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ON SURFACES CONSTRUCTED BY EVOLUTION ACCORDING TO QUASI FRAME

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Abstract. The present paper presents evolutions of spherical indicatrix of a space curve according to the quasi-frame. Then, some geometric properties of these surfaces constructed by evolutions have been obtained. At the end, illustrative examples of the spherical images of a space curve have been presented.
Keywords: space curve; spherical images; quasi-frame.

1. Introduction

The curves obtained with the help of a given space curve have been studied by many researchers. Bertrand curve pairs, involute-evolute curve pairs and spherical images of a space curve can be given as examples of these curves, [7]. For example, Korpinar [4] investigated the surfaces constructed by the binormal spherical image of a space curve. They derived the time evolution equations for the Frenet frame of binormal spherical image as a curve occurring on the sphere and gave some geometric properties of these surfaces such as fundamental forms and curvatures. The spherical image of the curve moving with time occurs on a sphere. In [6], time evolution equations of a space curve given with the quasi frame are obtained.

In this paper, we have found relations between the motion of curves and the motion of their spherical image. We have obtained the Frenet elements of the spherical images of the curve given with the quasi frame. Then we have derived some geometric properties of the surfaces constructed by the evolution of the spherical images of a space curve. At the end, we have given the illustrative examples of the spherical images of a space curve.

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2. Preliminaries

In this section, we present the Frenet frame and the quasi frame along a space curve which are given by Soliman in [6]. Also, we give some geometric properties for these frames.

Let $r = r(s)$ be a space curve parameterized with arc-length in $\mathbb{R}^3$. The Frenet frame of r consists of the vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ which are given by

$$\begin{aligned} \mathbf{T} &= r'(s), \\ \mathbf{N} &= \frac{r''(s)}{\|r''(s)\|}, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N}, \end{aligned}$$

where $\mathbf{T}$ is the tangent vector, $\mathbf{N}$ is the normal vector and $\mathbf{B}$ is the binormal vector of the curve r.

The curvature κ and the torsion τ are given by

$$\begin{aligned} \kappa &= \|r''(s)\|, \\ \tau &= \frac{\det(r', r'', r''')}{\|r''(s)\|^2}. \end{aligned}$$

The quasi frame of a space curve $r = r(s)$ which is parameterized with arc-length consists of the vectors $\mathbf{T}_q, \mathbf{N}_q, \mathbf{B}_q$. They are given by

$$\begin{aligned} \mathbf{T}_q &= \mathbf{T}, \\ \mathbf{N}_q &= \frac{\mathbf{T} \times \overrightarrow{\mathbf{k}}}{\left\|\mathbf{T} \times \overrightarrow{\mathbf{k}}\right\|}, \\ \mathbf{B}_q &= \mathbf{T} \times \mathbf{N}_q, \end{aligned}$$

where $\overrightarrow{\mathbf{k}}$ is the projection vector which can be chosen as $\overrightarrow{\mathbf{k}} = (1,0,0)$ or $\overrightarrow{\mathbf{k}} = (0,1,0)$ or $\overrightarrow{\mathbf{k}} = (0,0,1)$. In this paper, we choose the projection vector $\overrightarrow{\mathbf{k}} = (0,0,1)$. $\mathbf{N}_q$ and $\mathbf{B}_q$ are called the quasi normal vector and the quasi binormal vector, respectively.

Let θ be the angle between the normal $\mathbf{N}$ and the quasi normal $\mathbf{N}_q$. The quasi formulas are given by, [1],

$$\frac{\partial}{\partial s}\begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix},$$

where k_i are called the quasi curvatures $(1 \leq i \leq 3)$ which are given by

$$\begin{aligned} k_1 &= \kappa \cos\theta = \left\langle \mathbf{T}'_q, \mathbf{N}_q \right\rangle, \\ k_2 &= -\kappa \sin\theta = \left\langle \mathbf{T}'_q, \mathbf{B}_q \right\rangle, \\ k_3 &= \theta' + \tau = -\left\langle \mathbf{N}_q, \mathbf{B}'_q \right\rangle. \end{aligned}$$

3. The Spherical Images of a Space Curve

In this section, we give the representation of the Frenet frame, curvature and torsion for spherical images of the curve in terms of the quasi frame and curvatures of the curve.

Given a space curve r parameterized with arc-length in $\mathbb{R}^3$. Let $\mathbf{T}$ be the unit tangent vector of r. When we take $\overrightarrow{PQ} = \mathbf{T}$; while the moving point P is drawing the curve r, the moving point Q draws a curve on the unit sphere. This curve is called the spherical image of the tangent to the curve r. The spherical image of the normal and the binormal to the curve are defined similarly. Now we give these concepts according to the quasi frame of the curve.

