FACTA UNIVERSITATIS (NIŠ)
SER. MATH. INFORM. Vol. 35, No 2 (2020), 437–458
https://doi.org/10.22190/FUMI2002437D

NEW APPROACHES ON DUAL SPACE

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Abstract. In this paper, we have explained how to define the basic concepts of differential geometry on Dual space. To support this, dual tangent vectors that have $\overline{p}$ as dual point of application have been defined. Then, the dual analytic functions defined by Dimentberg have been examined in detail, and by using the derivative of the these functions, dual directional derivatives and dual tangent maps have been introduced.
Keywords: Dual space; dual tangent vectors; dual analytic functions; tangent maps.

1. Introduction and Basic Concepts

Sir Isaac Newton invented calculus in about 1665. The solution to the problem which he was interested in was too difficult for mathematics used in that time. For this reason, he found a new approach to mathematics. Also, he tried to compute the velocity of n object at any instant. Nowadays, many scientists tend to calculate the rate at which satellite's position changes according to time. A comparison of the change in one quantity to the simultaneous change in a second quantity is known as a rate of change. If both changes emerge in the course of an infinitely short period of time, the rate is called instantaneous. Then, the derivatives are important to the solution of the problems in calculus. Calculus has application fields in physics and engineering [1].

Dual numbers were defined by W. K. Clifford [3] (1845-1879) as a tool of his geometrical studies. Their first applications were given by Kotelnikov [9] and Study [13]. Dual variable functions were introduced by Dimentberg [4]. He investigated the analytic conditions of these functions, and by means of conditions, he described the derivative concept of these functions. In 1999, by using these dual analytic functions, Brodsky *et al.* [2] showed that the derivatives of products of two dual analytic functions with respect to dual variables are equal to moment-product derivative.

Received March 06, 2019; accepted April 07, 2019
2010 *Mathematics Subject Classification.* Primary 47L50; Secondary 53A40, 53A35

In recent years, dual numbers have been widely used in kinematics, dynamics, mechanism design, and field and group theories ([5], [6], [7], [8] and [12]). For example in kinematics, constraint manifolds of spatial mechanisms are explained using dual numbers system [10]. The aim of this study is to calculate the derivative of dual analytic functions with respect to dual vectors, by expanding the definition of the derivative in dual analytic function. After then, by using this derivative concept and dual analytic functions, the authors showed how to define vector fields and tangent maps on Dual space. These concepts will give us a new perspective in Dual space.

This paper is organized in the following way: In section II, the dual analytic functions defined by Dimentberg are introduced, and by using these functions, the partial derivatives of the functions $\overline{f} : D^n \to D$ are calculated.

In section III, dual tangent vectors are introduced, and the derivative of $\overline{f}$ with respect to dual tangent vectors is computed. For $1 \leq i \leq n$, it is shown that partial derivatives calculated in the second part is the derivative of $\overline{f}$ with respect to vectors e_i, where $e_i = (\delta_{i1}, ..., \delta_{in})$. Here, δ_{ij} is the Kronecker delta (0 if $i \neq j$, 1 if $i = j$).

In section IV, dual vector fields are introduced, and in the last section, the dual tangent map that sends the dual tangent vectors at dual point $\overline{p}$ to the dual tangent vectors at dual point $\overline{f}(\overline{p})$ is defined.

Now, we recall a brief summary of the theory of dual numbers and the fundamental concepts of Differential Geometry.

Let the set $\mathbb{R} \times \mathbb{R}$ be shown as D. On the set $D = \{\overline{x} = (x, x^*) \mid x, x^* \in \mathbb{R}\}$, two operators and equality are defined as follows.

$$\begin{aligned} \overline{x} \oplus \overline{y} &= (x + y, x^* + y^*), \\ \overline{x} \odot \overline{y} &= (xy, x^* y + x y^*), \\ \overline{x} &= \overline{y} \Longleftrightarrow x = y,\ x^* = y^*. \end{aligned}$$

The set D is called the dual numbers system and $(x, x^*) \in D$ is called a dual number. The dual numbers $(1, 0) = 1$ and $(0, 1) = \varepsilon$ are called the unit element of multiplication operation in D, and dual unit which satisfies the condition that $\varepsilon^2 = 0$, respectively. Also, the dual number $\overline{x} = (x, x^*)$ can be written as $\overline{x} = x + \varepsilon x^*$, and the set of all dual numbers is shown by

$$D = \left\{\overline{x} = x + \varepsilon x^* \mid x, x^* \in \mathbb{R},\ \varepsilon^2 = 0\right\}.$$

The set of

$$D^3 = \{\overline{v} = (\overline{v}_1, \overline{v}_2, \overline{v}_3) \mid \overline{v}_i \in D,\ 1 \leq i \leq 3\}$$

gives all triples of dual numbers. The element of D^3 is called as dual vectors and a dual vector can be written in the following form

$$\overline{v} = \overrightarrow{v} + \varepsilon \overrightarrow{v}^*,$$

where $\overrightarrow{v}$ and $\overrightarrow{v}^*$ are the vectors of $\mathbb{R}^3$. The addition and multiplication operations on D^3 are as below:

$$\begin{aligned} \overline{v}+\overline{w} &= \overrightarrow{v}+\overrightarrow{w}+\varepsilon\left(\overrightarrow{v}^*+\overrightarrow{w}^*\right), \\ \overline{\lambda}\overline{v} &= \lambda\overrightarrow{v}+\varepsilon\left(\lambda\overrightarrow{v}^*+\lambda^*\overrightarrow{v}\right), \end{aligned}$$

where $\overline{v}=\overrightarrow{v}+\varepsilon\overrightarrow{v}^*, \overline{w}=\overrightarrow{w}+\varepsilon\overrightarrow{w}^* \in D^3$ and $\overline{\lambda}=\lambda+\varepsilon\lambda^* \in D$. The set D^3 is a module over the ring D, and is called D-module or dual space.

The set of dual vectors on D^n is represented by

$$D^n=\{\overline{v}=(\overline{v}_1,...,\overline{v}_n) \mid \overline{v}_i \in D,\ 1\leq i\leq n\}.$$

These vectors can be given in the form

$$\overline{v}=\overrightarrow{v}+\varepsilon\overrightarrow{v}^*,$$

where $\overrightarrow{v}$ and $\overrightarrow{v}^*$ are the vectors of $\mathbb{R}^n$. On this set, the addition and multiplication are given as follows

$$\begin{aligned} \overline{v}+\overline{w} &= \overrightarrow{v}+\overrightarrow{w}+\varepsilon\left(\overrightarrow{v}^*+\overrightarrow{w}^*\right), \\ \overline{\lambda}\overline{v} &= \lambda\overrightarrow{v}+\varepsilon\left(\lambda\overrightarrow{v}^*+\lambda^*\overrightarrow{v}\right). \end{aligned}$$

