty. \end{cases}$$

By definition, the function $f \in \mathcal{H}$ is said to belong to the Hardy space \mathcal{H}^η , where $0 < \eta \leq \infty$, if the set $\{M_\eta(r, f) : r \in [0, 1)\}$ is bounded. Note that for $1 \leq \eta \leq \infty$, \mathcal{H}^η is a Banach space with the norm defined by $\|f\|_\eta = \lim_{r \rightarrow 1^-} M_\eta(r, f)$. Furthermore, \mathcal{H}^∞ is the class of bounded analytic functions in \mathcal{H} . We note that for $0 < p \leq q \leq \infty$, it can be shown that \mathcal{H}^q is a subset of \mathcal{H}^p (see[11]).

The following lemma due to Eeingenburg and Keogh [12] will be used in order derive our last main result.

Lemma 3.1. *Let $\alpha \in [0, 1)$. If the function $f \in \mathcal{C}(\alpha)$ is not of the form*

$$\begin{cases} f(z) = k + lz (1 - ze^{i\theta})^{2\alpha-1}, & \alpha \neq \frac{1}{2} \\ f(z) = k + l \log(1 - ze^{i\theta}), & \alpha = \frac{1}{2}. \end{cases}$$

for some $k, l \in \mathbb{C}$ and $\theta \in \mathbb{R}$, then the following statements hold:

- a.* *There exist $\delta = \delta(f) > 0$ such that $f' \in \mathcal{H}^{\delta + \frac{1}{2(1-\alpha)}}$.*
- b.* *If $\alpha \in [0, \frac{1}{2})$, then there exist $\tau = \tau(f) > 0$ such that $f \in \mathcal{H}^{\tau + \frac{1}{1-2\alpha}}$.*
- c.* *If $\alpha \geq \frac{1}{2}$, then $f \in \mathcal{H}^\infty$.*

Our main result concerning the Hardy space of the normalized hyper-Bessel function $F_{\gamma_d}(z)$ is as follows.

Theorem 3.1. *Let $\alpha \in [0, 1)$, $d=1,2,\dots$, $\rho = (d + 1)^{d+1}$ and $\mu = \prod_{i=1}^d (\gamma_i + 1)$. If*

$$\alpha \leq \frac{(2\rho\mu - 1)^3 - 8\rho^2\mu^2 + 12\rho\mu - 2}{(2\rho\mu - 1)^3 - 16\rho^2\mu^2 + 12\rho\mu - 2},$$

then the normalized hyper-Bessel function $F_{\gamma_d}(z)$ has the following properties:

i. $F_{\gamma_d}(z) \in \mathcal{H}^{\frac{1}{1-2\alpha}}$ for $\alpha \in [0, \frac{1}{2})$,

ii. $F_{\gamma_d}(z) \in \mathcal{H}^\infty$ for $\alpha \in (\frac{1}{2}, 1)$.

Proof. Gauss hypergeometric function is defined by

$$(3.1) \quad {}_2F_1(a, b, c; z) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

By using the definition of the Gauss hypergeometric function given by (3.1), it is easily shown that

$$\begin{aligned} k + \frac{lz}{(1 - ze^{i\theta})^{1-2\alpha}} &= k + lz {}_2F_1(1, (1 - 2\alpha), 1; ze^{i\theta}) \\ &= k + l \sum_{n \geq 0} \frac{(1 - 2\alpha)_n}{n!} e^{i\theta n} z^{n+1} \end{aligned}$$

for $k, l \in \mathbb{C}, \alpha \neq \frac{1}{2}$ and $\theta \in \mathbb{R}$. On the other hand, it can be easily shown that

$$\begin{aligned} k + l \log(1 - ze^{i\theta}) &= k - lz {}_2F_1(1, 1, 2; ze^{i\theta}) \\ &= k - l \sum_{n \geq 0} \frac{1}{n+1} e^{i\theta n} z^{n+1}. \end{aligned}$$

As a result, the function $F_{\gamma_d}(z)$ given by (2.4) is not of the forms $k + \frac{lz}{(1 - ze^{i\theta})^{1-2\alpha}}$ for $\alpha \neq \frac{1}{2}$ and $k + l \log(1 - ze^{i\theta})$ for $\alpha = \frac{1}{2}$, respectively. Also, it is known from Theorem 2.2 that, the function $F_{\gamma_d}(z)$ is convex of order α . Hence, by Lemma 3.1 the proof is completed. \square

By setting $d = 1$ and $\gamma_1 = \nu$ in the Theorem 3.1, we have the following:

Corollary 3.1. *Let $\alpha \in [0, 1)$. If*

$$\alpha \leq \frac{512\nu^3 + 1216\nu^2 + 968\nu + 261}{512\nu^3 + 1088\nu^2 + 712\nu + 133},$$

then the function $z \mapsto F_\nu$ has the following properties:

i. $F_\nu(z) \in \mathcal{H}^{\frac{1}{1-2\alpha}}$ for $\alpha \in [0, \frac{1}{2})$,

ii. $F_\nu(z) \in \mathcal{H}^\infty$ for $\alpha \in (\frac{1}{2}, 1)$.

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