Definition 3.1. Let $r = r(s)$ be a space curve parameterized with arc-length in $\mathbb{R}^3$. The following space curves lie on a unit sphere

$$\begin{aligned} r_1(s) &= \mathbf{T}_q(s), \\ r_2(s) &= \mathbf{N}_q(s), \\ r_3(s) &= \mathbf{B}_q(s) \end{aligned}$$

and they are called the spherical image of the tangent, the quasi normal and the quasi binormal to the curve, respectively.

3.1. Spherical Image of $\mathbf{T}_q$

Let $\{\mathbf{T}_q, \mathbf{N}_q, \mathbf{B}_q\}$ be the quasi frame of the curve $r = r(s)$ parameterized with arc-length and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame of the curve $r_1(s) = \mathbf{T}_q(s)$. The quasi curvatures of the curve r are denoted by k_1, k_2, k_3 and the curvature and the torsion of the curve r_1 are denoted by κ and τ, respectively.

Theorem 3.1. *The Frenet elements of r_1 can be given in terms of the quasi elements of r as follows:*

$$\begin{aligned} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} &= \begin{bmatrix} 0 & \frac{k_1}{\sqrt{k_1^2+k_2^2}} & \frac{k_2}{\sqrt{k_1^2+k_2^2}} \\ \frac{A_1}{\sqrt{U_1}} & \frac{B_1}{\sqrt{U_1}} & \frac{C_1}{\sqrt{U_1}} \\ \frac{K_1}{\sqrt{V_1}} & \frac{L_1}{\sqrt{V_1}} & \frac{M_1}{\sqrt{V_1}} \end{bmatrix} \begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} \\ \kappa &= (1 + \frac{K_1}{(k_1^2+k_2^2)^3})^{\frac{1}{2}}, \\ \tau &= \frac{W_1}{V_1}, \end{aligned}$$

where

$$\begin{aligned}
A_1 &= -\left(k_1^2+k_2^2\right)^2, \\
B_1 &= k_1'k_2^2 - k_1k_2k_2' - k_1^2k_2k_3 - k_2^3k_3, \\
C_1 &= k_1^3k_3 + k_1^2k_2' + k_1k_2^2k_3 - k_1k_1'k_2, \\
K_1 &= k_1^2k_3 + k_2^2k_3 + k_1k_2' - k_1'k_2, \\
L_1 &= -k_2\left(k_1^2+k_2^2\right), \\
M_1 &= k_1\left(k_1^2+k_2^2\right), \\
U_1 &= \left(k_1^2+k_2^2\right)^4 + \left(k_1^2+k_2^2\right)\left(k_1^2k_3 + k_2^2k_3 + k_1k_2' - k_1'k_2\right)^2, \\
V_1 &= \left(k_1^2+k_2^2\right)^3 + \left(k_1^2k_3 + k_2^2k_3 + k_1k_2' - k_1'k_2\right)^2, \\
W_1 &= 3\left(k_1\left(k_1'\right)^2k_2 + k_1'k_2'k_2^2 - k_1^2k_1'k_2' - k_1k_2\left(k_2'\right)^2\right) \\
&\quad + \left(k_1^2+k_2^2\right)\left(k_1k_2'' + k_1^2k_3' + k_2^2k_3' - k_1''k_2 - k_1k_1'k_3 - k_2k_2'k_3\right).
\end{aligned}$$

Proof. By simple calculations one can easily get the first, the second and the third derivatives of r_1 as follows:

$$\begin{aligned}
r_1'(s) &= k_1\mathbf{N}_q + k_2\mathbf{B}_q, \\
r_1''(s) &= -\left(k_1^2+k_2^2\right)\mathbf{T}_q + (k_1' - k_2k_3)\mathbf{N}_q + (k_1k_3 + k_2')\mathbf{B}_q, \\
r_1'''(s) &= -3(k_1k_1' + k_2k_2')\mathbf{T}_q + (k_1'' - 2k_2'k_3 - k_2k_3' - k_1k_3^2 - k_1k_2^2 - k_1^3)\mathbf{N}_q \\
&\quad + (k_2'' + 2k_1'k_3 + k_1k_3' - k_2k_3^2 - k_2k_1^2 - k_2^3)\mathbf{B}_q.
\end{aligned}$$

Then, it is easy to compute the following:

$$\begin{aligned}
\|r_1'\| &= \sqrt{k_1^2+k_2^2}, \\
r_1' \times r_1'' &= K_1\mathbf{T}_q + L_1\mathbf{N}_q + M_1\mathbf{B}_q, \\
\|r_1' \times r_1''\| &= \sqrt{V_1}, \\
\det(r_1', r_1'', r_1''') &= \langle r_1' \times r_1'', r_1''' \rangle = W_1.
\end{aligned}$$