The set D^n is a module over the ring D. On the other hand, since $\overrightarrow{v}$ and $\overrightarrow{v}^*$ are the vectors of $\mathbb{R}^n$, we can write the equalities below

$$\overrightarrow{v}=v_1e_1+...+v_ne_n,$$

and

$$\overrightarrow{v}^*=v_1^*e_1+...+v_n^*e_n,$$

where $e_i=(\delta_{i1},...,\delta_{in})$ for $1\leq i\leq n$. Thus, we have

$$\begin{aligned} \overline{v} &= \overrightarrow{v}+\varepsilon\overrightarrow{v}^* \\ &= v_1e_1+...+v_ne_n+\varepsilon\left(v_1^*e_1+...+v_n^*e_n\right) \\ &= \left(v_1+\varepsilon v_1^*\right)e_1+...+\left(v_n+\varepsilon v_n^*\right)e_n \\ &= \overline{v}_1e_1+...+\overline{v}_ne_n. \end{aligned}$$

For $1\leq i\leq n$, let $x_i:\mathbb{R}^n\to\mathbb{R}$ be the function that sends each point $p=(p_1,...,p_n)$ to its ith coordinate p_i. Then $x_1,...,x_n$ are the natural coordinate functions of $\mathbb{R}^n$. On the set

$$T_p\mathbb{R}^n=\{p\}\times\mathbb{R}^n=\{(p,\overrightarrow{v}) \mid \overrightarrow{v}\in\mathbb{R}^n\},$$

addition and scalar product operators are defined as follows, respectively.

$+:T_p\mathbb{R}^n\times T_p\mathbb{R}^n\to T_p\mathbb{R}^n$, for $(p,\overrightarrow{v}),\ (p,\overrightarrow{w})$ defined as

$$(p,\overrightarrow{v})+(p,\overrightarrow{w})=(p,\overrightarrow{v}+\overrightarrow{w}).$$

$\cdot : \mathbb{R} \times T_p\mathbb{R}^n \to T_p\mathbb{R}^n$, for $\lambda \in \mathbb{R}$ and $(p, \overrightarrow{v})$ defined as

$$\lambda\,(p, \overrightarrow{v}) = (p, \lambda \overrightarrow{v})\,.$$

In this case, the set $(T_p\mathbb{R}^n, +, (\mathbb{R}, +, \cdot), \cdot)$ is a vector space otherwise known as a tangent space. The element $\overrightarrow{v}_p = (p, \overrightarrow{v})$ is called a tangent vector to $\mathbb{R}^n$ at p.

A real-valued function of f on $\mathbb{R}^n$ is differentiable proved all mixed partial derivatives of f exist and are continuous.

For $1 \leq j \leq n$, if the functions $f_j : \mathbb{R}^n \to \mathbb{R}$ are differentiable, then the function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable.

Let f be a differentiable real-valued function on $\mathbb{R}^n$. Gradient of the function f is defined as

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right).$$

Definition 1.1. Let f be a differentiable real-valued function on $\mathbb{R}^n$ and $\overrightarrow{v}_p$ be a tangent vector to $\mathbb{R}^n$. Then, the number

$$\overrightarrow{v}_p\,[f] = \frac{d}{dt} f\,(p + t\overrightarrow{v})\,|_{t=0}$$

is called the derivative of f with respect to $\overrightarrow{v}_p$.

A vector field is a function that assigns to each point p of $\mathbb{R}^n$ a tangent vector $\overrightarrow{v}_p$ to $\mathbb{R}^n$ at p.

Definition 1.2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. For every $p \in \mathbb{R}^n$, the function $f_{*p} : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ is defined as follows:

$$f_{*p}\,(\overrightarrow{v}_p) = (\overrightarrow{v}_p\,[f_1], ..., \overrightarrow{v}_p\,[f_m])\,|_{f(p)}\,.$$

This function is called tangent map of f.

For the vectors $\overrightarrow{v} = (v_1, ..., v_n)$ and $\overrightarrow{w} = (w_1, ..., w_n)$, the inner product on $\mathbb{R}^n$ is given by

$$\overrightarrow{v} \cdot \overrightarrow{w} = v_1 w_1 + ... + v_n w_n.$$

For more details, we refer the readers to [11].

2. Derivative of Dual Analytic Functions

Let $\overline{x} = x + \varepsilon x^*$ be a dual number. A dual variable function $\overline{f} : D \to D$ is defined as follows:

$$\overline{f}\,(\overline{x}) = f\,(x, x^*) + \varepsilon f^o\,(x, x^*)\,,$$

where f and f^o are real functions with two real variables x and x^*. Dimentberg comprehensively investigated the properties of dual functions. He showed that the analytic conditions of dual functions are

$$\frac{\partial f}{\partial x^*} = 0 \text{ and } \frac{\partial f^o}{\partial x^*} = \frac{\partial f}{\partial x}. \tag{2.1}$$

From the above first condition, the function f is a function which has only variable x, i.e.,

$$f(x, x^*) = f(x)$$

and the second implies that the function f^o is as below expression

$$f^o(x, x^*) = x^* \frac{\partial f}{\partial x} + \widetilde{f}(x),$$

where $\widetilde{f}(x)$ is a certain function of x. General notation of dual analytic function is given by following equality

$$\overline{f}(\overline{x}) = \overline{f}(x + \varepsilon x^*) = f(x) + \varepsilon \left(x^* \frac{df}{dx} + \widetilde{f}(x) \right). \tag{2.2}$$

For $x^* = 0$, the function must be written in the form

$$\overline{f}(\overline{x}) = \overline{f}(x + \varepsilon x^*) = f(x) + \varepsilon \widetilde{f}(x).$$

The derivative of the dual analytic function $\overline{f}$ is defined by

$$\begin{aligned} \frac{d\overline{f}}{d\overline{x}} &= \frac{df}{dx} + \varepsilon \frac{d}{dx} \left(x^* \frac{df}{dx} + \widetilde{f}(x) \right) \\ &= \frac{df}{dx} + \varepsilon \left(x^* \frac{d^2 f}{dx^2} + \frac{d\widetilde{f}}{dx} \right). \end{aligned} \tag{2.3}$$

It is seen that the derivative of the function $\overline{f}$ with respect to dual variable $\overline{x}$ is equal to the derivative with respect to real variable x [4]. Now, we shall study dual analytic functions $\overline{f} : D^n \to D$, i.e.,

$$\overline{f}(\overline{x}) = \overline{f}(x + \varepsilon x^*) = f(x_1, ..., x_n, x_1^*, ..., x_n^*) + \varepsilon f^o(x_1, ..., x_n, x_1^*, ..., x_n^*),$$

where $x = (x_1, ..., x_n)$ and $x^* = (x_1^*, ..., x_n^*)$. Using the above equalities (2.1), the analytic conditions of this function can be given

$$\frac{\partial f}{\partial x_i^*} = 0 \text{ and } \frac{\partial f^o}{\partial x_i^*} = \frac{\partial f}{\partial x_i}, \qquad (1 \le i \le n).$$

In that case, general expression of the dual analytic functions is defined as follows:

$$\overline{f}(\overline{x}) = f(x_1, ..., x_n) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial f}{\partial x_i} + \widetilde{f}(x_1, ..., x_n) \right).$$