Using the Frenet formulas, one can easily obtain the Frenet elements of r_1 in terms of the quasi elements of r as indicated in the theorem. □

3.2. Spherical Image of $\mathbf{N}_q$

Let $\left\{\mathbf{T}_q, \mathbf{N}_q, \mathbf{B}_q\right\}$ be the quasi frame of the curve $r = r(s)$ parameterized with arc-length and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame of the curve $r_2(s) = \mathbf{N}_q(s)$. The quasi curvatures of the curve r are denoted by k_1, k_2, k_3 and the curvature and the torsion of the curve r_2 are denoted by κ and τ, respectively.

Theorem 3.2. *The Frenet elements of r_2 can be given in terms of the quasi ele-*

ments of r *as follows:*

$$\begin{aligned}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix} &= \begin{bmatrix}\frac{-k_1}{\sqrt{k_1^2+k_3^2}} & 0 & \frac{k_3}{\sqrt{k_1^2+k_3^2}}\\ \frac{A_2}{\sqrt{U_2}} & \frac{B_2}{\sqrt{U_2}} & \frac{C_2}{\sqrt{U_2}}\\ \frac{K_2}{\sqrt{V_2}} & \frac{L_2}{\sqrt{V_2}} & \frac{M_2}{\sqrt{V_2}}\end{bmatrix}\begin{bmatrix}\mathbf{T}_q\\ \mathbf{N}_q\\ \mathbf{B}_q\end{bmatrix}\\ \kappa &= (1+\frac{K_2}{\left(k_1^2+k_3^2\right)^3})^{\frac{1}{2}},\\ \tau &= \frac{W_2}{V_2},\end{aligned}$$

where

$$\begin{aligned}A_2 &= k_1k_3k_3' - k_1^2k_2k_3 - k_1'k_3^2 - k_2k_3^3,\\ B_2 &= -k_1^2\left(k_1^2+k_3^2\right),\\ C_2 &= k_1^2k_3' - k_1k_1'k_3 - k_1^3k_2 - k_1k_2k_3^2-,\\ K_2 &= k_3\left(k_1^2+k_3^2\right),\\ L_2 &= k_1k_3' - k_1'k_3 - k_1^2k_2 - k_2k_3^2,\\ M_2 &= k_1\left(k_1^2+k_3^2\right),\\ U_2 &= \left(k_1^2+k_3^2\right)^4 + \left(k_1^2+k_3^2\right)\left(k_1k_3' - k_1'k_3 - k_1^2k_2 - k_2k_3^2\right)^2,\\ V_2 &= \left(k_1^2+k_3^2\right)^3 + \left(k_1k_3' - k_1'k_3 - k_1^2k_2 - k_2k_3^2\right)^2,\\ W_2 &= 3\left(k_1\left(k_1'\right)^2k_3 + k_1'k_3'k_3^2 - k_1^2k_3'k_3' - k_1k_3\left(k_3'\right)^2\right)\\ &\quad + \left(k_1^2+k_3^2\right)\left(k_1k_3'' - k_1''k_3 - k_2'k_3^2 - k_1^2k_2' + k_2k_3'k_3 + k_1k_1'k_2\right).\end{aligned}$$

Proof. The calculations can be made similar to the proof of the first theorem. □

3.3. Spherical Image of $\mathbf{B}_q$

Let $\left\{\mathbf{T}_q, \mathbf{N}_q, \mathbf{B}_q\right\}$ be the quasi frame of the curve $r = r(s)$ parameterized with arc-length and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame of the curve $r_3(s) = \mathbf{B}_q(s)$. The quasi curvatures of the curve r are denoted by k_1, k_2, k_3 and the curvature and the torsion of the curve r_3 are denoted by κ and τ, respectively.

Theorem 3.3. *The Frenet elements of* r_3 *can be given in terms of the quasi elements of* r *as follows:*

$$\begin{aligned}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix} &= \begin{bmatrix}\frac{-k_2}{\sqrt{k_2^2+k_3^2}} & 0 & \frac{-k_3}{\sqrt{k_2^2+k_3^2}}\\ \frac{A_3}{\sqrt{U_3}} & \frac{B_3}{\sqrt{U_3}} & \frac{C_3}{\sqrt{U_3}}\\ \frac{K_3}{\sqrt{V_3}} & \frac{L_3}{\sqrt{V_3}} & \frac{M_3}{\sqrt{V_3}}\end{bmatrix}\begin{bmatrix}\mathbf{T}_q\\ \mathbf{N}_q\\ \mathbf{B}_q\end{bmatrix}\\ \kappa &= (1+\frac{K_3}{\left(k_2^2+k_3^2\right)^3})^{\frac{1}{2}},\\ \tau &= \frac{W_3}{V_3},\end{aligned}$$