If the equality (2.2) is used, then the partial derivatives of these dual analytic functions are given by

$$\frac{\partial \overline{f}}{\partial \overline{x}_j} = \frac{\partial f}{\partial x_j} + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial \widetilde{f}}{\partial x_j} \right),$$

where $1 \leq j \leq n$. Similarly, the partial derivatives of the function $\overline{f}$ according to dual variables $\overline{x}_j$ are reduced to the partial derivatives according to real variables x_j. For the general dual functions $\overline{f} : D^n \to D^m$, if the functions

$$\overline{f}_k : D^n \to D, \quad (1 \leq k \leq m)$$

are dual analytic functions, then the dual function $\overline{f}$ is a dual analytic function, and the set of the dual analytic functions is shown by

$$C\left(D^n, D^m\right) = \left\{ \overline{f} \mid \overline{f} : D^n \to D^m \text{ is a dual analytic function} \right\}.$$

For the dual-valued analytic functions on D^n, the following equalities can be defined

$$\begin{aligned} \left(\overline{f} + \overline{g}\right)(\overline{x}) &= \overline{f}(\overline{x}) + \overline{g}(\overline{x}) \\ &= f(x) + g(x) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \left(\frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i} \right) + \widetilde{f}(x) + \widetilde{g}(x) \right), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \left(\overline{\lambda f}\right)(\overline{x}) &= \overline{\lambda f}(\overline{x}) \\ &= \lambda f(x) + \varepsilon \left(\lambda \left(\sum_{i=1}^{n} x_i^* \frac{\partial f}{\partial x_i} \right) + \lambda^* f(x) + \lambda \widetilde{f}(x) \right) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \left(\overline{f} \cdot \overline{g}\right)(\overline{x}) &= \overline{f}(\overline{x}) \cdot \overline{g}(\overline{x}) \\ &= f(x) g(x) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \left(\frac{\partial (fg)}{\partial x_i} \right) + f(x) \widetilde{g}(x) + g(x) \widetilde{f}(x) \right), \end{aligned} \tag{2.6}$$

where $\overline{x} = x + \varepsilon x^* = (x_1, ..., x_n) + \varepsilon (x_1^*, ..., x_n^*)$. It is clear that the above equations are the dual analytic functions.

Let $\overline{p} = p + \varepsilon p^*$ be a dual point of D^n, and $\overline{v} = \overrightarrow{v} + \varepsilon \overrightarrow{v}^*$ be a dual vector to D^n. The equation of dual straight line is given by

$$\begin{aligned} \overline{\alpha}\left(\overline{t}\right) &= p + t\overrightarrow{v} + \varepsilon \left(t^* \overrightarrow{v} + p^* + t\overrightarrow{v}^* \right) \\ &= \alpha(t) + \varepsilon \left(t^* \alpha'(t) + \widetilde{\alpha}(t) \right). \end{aligned} \tag{2.7}$$

It is seen that the equality (2.7) is a dual analytic function.

Definition 2.1. Let $x_1, ..., x_n, x_1^*, ..., x_n^*$ be coordinate functions of $\mathbb{R}^{2n}$. For $1 \leq i \leq n$, these functions from $\mathbb{R}^{2n}$ to $\mathbb{R}$ are given as follows:

$$x_i\left(\widetilde{p}\right) = p_i, \quad x_i^*\left(\widetilde{p}\right) = p_i^*,$$

where $\widetilde{p} = (p_1, ..., p_n, p_1^*, ..., p_n^*)$ is a point of $\mathbb{R}^{2n}$. In this case, dual coordinate functions $\overline{x}_i : D^n \to D$ are defined by

$$\begin{aligned} \overline{x}_i\left(\overline{p}\right) &= x_i\left(\widetilde{p}\right) + \varepsilon x_i^*\left(\widetilde{p}\right) \\ &= p_i + \varepsilon p_i^* \\ &= \overline{p}_i, \end{aligned}$$

where $\overline{p} = (p_1 + \varepsilon p_1^*, ..., p_n + \varepsilon p_n^*) = (p_1, ..., p_n) + \varepsilon\left(p_1^*, ..., p_n^*\right) = p + \varepsilon p^*$ is a point of D^n.

The above definition shows how to implement the dual point in the dual analytic functions. For example, for the dual-valued analytic functions on D^n, the following equalities can be written

$$\begin{aligned} \overline{f}\left(\overline{p}\right) &= f\left(\widetilde{p}\right) + \varepsilon\left(\sum_{i=1}^{n} p_i^* \frac{\partial f}{\partial x_i}\left(\widetilde{p}\right) + \widetilde{f}\left(\widetilde{p}\right)\right) \\ &= f\left(\widetilde{p}\right) + \varepsilon f^o\left(\widetilde{p}\right), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \overline{f}}{\partial \overline{x}_j}\left(\overline{p}\right) &= \frac{\partial f}{\partial x_j}\left(\widetilde{p}\right) + \varepsilon\left(\sum_{i=1}^{n} x_i^* \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial \widetilde{f}}{\partial x_j}\right)\left(\widetilde{p}\right) \\ &= \frac{\partial f}{\partial x_j}\left(\widetilde{p}\right) + \varepsilon\left(\sum_{i=1}^{n} p_i^* \frac{\partial^2 f}{\partial x_i \partial x_j}\left(\widetilde{p}\right) + \frac{\partial \widetilde{f}}{\partial x_j}\left(\widetilde{p}\right)\right) \\ &= \frac{\partial f}{\partial x_j}\left(\widetilde{p}\right) + \varepsilon \frac{\partial f^o}{\partial x_j}\left(\widetilde{p}\right). \end{aligned}$$

Definition 2.2. Let $\overline{f}$ and $\overline{g}$ be dual-valued analytic functions on D. Composition of the dual analytic functions $\overline{f}$ and $\overline{g}$ is determined by

$$\begin{aligned} \overline{f} \circ \overline{g} &: D \to D \\ \left(\overline{f} \circ \overline{g}\right)\left(\overline{x}\right) &= \overline{f}\left(\overline{g}\left(\overline{x}\right)\right), \end{aligned} \tag{2.8}$$

where

$$\overline{f}\left(\overline{g}\left(\overline{x}\right)\right) = \left(f \circ g\right)(x) + \varepsilon\left(x^*\left(f \circ g\right)'(x) + \widetilde{g}\,(x)\left(f' \circ g\right)(x) + \left(\widetilde{f} \circ g\right)(x)\right). \tag{2.9}$$

If $(f \circ g)(x) = h(x)$ and $\widetilde{g}(x)(f' \circ g)(x) + \left(\widetilde{f} \circ g\right)(x) = \widetilde{h}(x)$ are taken,

$$\left(\overline{f} \circ \overline{g}\right)(\overline{x}) = h(x) + \varepsilon\left(x^* h'(x) + \widetilde{h}(x)\right)$$

can be written. This formula demonstrates that the dual function $\overline{f} \circ \overline{g}$ is a dual analytic function. The explanation on how to calculate the derivative of this function is given in the following theorem.

Theorem 2.1. *Let $\overline{f}$ and $\overline{g}$ be dual-valued analytic functions on D. The derivative of the dual analytic composite function is given by*

$$\frac{d}{d\overline{x}}\left(\overline{f} \circ \overline{g}\right)(\overline{x}) = \frac{d\overline{g}}{d\overline{x}}(\overline{x})\frac{d\overline{f}}{d\overline{x}}(\overline{g}(\overline{x})).$$

Proof. Since $\overline{f}$ and $\overline{g}$ are the dual analytic functions,

$$\overline{f}(\overline{x}) = f(x) + \varepsilon\left(x^* f'(x) + \widetilde{f}(x)\right)$$

and

$$\overline{g}(\overline{x}) = g(x) + \varepsilon\left(x^* g'(x) + \widetilde{g}(x)\right)$$

can be written. We know that the derivative of the dual analytic functions are attained by the following equalities:

$$\frac{d\overline{f}}{d\overline{x}} = f'(x) + \varepsilon\left(x^* f''(x) + \widetilde{f}'(x)\right)$$

and

$$\frac{d\overline{g}}{d\overline{x}} = g'(x) + \varepsilon\left(x^* g''(x) + \widetilde{g}'(x)\right).$$

Moreover, since the dual function $\overline{f} \circ \overline{g}$ is the dual analytic function, by using defined derivative of the dual analytic functions, the below equality is obtained

$$\begin{aligned}
\frac{d}{d\overline{x}}\left(\overline{f} \circ \overline{g}\right)(\overline{x}) &= \frac{d}{dx}(f \circ g)(x) \\
&\quad + \varepsilon\frac{d}{dx}\left(x^*(f \circ g)'(x) + \widetilde{g}(x)(f' \circ g)(x) + \left(\widetilde{f} \circ g\right)(x)\right) \\
&= \left(g'(x) + \varepsilon\left(x^* g''(x)\right) + \widetilde{g}'(x)\right) \\
&\quad \cdot \left(f'(g(x)) + \varepsilon\left(\left(x^* g'(x) + \widetilde{g}(x)\right) f''(g(x)) + \widetilde{f}'(g(x))\right)\right) \\
&= \frac{d\overline{g}}{d\overline{x}}(\overline{x})\frac{d\overline{f}}{d\overline{x}}(\overline{g}(\overline{x})).
\end{aligned}$$

□

3. Directional Derivatives on Dual Space

Let $\overline{p} = p + \varepsilon p^*$ be a dual point of D^n and $\overline{v} = \overrightarrow{v} + \varepsilon \overrightarrow{v}^*$ be dual vector to D^n. A dual tangent vector that has $\overline{p}$ as point of application is given as following equality

$$\overline{v}_{\overline{p}} = \overrightarrow{v}_{\widetilde{p}} + \varepsilon \overrightarrow{v}^*_{\widetilde{p}},$$

where $\widetilde{p}$ is the point of $\mathbb{R}^{2n}$. The set of all the dual tangent vectors is shown by

$$T_{\overline{p}}D^n = \left\{ \overline{v}_{\overline{p}} \mid \overline{v}_{\overline{p}} = \overrightarrow{v}_{\widetilde{p}} + \varepsilon \overrightarrow{v}^*_{\widetilde{p}},\ \widetilde{p} \in \mathbb{R}^{2n};\ \overrightarrow{v}_{\widetilde{p}}, \overrightarrow{v}^*_{\widetilde{p}} \in T_{\widetilde{p}}\mathbb{R}^n \right\}.$$

Since the tangent vectors of $T_{\widetilde{p}}\mathbb{R}^n$ are written in the form $\overrightarrow{v}_{\widetilde{p}} = (\widetilde{p}, \overrightarrow{v})$, the dual tangent vectors can be determined by

$$\overline{v}_{\overline{p}} = (\overline{p}, \overline{v}) = (\widetilde{p}, \overrightarrow{v}) + \varepsilon (\widetilde{p}, \overrightarrow{v}^*).$$

On the set $T_{\overline{p}}D^n$, we can define the following operations:

$+ : T_{\overline{p}}D^n \times T_{\overline{p}}D^n \to T_{\overline{p}}D^n$, for $(\overline{p}, \overline{v})$, $(\overline{p}, \overline{w})$ defined as

$$(\overline{p}, \overline{v}) + (\overline{p}, \overline{w}) = (\overline{p}, \overline{v} + \overline{w}) = (\widetilde{p}, \overrightarrow{v} + \overrightarrow{w}) + \varepsilon (\widetilde{p}, \overrightarrow{v}^* + \overrightarrow{w}^*).$$

$\cdot : D \times T_{\overline{p}}D^n \to T_{\overline{p}}D^n$, for $\overline{\lambda}$, $(\overline{p}, \overline{v})$ defined as

$$\overline{\lambda} \cdot (\overline{p}, \overline{v}) = \left(\overline{p}, \overline{\lambda}\overline{v}\right) = (\widetilde{p}, \lambda \overrightarrow{v}) + \varepsilon (\widetilde{p}, \lambda^* \overrightarrow{v} + \lambda \overrightarrow{v}^*).$$

Taken into account the above operations, the set $\{T_{\overline{p}}D^n, +, (D, \oplus, \odot), \cdot\}$ is a D-module and is called a dual tangent space. Besides, since $\overrightarrow{v}$ and $\overrightarrow{v}^*$ are the vectors of $\mathbb{R}^n$, $\overline{v}_{\overline{p}} = (\overline{p}, \overline{v})$ can be written in the form

$$\begin{aligned}
(\overline{p}, \overline{v}) &= (\widetilde{p}, \overrightarrow{v}) + \varepsilon (\widetilde{p}, \overrightarrow{v}^*) \\
&= (\widetilde{p}, v_1 e_1 + ... + v_n e_n) + \varepsilon (\widetilde{p}, v_1^* e_1 + ... + v_n^* e_n) \\
&= v_1 (\widetilde{p}, e_1) + ... + v_n (\widetilde{p}, e_n) + \varepsilon (v_1^* (\widetilde{p}, e_1) + ... + v_n^* (\widetilde{p}, e_n)) \\
&= (v_1 + \varepsilon v_1^*) e_{1\widetilde{p}} + ... + (v_n + \varepsilon v_n^*) e_{n\widetilde{p}} \\
&= \overline{v}_1 e_{1\overline{p}} + ... + \overline{v}_n e_{n\overline{p}}, \qquad (3.1)
\end{aligned}$$

where $e_{i\overline{p}} = e_{i\widetilde{p}} + \varepsilon 0_{\widetilde{p}} = e_{i\widetilde{p}}$, for $1 \leq i \leq n$. On the other hand, let us assume that

$$\sum_{i=1}^{n} \overline{\lambda}_i e_{i\overline{p}} = 0_{\overline{p}}, \qquad (3.2)$$

where $\overline{\lambda}_i = \lambda_i + \varepsilon \lambda_i^*$ is a dual number, for $1 \leq i \leq n$. Expanding the equality (3.2), we obtain the following equalities:

$$\lambda_1 e_{1\widetilde{p}} + ... + \lambda_n e_{n\widetilde{p}} + \varepsilon (\lambda_1^* e_{1\widetilde{p}} + ... + \lambda_n^* e_{n\widetilde{p}}) = 0_{\widetilde{p}} + \varepsilon 0_{\widetilde{p}}$$

and

$$(\widetilde{p}, \lambda_1 e_1 + ... + \lambda_n e_n) + \varepsilon\,(\widetilde{p}, \lambda_1^* e_1 + ... + \lambda_n^* e_n) = (\widetilde{p}, 0) + \varepsilon\,(\widetilde{p}, 0)\,.$$