where

$$\begin{aligned}
A_3 &= k_2k_3k_3' + k_1k_2^2k_3 + k_1k_3^3 - k_2'k_3^2,\\
B_3 &= k_2k_2'k_3 - k_1k_2k_3^2 - k_1k_2^3 - k_3'k_2^2,\\
C_3 &= -\left(k_2^2+k_3^2\right)^2,\\
K_3 &= k_3\left(k_2^2+k_3^2\right),\\
L_3 &= -k_2\left(k_2^2+k_3^2\right),\\
M_3 &= k_1^2k_2 + k_1k_3^2 + k_2k_3' - k_2'k_3,\\
U_3 &= \left(k_2^2+k_3^2\right)^4 + \left(k_2^2+k_3^2\right)\left(k_2k_3' - k_2'k_3 + k_1k_2^2 + k_1k_3^2\right)^2,\\
V_3 &= \left(k_2^2+k_3^2\right)^3 + \left(k_2k_3' - k_2'k_3 + k_1k_2^2 + k_1k_3^2\right)^2,\\
W_3 &= 3\left(k_2\left(k_2'\right)^2k_3 + k_2'k_3'k_3^2 - k_2^2k_2'k_3' - k_2k_3\left(k_3'\right)^2\right)\\
&\quad + \left(k_2^2+k_3^2\right)\left(k_1'k_2^2 + k_1'k_3^2 + k_3''k_2 - k_3k_2'' - k_1k_2k_2' - k_1k_3k_3'\right).
\end{aligned}$$

Proof. The calculations can be made similar to the proof of the first theorem. □

An evolving curve can be thought as a collection of curves parameterized by time. This means that each curve in the collection has a space parameter s and a time parameter t, [3]. The following definitions can be given according to quasi frame in $\mathbb{R}^3$ considering references [6] and [7].

$$\frac{\partial}{\partial s}\begin{bmatrix}\mathbf{T}_q\\ \mathbf{N}_q\\ \mathbf{B}_q\end{bmatrix} = \begin{bmatrix}0 & k_1 & k_2\\ -k_1 & 0 & k_3\\ -k_2 & -k_3 & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}_q\\ \mathbf{N}_q\\ \mathbf{B}_q\end{bmatrix}, \tag{3.1}$$

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{T}_q\\ \mathbf{N}_q\\ \mathbf{B}_q\end{bmatrix} = \begin{bmatrix}0 & \lambda & \mu\\ -\lambda & 0 & \nu\\ -\mu & -\nu & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}_q\\ \mathbf{N}_q\\ \mathbf{B}_q\end{bmatrix}. \tag{3.2}$$

Applying the compatibility condition

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{T}_q\\ \mathbf{N}_q\\ \mathbf{B}_q\end{bmatrix} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\begin{bmatrix}\mathbf{T}_q\\ \mathbf{N}_q\\ \mathbf{B}_q\end{bmatrix},$$

in the light of the equations (3.1) and (3.2) one can easily get

$$\begin{bmatrix} 0 & \left(\frac{\partial k_1}{\partial t} - \nu k_2 + \mu k_3 - \frac{\partial \lambda}{\partial s}\right) & \left(\frac{\partial k_2}{\partial t} + \nu k_1 - \lambda k_3 + \frac{\partial \mu}{\partial s}\right)\\ -\left(\frac{\partial k_1}{\partial t} - \nu k_2 + \mu k_3 - \frac{\partial \lambda}{\partial s}\right) & 0 & \left(\frac{\partial k_3}{\partial t} - \mu k_1 + \lambda k_2 - \frac{\partial \nu}{\partial s}\right)\\ -\left(\frac{\partial k_2}{\partial t} + \nu k_1 - \lambda k_3 + \frac{\partial \mu}{\partial s}\right) & -\left(\frac{\partial k_3}{\partial t} - \mu k_1 + \lambda k_2 - \frac{\partial \nu}{\partial s}\right) & 0 \end{bmatrix} = 0_{3\times 3}.$$

Thus, the compatibility condition becomes

$$\begin{aligned}\frac{\partial k_1}{\partial t} &= \nu k_2 - \mu k_3 + \frac{\partial \lambda}{\partial s},\\ \frac{\partial k_2}{\partial t} &= \lambda k_1 - \nu k_3 - \frac{\partial \mu}{\partial s},\\ \frac{\partial k_3}{\partial t} &= \mu k_1 - \lambda k_2 + \frac{\partial \nu}{\partial s}.\end{aligned}$$

4. Surfaces Constructed by the Evolution of the Spherical Images of a Space Curve

In this section, we study the surfaces constructed by the evolution of the spherical image of the tangent, spherical image of the quasi normal and spherical image of the quasi binormal to the curve.