If the equality property of dual numbers is used, the second formula implies that

$$\lambda_1 e_1 + ... + \lambda_n e_n = 0$$

and

$$\lambda_1^* e_1 + ... + \lambda_n^* e_n = 0.$$

Since the set $\{e_1, ..., e_n\}$ is linear independent, we have

$$\lambda_1 = ... = \lambda_n = 0$$

and

$$\lambda_1^* = ... = \lambda_n^* = 0.$$

If we consider the equations (3.1) and (3.2), it is seen that

$$T_{\overline{p}} D^n = Sp\,\{e_{1\overline{p}}, ..., e_{n\overline{p}}\}\,.$$

Consequently, each element of $T_{\overline{p}} D^n$ can be written as a linear combination of element of the set $\{e_{1\overline{p}}, ..., e_{n\overline{p}}\}$, and this set is known as a standard base of $T_{\overline{p}} D^n$.

Definition 3.1. Let $\overline{f}$ be a dual-valued analytic function on D^n and $\overline{v}_{\overline{p}}$ be a dual tangent vector to D^n. The dual number

$$\frac{d}{d\overline{t}} \overline{f}\left(\overline{p} + \overline{t}\overline{v}\right) |_{\overline{t}=0} \tag{3.3}$$

is called a derivative of $\overline{f}$ with respect to $\overline{v}_{\overline{p}}$ and is denoted by

$$\overline{v}_{\overline{p}}\left[\overline{f}\right] = \frac{d}{d\overline{t}} \overline{f}\left(\overline{p} + \overline{t}\overline{v}\right) |_{\overline{t}=0}\,.$$

For example, we calculate $\overline{v}_{\overline{p}}\left[\overline{f}\right]$ for the dual analytic function $\overline{f} = x_1^2 + x_2 x_3 + \varepsilon\,(2x_1 x_1^* + x_2^* x_3 + x_3^* x_2)$ with $\overline{p} = (1, 0, -1) + \varepsilon\,(-1, 2, 1)$ and $\overline{v} = \overrightarrow{v} + \varepsilon \overrightarrow{v}^* = (1, 5, 3) + \varepsilon\,(-1, 0 - 1)$. Then

$$\overline{p} + \overline{t}\overline{v} = (1 + t, 5t, -1 + 3t) + \varepsilon\,(-t + t^* - 1, 5t^* + 2, -t + 3t^* + 1)$$

is computed. Because of

$$\overline{f} = x_1^2 + x_2 x_3 + \varepsilon\,(2x_1 x_1^* + x_2^* x_3 + x_3^* x_2)\,,$$

we have

$$\overline{f}\left(\overline{p} + \overline{t}\overline{v}\right) = 16t^2 - 3t + 1 + \varepsilon\left((32t - 3)\,t^* - 7t^2 + 7t - 4\right).$$

Now, the derivative of the function $\overline{f}$ according to $\overline{t}$ is calculated as below:

$$\frac{d}{d\overline{t}}\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)=32t-3+\varepsilon\left(32t^{*}-14t+7\right).$$

Then, we obtain $\overline{v}_{\overline{p}}\left[\overline{f}\right]=-3+7\varepsilon$ at $\overline{t}=t+\varepsilon t^{*}=0+\varepsilon 0$.

This definition appears to be the same as the directional derivatives defined in Euclidean space. However, both definition are different. The following theorem shows how to calculate $\overline{v}_{\overline{p}}\left[\overline{f}\right]$, by using the partial derivatives of the dual analytic function $\overline{f}$ at point $\overline{p}$ on dual space D^{n}.

Theorem 3.1. *Let $\overline{f}=f+\varepsilon f^{o}$ be a dual-valued analytic function on D^{n} and $\overline{v}_{\overline{p}}=\overrightarrow{v}_{\widetilde{p}}+\varepsilon\overrightarrow{v}_{\widetilde{p}}^{*}$ be a dual tangent vector to D^{n}. Then, the dual directional derivatives are*

$$\overline{v}_{\overline{p}}\left[\overline{f}\right]=\left(\nabla f\right)_{(\widetilde{p})}\cdot\overrightarrow{v}+\varepsilon\left(\left(\nabla f^{o}\right)_{(\widetilde{p})}\cdot\overrightarrow{v}+\left(\nabla f\right)_{(\widetilde{p})}\cdot\overrightarrow{v}^{*}\right),$$

where

$$\left(\nabla f\right)_{(\widetilde{p})}=\left(\frac{\partial f}{\partial x_{1}}\left(\widetilde{p}\right),...,\frac{\partial f}{\partial x_{n}}\left(\widetilde{p}\right)\right)$$

and

$$\begin{aligned}\left(\nabla f^{o}\right)_{(\widetilde{p})} &= \left(\frac{\partial f^{o}}{\partial x_{1}}\left(\widetilde{p}\right),...,\frac{\partial f^{o}}{\partial x_{n}}\left(\widetilde{p}\right)\right)\\ &= \left(\sum_{i=1}^{n}p_{i}^{*}\frac{\partial^{2}f}{\partial x_{i}\partial x_{1}}\left(\widetilde{p}\right)+\frac{\partial\widetilde{f}}{\partial x_{1}}\left(\widetilde{p}\right),...,\sum_{i=1}^{n}p_{i}^{*}\frac{\partial^{2}f}{\partial x_{i}\partial x_{n}}\left(\widetilde{p}\right)+\frac{\partial\widetilde{f}}{\partial x_{n}}\left(\widetilde{p}\right)\right).\end{aligned}$$

Proof. As it is has already been known, the directional derivatives are defined by

$$\overline{v}_{\overline{p}}\left[\overline{f}\right]=\frac{d}{d\overline{t}}\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)\mid_{\overline{t}=0}.$$

Since $\overline{x}_{1},...,\overline{x}_{n}$ are the dual coordinate functions of D^{n}, we can write

$$\begin{aligned}\overline{p}+\overline{t}\overline{v} &= \left(\overline{p}_{1}+\overline{t}\overline{v}_{1},...,\overline{p}_{n}+\overline{t}\overline{v}_{n}\right)\\ &= \left(\overline{x}_{1}\left(\overline{p}+\overline{t}\overline{v}\right),...,\overline{x}_{n}\left(\overline{p}+\overline{t}\overline{v}\right)\right),\end{aligned}$$

where the expressions $\overline{x}_{i}\left(\overline{p}+\overline{t}\overline{v}\right)$ can be written in the form

$$\overline{x}_{i}\left(\overline{p}+\overline{t}\overline{v}\right)=p_{i}+tv_{i}+\varepsilon\left(t^{*}v_{i}+p_{i}^{*}+tv_{i}^{*}\right)$$

for $1\leq i\leq n$. Since these functions are the dual analytic functions, the derivative of these functions is as in the following equality

$$\begin{aligned}\frac{d\overline{x}_{i}\left(\overline{p}+\overline{t}\overline{v}\right)}{d\overline{t}} &= \frac{d}{dt}\left(p_{i}+tv_{i}\right)+\varepsilon\frac{d}{dt}\left(\left(t^{*}v_{i}+p_{i}^{*}+tv_{i}^{*}\right)\right)\\ &= v_{i}+\varepsilon v_{i}^{*}.\end{aligned}\tag{3.4}$$