4.1. Surfaces Constructed Using the Spherical Image of the Tangent

The equation of surfaces constructed by the spherical image of the tangent is given by

$$\Psi = \mathbf{T}_q(s,t).$$

Theorem 4.1. *Under the assumption $\mu k_1 - \lambda k_2 > 0$, the Gaussian curvature K_1, the mean curvature H_1 and the principal curvatures k_{11} and k_{21} of Ψ are given by*

$$K_1 = 1,\ H_1 = -1,\ k_{11} = -1,\ k_{21} = -1.$$

Proof. The tangent space to the surface is spanned by

$$\begin{aligned}\Psi_s &= k_1\mathbf{N}_q + k_2\mathbf{B}_q,\\ \Psi_t &= \lambda\mathbf{N}_q + \mu\mathbf{B}_q,\end{aligned} \tag{4.1}$$

where the lower indices show partial differentiation. Then the unit normal to Ψ is given by

$$\mathbf{N}_\Psi = \frac{\Psi_s \times \Psi_t}{\|\Psi_s \times \Psi_t\|} = \mathbf{T}_q.$$

Using the equations (3.1), (3.2) and (4.1), the second order derivatives are calculated and given by

$$\begin{aligned}\Psi_{ss} &= -\left(k_1^2 + k_2^2\right)\mathbf{T}_q + \left((k_1)_s - k_2k_3\right)\mathbf{N}_q + \left((k_2)_s + k_1k_3\right)\mathbf{B}_q,\\ \Psi_{tt} &= -\left(\lambda^2 + \mu^2\right)\mathbf{T}_q + \left(\lambda_t - \mu\nu\right)\mathbf{N}_q + (\mu_t + \lambda\nu)\mathbf{B}_q,\\ \Psi_{st} &= -(\lambda k_1 + \mu k_2)\mathbf{T}_q + \left(\lambda_s - \mu k_3\right)\mathbf{N}_q + (\lambda k_3 - \mu_s)\mathbf{B}_q.\end{aligned}$$

The components of the first fundamental form $g_{ij}, (1 \le i,j \le 2)$ are obtained as follows:

$$\begin{aligned}g_{11} &= \langle\Psi_s, \Psi_s\rangle = k_1^2 + k_2^2,\\ g_{12} &= \langle\Psi_s, \Psi_t\rangle = \lambda k_1 + \mu k_2,\\ g_{22} &= \langle\Psi_t, \Psi_t\rangle = \lambda^2 + \mu^2.\end{aligned}$$

The components of the second fundamental form $l_{ij}, (1 \leq i, j \leq 2)$ are obtained as follows:

$$\begin{aligned} l_{11} &= \langle \Psi_{ss}, \mathbf{N}_\Psi \rangle = -\left(k_1^2 + k_2^2\right), \\ l_{12} &= \langle \Psi_{st}, \mathbf{N}_\Psi \rangle = -\left(\lambda k_1 + \mu k_2\right), \\ l_{22} &= \langle \Psi_{tt}, \mathbf{N}_\Psi \rangle = -\left(\lambda^2 + \mu^2\right). \end{aligned}$$

Thus, we get the following equalities:

$$\begin{aligned} K_1 &= \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} = 1, \\ H_1 &= \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2\left(g_{11}g_{22} - g_{12}^2\right)} = -1, \\ k_{11} &= H_1 + \sqrt{H_1^2 - K_1} = -1, \\ k_{21} &= H_1 - \sqrt{H_1^2 - K_1} = -1. \end{aligned}$$

□

4.2. Surfaces Constructed Using the Spherical Image of the Quasi Normal

The equation of surfaces constructed by the spherical image of the quasi normal is given by

$$\phi = \mathbf{N}_q(s,t).$$

Theorem 4.2. *Under the assumption* $\nu k_1 - \lambda k_3 > 0$, *the Gaussian curvature* K_2, *the mean curvature* H_2 *and the principal curvatures* k_{12} *and* k_{22} *of* ϕ *are given by*

$$K_2 = 1,\ H_2 = -1,\ k_{12} = -1,\ k_{22} = -1.$$

Proof. The tangent space to the surface is spanned by

$$\begin{aligned} \phi_s &= -k_1\mathbf{T}_q + k_3\mathbf{B}_q, \\ \phi_t &= -\lambda\mathbf{T}_q + \nu\mathbf{B}_q, \end{aligned} \tag{4.2}$$

where the lower indices show partial differentiation. Then the unit normal to ϕ is given by

$$\mathbf{N}_\phi = \frac{\phi_s \times \phi_t}{\|\phi_s \times \phi_t\|} = \mathbf{N}_q.$$

Using the equations (3.1), (3.2) and (4.2), the second order derivatives are calculated and given by

$$\begin{aligned} \phi_{ss} &= -\left((k_1)_s + k_2k_3\right)\mathbf{T}_q - \left(k_1^2 + k_3^2\right)\mathbf{N}_q + \left((k_3)_s - k_1k_2\right)\mathbf{B}_q, \\ \phi_{tt} &= -\left(\lambda_t + \mu\nu\right)\mathbf{T}_q - \left(\lambda^2 + \nu^2\right)\mathbf{N}_q + (\nu_t - \lambda\mu)\mathbf{B}_q, \\ \phi_{st} &= -\left(\lambda_s + \nu k_2\right)\mathbf{T}_q - \left(\lambda k_1 + \nu k_3\right)\mathbf{N}_q + (\nu_s - \lambda k_2)\mathbf{B}_q. \end{aligned}$$

The components of the first fundamental form $g_{ij}, (1 \leq i, j \leq 2)$ are obtained as follows:

$$\begin{aligned} g_{11} &= \langle \phi_s, \phi_s \rangle = k_1^2 + k_3^2, \\ g_{12} &= \langle \phi_s, \phi_t \rangle = \lambda k_1 + \nu k_3, \\ g_{22} &= \langle \phi_t, \phi_t \rangle = \lambda^2 + \nu^2. \end{aligned}$$

The components of the second fundamental form $l_{ij}, (1 \leq i, j \leq 2)$ are obtained as follows:

$$\begin{aligned} l_{11} &= \langle \phi_{ss}, \mathbf{N}_\phi \rangle = -\left(k_1^2 + k_3^2\right), \\ l_{12} &= \langle \phi_{st}, \mathbf{N}_\phi \rangle = -\left(\lambda k_1 + \nu k_3\right), \\ l_{22} &= \langle \phi_{tt}, \mathbf{N}_\phi \rangle = -\left(\lambda^2 + \nu^2\right). \end{aligned}$$

Thus, we get the following equalities:

$$\begin{aligned} K_2 &= \frac{l_{11} l_{22} - l_{12}^2}{g_{11} g_{22} - g_{12}^2} = 1, \\ H_2 &= \frac{l_{11} g_{22} - 2 l_{12} g_{12} + l_{22} g_{11}}{2\left(g_{11} g_{22} - g_{12}^2\right)} = -1, \\ k_{12} &= H_2 + \sqrt{H_2^2 - K_2} = -1, \\ k_{22} &= H_2 - \sqrt{H_2^2 - K_2} = -1. \end{aligned}$$

□

4.3. Surfaces Constructed Using the Spherical Image of the Quasi Binormal

The equation of surfaces constructed by the spherical image of the quasi binormal is given by

$$\varphi = \mathbf{B}_q(s, t).$$

Theorem 4.3. *Under the assumption $\nu k_2 - \mu k_3 > 0$, the Gaussian curvature K_3, the mean curvature H_3 and the principal curvatures k_{13} and k_{23} of φ are given by*

$$K_3 = 1, \; H_3 = -1, \; k_{13} = -1, \; k_{23} = -1.$$

Proof. The tangent space to the surface is spanned by

$$\begin{aligned} \varphi_s &= -k_2 \mathbf{T}_q - k_3 \mathbf{N}_q, \\ \varphi_t &= -\mu \mathbf{T}_q - \nu \mathbf{N}_q, \end{aligned} \tag{4.3}$$

where the lower indices show partial differentiation. Then the unit normal to φ is given by

$$\mathbf{N}_\varphi = \frac{\varphi_s \times \varphi_t}{\|\varphi_s \times \varphi_t\|} = \mathbf{B}_q.$$

Using the equations (3.1), (3.2) and (4.3), the second order derivatives are calculated and given by

$$\begin{aligned}
\varphi_{ss} &= \left(k_1 k_3 - (k_2)_s\right)\mathbf{T}_q - \left((k_3)_s + k_1 k_2\right)\mathbf{N}_q - \left(k_2^2 + k_3^2\right)\mathbf{B}_q, \\
\varphi_{tt} &= \left(\lambda\nu - \mu_t\right)\mathbf{T}_q - (\nu_t + \lambda\mu)\mathbf{N}_q - \left(\mu^2 + \nu^2\right)\mathbf{B}_q, \\
\varphi_{st} &= \left(\mu_s + \nu k_1\right)\mathbf{T}_q - (\nu_s - \mu k_1)\mathbf{N}_q - (\mu k_2 + \nu k_3)\mathbf{B}_q.
\end{aligned}$$