Due to

$$\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)=\overline{f}\left(\overline{x}_1\left(\overline{p}+\overline{t}\overline{v}\right),...,\overline{x}_n\left(\overline{p}+\overline{t}\overline{v}\right)\right),$$

the derivative of dual analytic composite functions is

$$\frac{d}{d\overline{t}}\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)=\frac{\partial\overline{f}}{\partial\overline{x}_1}\mid_{\overline{p}+\overline{t}\overline{v}}\frac{d\overline{x}_1}{d\overline{t}}\mid_{\overline{t}}+...+\frac{\partial\overline{f}}{\partial\overline{x}_n}\mid_{\overline{p}+\overline{t}\overline{v}}\frac{d\overline{x}_n}{d\overline{t}}\mid_{\overline{t}}.$$

In this case, for $\overline{t}=0$, we find

$$\frac{d}{d\overline{t}}\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)\mid_{\overline{t}=0}=\frac{\partial\overline{f}}{\partial\overline{x}_1}\mid_{\overline{p}}\frac{d\overline{x}_1}{d\overline{t}}\mid_{\overline{t}=0}+...+\frac{\partial\overline{f}}{\partial\overline{x}_n}\mid_{\overline{p}}\frac{d\overline{x}_n}{d\overline{t}}\mid_{\overline{t}=0}. \tag{3.5}$$

Since $\overline{f}$ is the dual-valued analytic function on D^n, the partial derivatives of this function are

$$\frac{\partial\overline{f}}{\partial\overline{x}_j}=\frac{\partial f}{\partial x_j}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2 f}{\partial x_i\partial x_j}+\frac{\partial\widetilde{f}}{\partial x_j}\right)\quad(1\leq j\leq n).$$

For $\overline{p}\in D^n$, we can express

$$\begin{aligned}\frac{\partial\overline{f}}{\partial\overline{x}_j}\left(\overline{p}\right) &= \frac{\partial f}{\partial x_j}\left(\widetilde{p}\right)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2 f}{\partial x_i\partial x_j}+\frac{\partial\widetilde{f}}{\partial x_j}\right)\left(\widetilde{p}\right)\\ &= \frac{\partial f}{\partial x_j}\left(\widetilde{p}\right)+\varepsilon\left(\sum_{i=1}^{n}p_i^*\frac{\partial^2 f}{\partial x_i\partial x_j}\left(\widetilde{p}\right)+\frac{\partial\widetilde{f}}{\partial x_j}\left(\widetilde{p}\right)\right)\\ &= \frac{\partial f}{\partial x_j}\left(\widetilde{p}\right)+\varepsilon\frac{\partial f^o}{\partial x_j}\left(\widetilde{p}\right).\end{aligned} \tag{3.6}$$

By substituting (3.6) and (3.4) into (3.5), we have

$$\begin{aligned}\frac{d}{d\overline{t}}\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right) \mid_{\overline{t}=0} &= \frac{\partial f}{\partial x_1}\left(\widetilde{p}\right)v_1+...+\frac{\partial f}{\partial x_n}\left(\widetilde{p}\right)v_n\\ &+\varepsilon\left(\frac{\partial f^o}{\partial x_1}\left(\widetilde{p}\right)v_1+...+\frac{\partial f^o}{\partial x_n}\left(\widetilde{p}\right)v_n+\frac{\partial f}{\partial x_1}\left(\widetilde{p}\right)v_1^*+...+\frac{\partial f}{\partial x_n}\left(\widetilde{p}\right)v_n^*\right).\end{aligned} \tag{3.7}$$

Thus, the dual directional derivatives are obtained from (3.7) as

$$\overline{v}_{\overline{p}}\left[\overline{f}\right]=\left(\nabla f\right)_{(\widetilde{p})}\cdot\overrightarrow{v}+\varepsilon\left(\left(\nabla f^o\right)_{(\widetilde{p})}\cdot\overrightarrow{v}+\left(\nabla f\right)_{(\widetilde{p})}\cdot\overrightarrow{v}^*\right).$$

□

Using this theorem, we recalculate $\overline{v}_{\overline{p}}\left[\overline{f}\right]$ for the example above. Due to

$$\overline{f}=f+\varepsilon f^o=x_1^2+x_2x_3+\varepsilon\left(2x_1x_1^*+x_2^*x_3+x_3^*x_2\right),$$

we get

$$f = x_1^2 + x_2 x_3 \text{ and } f^o = 2x_1 x_1^* + x_2^* x_3 + x_3^* x_2.$$

At the point $\widetilde{p}$, since

$$x_1(\widetilde{p}) = 1,\ x_2(\widetilde{p}) = 0,\ x_3(\widetilde{p}) = -1,\ x_1^*(\widetilde{p}) = -1,\ x_2^*(\widetilde{p}) = 2,\ x_3^*(\widetilde{p}) = 1,$$

the following equalities are obtained

$$(\nabla f)_{(\widetilde{p})} \cdot \overrightarrow{v} = -3,\ (\nabla f^o)_{(\widetilde{p})} \cdot \overrightarrow{v} = 9, \text{ and } (\nabla f)_{(\widetilde{p})} \cdot \overrightarrow{v}^* = -2.$$

By the theorem

$$\overline{v}_{\overline{p}}\left[\overline{f}\right] = -3 + \varepsilon(9-2) = -3 + 7\varepsilon$$

as before.

Throughout this paper, we will use the following notations:

$$(\nabla f)_{(\widetilde{p})} \cdot \overrightarrow{v} = \overrightarrow{v}_{\widetilde{p}}[f],\quad (\nabla f^o)_{(\widetilde{p})} \cdot \overrightarrow{v} = \overrightarrow{v}_{\widetilde{p}}[f^o],\quad (\nabla f)_{(\widetilde{p})} \cdot \overrightarrow{v}^* = \overrightarrow{v}^*_{\widetilde{p}}[f].$$

In this case, dual directional derivatives are shown by

$$\overline{v}_{\overline{p}}\left[\overline{f}\right] = \overrightarrow{v}_{\widetilde{p}}[f] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^o] + \overrightarrow{v}^*_{\widetilde{p}}[f]\right).$$

Thus, the following theorem can be given.

Theorem 3.2. *Let $\overline{f} = f + \varepsilon f^o$ and $\overline{g} = g + \varepsilon g^o$ be dual-valued analytic functions on D^n and $\overline{v}_{\overline{p}} = \overrightarrow{v}_{\widetilde{p}} + \varepsilon \overrightarrow{v}^*_{\widetilde{p}}$ be dual tangent vector to D^n. Then*

(1) $\overline{v}_{\overline{p}}\left[\overline{f} + \overline{g}\right] = \overline{v}_{\overline{p}}\left[\overline{f}\right] + \overline{v}_{\overline{p}}[\overline{g}]$.

(2) $\overline{v}_{\overline{p}}\left[\overline{f}\,\overline{g}\right] = \overline{v}_{\overline{p}}\left[\overline{f}\right]\overline{g}(\overline{p}) + \overline{f}(p)\,\overline{v}_{\overline{p}}[\overline{g}]$.

Proof. (1) From the above theorem, we know that

$$\overline{v}_{\overline{p}}\left[\overline{f}\right] = \overrightarrow{v}_{\widetilde{p}}[f] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^o] + \overrightarrow{v}^*_{\widetilde{p}}[f]\right).$$

In that case, we have

$$\begin{aligned}
\overline{v}_{\overline{p}}\left[\overline{f} + \overline{g}\right] &= \overrightarrow{v}_{\widetilde{p}}[f+g] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^o + g^o] + \overrightarrow{v}^*_{\widetilde{p}}[f+g]\right) \\
&= \overrightarrow{v}_{\widetilde{p}}[f] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^o] + \overrightarrow{v}^*_{\widetilde{p}}[f]\right) + \overrightarrow{v}_{\widetilde{p}}[g] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[g^o] + \overrightarrow{v}^*_{\widetilde{p}}[g]\right) \\
&= \overline{v}_{\overline{p}}\left[\overline{f}\right] + \overline{v}_{\overline{p}}[\overline{g}].
\end{aligned}$$

(2) From (2.6), the function

$$\begin{aligned}
\left(\overline{f} \cdot \overline{g}\right)(\overline{x}) &= \overline{f}(\overline{x}) \cdot \overline{g}(\overline{x}) \\
&= f(x)\,g(x) + \varepsilon\left(\sum_{i=1}^{n} x_i^*\left(\frac{\partial(fg)}{\partial x_i}\right) + f(x)\,\widetilde{g}(x) + g(x)\,\widetilde{f}(x)\right)
\end{aligned}$$

is a dual analytic function. If this dual analytic function is shown as below

$$\begin{aligned}\left(\overline{f}\overline{g}\right)(\overline{x}) &= \overline{f}(\overline{x})\,\overline{g}(\overline{x}) \\ &= fg+\varepsilon\left(fg^{o}+gf^{o}\right),\end{aligned}$$

the following equality is obtained

$$\begin{aligned}\overline{v}_{\overline{p}}\left[\overline{f}\overline{g}\right] &= \overrightarrow{v}_{\widetilde{p}}[fg]+\varepsilon\left(\overrightarrow{v}_{\widetilde{p}}\left[fg^{o}+gf^{o}\right]+\overrightarrow{v}^{*}_{\widetilde{p}}[fg]\right) \\ &= \overrightarrow{v}_{\widetilde{p}}[f]\,g(\widetilde{p})+f(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g] \\ &\quad +\varepsilon\left(\begin{array}{l}\overrightarrow{v}_{\widetilde{p}}[f]\,g^{o}(\widetilde{p})+f(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g^{o}]+\overrightarrow{v}_{\widetilde{p}}[f^{o}]\,g(\widetilde{p}) \\ +f^{o}(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g]+g(\widetilde{p})\,\overrightarrow{v}^{*}_{\widetilde{p}}[f]+f(\widetilde{p})\,\overrightarrow{v}^{*}_{\widetilde{p}}[g]\end{array}\right) \\ &= \overrightarrow{v}_{\widetilde{p}}[f]\,g(\widetilde{p})+\varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f]\,g^{o}(\widetilde{p})+\overrightarrow{v}_{\widetilde{p}}[f^{o}]\,g(\widetilde{p})+\overrightarrow{v}^{*}_{\widetilde{p}}[f]\,g(\widetilde{p})\right) \\ &\quad +f(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g]+\varepsilon\left(f^{o}(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g]+f(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g^{o}]+f(\widetilde{p})\,\overrightarrow{v}^{*}_{\widetilde{p}}[g]\right) \\ &= \left(\overrightarrow{v}_{\widetilde{p}}[f]+\varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^{o}]+\overrightarrow{v}^{*}_{\widetilde{p}}[f]\right)\right)\left(g(\widetilde{p})+\varepsilon g^{o}(\widetilde{p})\right) \\ &\quad +\left(f(\widetilde{p})+\varepsilon f^{o}(\widetilde{p})\right)\left(\overrightarrow{v}_{\widetilde{p}}[g]+\varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[g^{o}]+\overrightarrow{v}^{*}_{\widetilde{p}}[g]\right)\right) \\ &= \overline{v}_{\overline{p}}\left[\overline{f}\right]\overline{g}(\overline{p})+\overline{f}(p)\,\overline{v}_{\overline{p}}[\overline{g}]\,.\end{aligned}$$

The equalities (1) and (2) show that the dual directional derivatives satisfy linear and Leibniz rules. □

Definition 3.2. Let $\overline{f}=f+\varepsilon f^{o}$ be dual-valued analytic function on D^{n} and $\overline{v}_{\overline{p}}=\overrightarrow{v}_{\widetilde{p}}+\varepsilon\overrightarrow{v}^{*}_{\widetilde{p}}$ be dual tangent vector to D^{n}. The expression

$$\overline{v}_{\overline{p}}:C\left(D^{n},D\right)\rightarrow D$$

$$\overline{v}_{\overline{p}}\left(\overline{f}\right)=\overline{v}_{\overline{p}}\left[\overline{f}\right]=\overrightarrow{v}_{\widetilde{p}}[f]+\varepsilon\left(\overrightarrow{v}_{\widetilde{p}}\left[f^{o}\right]+\overrightarrow{v}^{*}_{\widetilde{p}}[f]\right)$$

can be defined as an operator.

In section II, we showed that each element of $T_{\overline{p}}D^{n}$ can be written as a linear combination of element of the set $\{e_{1\overline{p}},...,e_{n\overline{p}}\}$. In this case, for $1\leq i\leq n$, the below equality

$$\begin{aligned}e_{i\overline{p}}\left[\overline{f}\right] &= e_{i\widetilde{p}}[f]+\varepsilon e_{i\widetilde{p}}\left[f^{o}\right] \\ &= \frac{\partial f}{\partial x_{i}}(\widetilde{p})+\varepsilon\frac{\partial f^{o}}{\partial x_{i}}(\widetilde{p}) \\ &= \frac{\partial f}{\partial x_{i}}(\widetilde{p})+\varepsilon\left(\sum_{j=1}^{n}p_{j}^{*}\frac{\partial^{2}f}{\partial x_{j}\partial x_{i}}(\widetilde{p})+\frac{\partial\widetilde{f}}{\partial x_{i}}(\widetilde{p})\right) \\ &= \frac{\partial f}{\partial x_{i}}(\widetilde{p})+\varepsilon\left(\sum_{j=1}^{n}x_{j}^{*}\frac{\partial^{2}f}{\partial x_{j}\partial x_{i}}+\frac{\partial\widetilde{f}}{\partial x_{i}}\right)(\widetilde{p}) \\ &= \frac{\partial\overline{f}}{\partial\overline{x}_{i}}(\overline{p})\end{aligned}\tag{3.8}$$

can be written. For every $\overline{p} \in D^n$, since the above equality is correct, we get the following equality:

$$e_i\left[\overline{f}\right] = e_i\left[f\right] + \varepsilon e_i\left[f^o\right] = \frac{\partial f}{\partial x_i} + \varepsilon\frac{\partial f^o}{\partial x_i} = \frac{\partial \overline{f}}{\partial \overline{x}_i}. \tag{3.9}$$

The equality (3.9) is shown that the partial derivatives of the dual analytic function $\overline{f}$ according to dual variables $\overline{x}_i$ are equal to the derivative of $\overline{f}$ with respect to vectors e_i.