The components of the first fundamental form $g_{ij}, (1 \leq i, j \leq 2)$ are obtained as follows:

$$\begin{aligned}
g_{11} &= \langle \varphi_s, \varphi_s \rangle = k_2^2 + k_3^2, \\
g_{12} &= \langle \varphi_s, \varphi_t \rangle = \mu k_2 + \nu k_3, \\
g_{22} &= \langle \varphi_t, \varphi_t \rangle = \mu^2 + \nu^2.
\end{aligned}$$

The components of the second fundamental form $l_{ij}, (1 \leq i, j \leq 2)$ are obtained as follows:

$$\begin{aligned}
l_{11} &= \langle \varphi_{ss}, \mathbf{N}_\varphi \rangle = -\left(k_2^2 + k_3^2\right), \\
l_{12} &= \langle \varphi_{st}, \mathbf{N}_\varphi \rangle = -\left(\mu k_2 + \nu k_3\right), \\
l_{22} &= \langle \varphi_{tt}, \mathbf{N}_\varphi \rangle = -\left(\mu^2 + \nu^2\right).
\end{aligned}$$

Thus, we get the following equalities:

$$\begin{aligned}
K_3 &= \frac{l_{11} l_{22} - l_{12}^2}{g_{11} g_{22} - g_{12}^2} = 1, \\
H_3 &= \frac{l_{11} g_{22} - 2 l_{12} g_{12} + l_{22} g_{11}}{2\left(g_{11} g_{22} - g_{12}^2\right)} = -1, \\
k_{13} &= H_3 + \sqrt{H_3^2 - K_3} = -1, \\
k_{23} &= H_3 - \sqrt{H_3^2 - K_3} = -1.
\end{aligned}$$

□

5. Examples

We give two illustrative examples to the spherical images of a regular space curve according to quasi frame.

Example 5.1. Let us consider the space curve α which is defined by

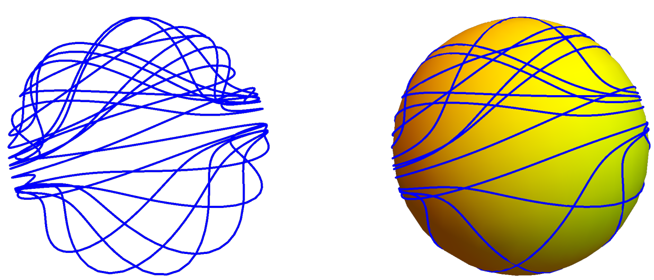
$$\begin{aligned}\alpha &: \mathbb{R} \longrightarrow \mathbb{R}^3,\\ \alpha(t) &= ((2+\cos t+\sin t)\sin t\cos(\sin(10t)),\\ &\quad (2+\cos t+\sin t)\sin t\sin(\sin(10t)),\\ &\quad (2+\cos t+\sin t)\cos t).\end{aligned}$$

Calculating the first derivative of α, one can easily see that

$$\|\alpha'(t)\| \neq 0$$

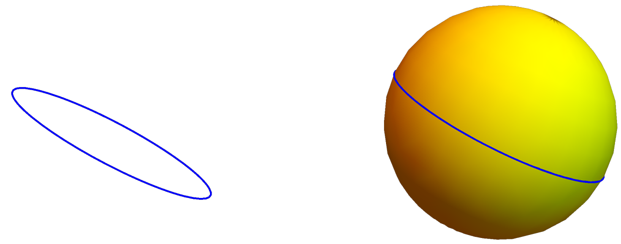
for all $t \in \mathbb{R}$. So we can say that α is a regular space curve. In the light of the quasi formulas, one can easily obtain the quasi frame $\{\mathbf{T}_q, \mathbf{N}_q, \mathbf{B}_q\}$ of α. The graphics of the curve α and its spherical images are given below.

FIG. 5.1: The curve α



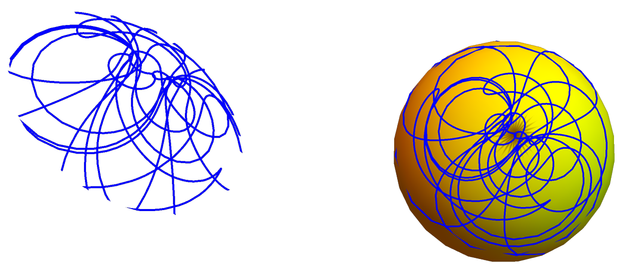
(a) (b)

FIG. 5.2: The spherical image of tangent of curve α



(a) (b)

Fig. 5.3: The spherical image of the quasi-normal of curve α



(a) (b)

Fig. 5.4: The spherical image of the quasi-binormal of curve α

Example 5.2. Let us consider the space curve β which is defined in [5] by

$$\begin{aligned} \beta &: \mathbb{R} \longrightarrow \mathbb{R}^3, \\ \beta(t) &= (-\frac{18}{5}\sin(-\frac{t}{4}) + \frac{2}{45}\sin(\frac{9t}{4}), -\frac{18}{5}\cos(-\frac{t}{4}) + \frac{2}{45}\cos(\frac{9t}{4}), \frac{3}{5}\cos t). \end{aligned}$$

Calculating the first derivative of β, one can easily see that

$$\left\|\beta'(t)\right\| \neq 0$$

for all $t \in \mathbb{R}$. So we can say that β is a regular space curve. In the light of the quasi formulas, one can easily obtain the quasi frame $\left\{\overline{\mathbf{T}}_q, \overline{\mathbf{N}}_q, \overline{\mathbf{B}}_q\right\}$ of β. The graphics of the curve β and its spherical images are given below.

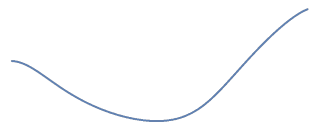
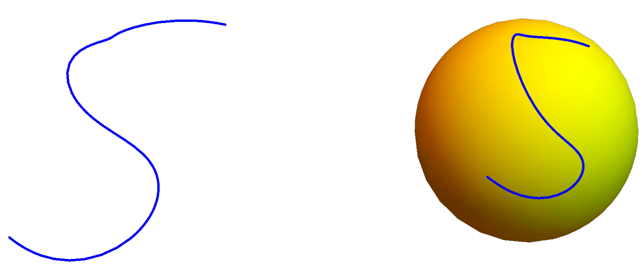
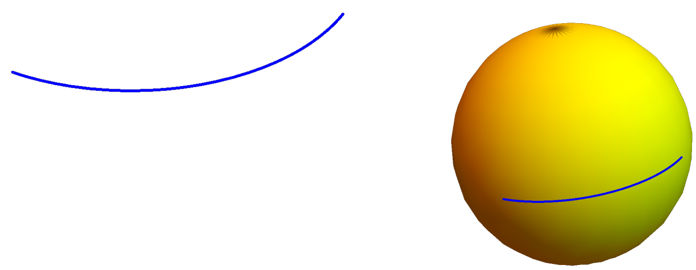


FIG. 5.5: The curve β



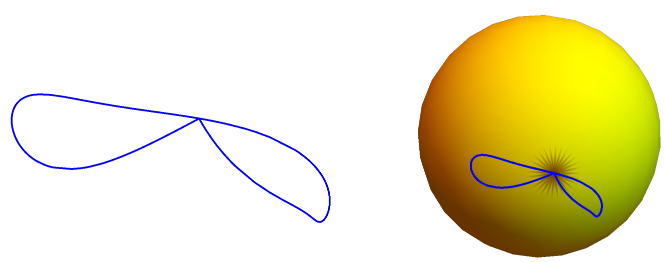
(a) (b)

FIG. 5.6: The spherical image of tangent of curve β



(a) (b)

FIG. 5.7: The spherical image of the quasi-normal of curve β



(a) (b)

FIG. 5.8: The spherical image of the quasi-binormal of curve β

REFERENCES

1. M. DEDE, C. EKICI and A. GORGULU: *Directional q-frame along a space curve*. Int Jour Adv Res Comp Scie Soft Eng IJARCSSE. **5(12)** (2015), 775-780.

2. M. Dede, C. Ekici and H. Tozak: *Directional tubular surfaces*. International Journal of Algebra. **9(12)** (2015), 527-535.
3. R. A. Hussien and G. M. Samah: *Generated Surfaces via Inextensible Flows of Curves*. Journal of Applied Mathematics. (2016).
4. T. Korpinar and E. Turhan: *Time Evolution Equations for Surfaces Generated via Binormal Spherical Image in Terms of Inextensible Flows*. Journal of Dynamical Systems and Geometric Theories. **12(2)** (2014), 145-157.
5. L. Kula, N. Ekmekci, Y. Yayli and K. Ilarslan: *Characterizations of Slant Helices in Euclidean 3-Space*. Turk. J. Math.. **34(2)** (2010), 261–273.
6. M. A. Soliman: *Evolutions of the Ruled Surfaces via the Evolution of Their Directrix Using Quasi Frame along a Space Curve*. Journal of Applied Mathematics and Physics. (2018).
7. M. A. Soliman, N. H. Abdel-All, R. A. Hussien and T. Y. Shaker: *Surfaces Generated via the Evolution of Spherical Image of a Space Curve*. Korean Journal of Mathematics. **26(3)** (2018), 425-437.

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