Definition 3.3. Let $\overline{f} = f + \varepsilon f^o$ be dual-valued analytic function on D^n. Differential of $\overline{f}$ is shown as $d\overline{f}$ and is defined as the following equality

$$d\overline{f}\left(\overline{v}_{\overline{p}}\right) = \overline{v}_{\overline{p}}\left[\overline{f}\right] = \overrightarrow{v}_{\widetilde{p}}\left[f\right] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}\left[f^o\right] + \overrightarrow{v}^*_{\widetilde{p}}\left[f\right]\right).$$

If the above definition is considered, since the dual identity function is defined as

$$\overline{I}\left(\overline{x}\right) = \overline{x} = x + \varepsilon x^* = x + \varepsilon\left(x'x^* + 0\left(x\right)\right),$$

$\overline{I}\left(\overline{x}\right)$ is the dual analytic function and, for $1 \leq i \leq n$,

$$\begin{aligned} d\overline{x}_i\left(\overline{v}_{\overline{p}}\right) &= \overline{v}_{\overline{p}}\left[\overline{x}_i\right] = \overrightarrow{v}_{\widetilde{p}}\left[x_i\right] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}\left[x_i^*\right] + \overrightarrow{v}^*_{\widetilde{p}}\left[x_i\right]\right) \\ &= v_i + \varepsilon v_i^* \end{aligned}$$

is calculated. In this case, it is seen that

$$d\overline{x}_i\left(\overline{v}_{\overline{p}}\right) = \overline{v}_i.$$

On the other hand, assuming that $x_1^*, ..., x_n^*$ are not dependent on $x_1, ..., x_n$, the following equality can be written

$$\begin{aligned} d\overline{x}_i &= dx_i + \varepsilon dx_i^* \\ &= dx_i\left(1 + \varepsilon\frac{dx_i^*}{dx_i}\right) \\ &= dx_i\ [4]. \end{aligned}$$

In this case, $d\overline{x}_i\left(\overline{v}_{\overline{p}}\right)$ can be rewritten as follows

$$\begin{aligned} d\overline{x}_i\left(\overline{v}_{\overline{p}}\right) &= dx_i\left(\overrightarrow{v}_{\widetilde{p}}\right) + \varepsilon dx_i\left(\overrightarrow{v}^*_{\widetilde{p}}\right) \\ &= v_i + \varepsilon v_i^*. \end{aligned}$$

Thus, $d\overline{x}_i$ is the ith coordinate functions of the dual vector $v + \varepsilon v^*$ while $\overline{x}_i$ is the ith coordinate functions of the dual point $\overline{p} = p + \varepsilon p^*$.

Let us consider that $g_{ij} = e_i \cdot e_j$, where $1 \leq i, j \leq n$. In this case, the dual inner product on D^n is shown by

$$G = g_{ij} d\overline{x}_i d\overline{x}_j.$$

For the dual vectors $\overline{v}=\overrightarrow{v}+\varepsilon\overrightarrow{v}^{*}$, $\overline{w}=\overrightarrow{w}+\varepsilon\overrightarrow{w}^{*}\in D^{3}$, the dual inner product is

$$\begin{aligned}
G(\overline{v},\overline{w}) &= g_{ij}d\overline{x}_{i}(\overline{v})\,d\overline{x}_{j}(\overline{w})\\
&= d\overline{x}_{1}(\overline{v})\,d\overline{x}_{1}(\overline{w})+d\overline{x}_{2}(\overline{v})\,d\overline{x}_{2}(\overline{w})+d\overline{x}_{3}(\overline{v})\,d\overline{x}_{3}(\overline{w})\\
&= (dx_{1}(\overrightarrow{v})+\varepsilon dx_{1}(\overrightarrow{v}^{*}))(dx_{1}(\overrightarrow{w})+\varepsilon dx_{1}(\overrightarrow{w}^{*}))\\
&\quad+(dx_{2}(\overrightarrow{v})+\varepsilon dx_{2}(\overrightarrow{v}^{*}))(dx_{2}(\overrightarrow{w})+\varepsilon dx_{2}(\overrightarrow{w}^{*}))\\
&\quad+(dx_{3}(\overrightarrow{v})+\varepsilon dx_{3}(\overrightarrow{v}^{*}))(dx_{3}(\overrightarrow{w})+\varepsilon dx_{3}(\overrightarrow{w}^{*}))\\
&= \overrightarrow{v}\cdot\overrightarrow{w}+\varepsilon(\overrightarrow{v}\cdot\overrightarrow{w}^{*}+\overrightarrow{v}^{*}\cdot\overrightarrow{w}).
\end{aligned}$$

This inner product shows how to define inner product studied on D^{3} in many articles.

4. Vector Fields on Dual Space

A dual vector field is a dual function that assigns to each dual point $\overline{p}=p+\varepsilon p^{*}\in D^{n}$ a dual tangent vector $\overline{X}_{\overline{p}}=\overrightarrow{X}_{\widetilde{p}}+\varepsilon\overrightarrow{X}^{*}_{\widetilde{p}}$ to D^{n}, i.e., for every $\overline{p}=p+\varepsilon p^{*}\in D^{n}$, the dual vector field is defined as below expression

$$\begin{aligned}
\overline{X} &: D^{n}\to TD^{n}\\
\overline{X}(\overline{p}) &= \overline{X}_{\overline{p}}=\overrightarrow{X}_{\widetilde{p}}+\varepsilon\overrightarrow{X}^{*}_{\widetilde{p}},
\end{aligned}$$

where $\overline{X}=X+\varepsilon X^{*}$. For $1\leq i\leq n$, let $\overline{a}_{i}=a_{i}+\varepsilon a_{i}^{o}$ be dual analytic function. In this case,

$$\overline{X}=(a_{1},...,a_{n})+\varepsilon(a_{1}^{o},...,a_{1}^{o}) \tag{4.1}$$

is a dual vector field on D^{n}. For each point $\overline{p}$ of D^{n}, the equality (4.1) is given in the form

$$\overline{X}(\overline{p})=(a_{1}(\widetilde{p}),...,a_{n}(\widetilde{p}))+\varepsilon(a_{1}^{o}(\widetilde{p}),...,a_{n}^{o}(\widetilde{p})).$$

Here, since $\overline{a}_{i}=a_{i}+\varepsilon a_{i}^{o}$ is the dual-valued analytic function on D^{n}, it can be written as follows

$$\begin{aligned}
\overline{a}_{i}(\overline{x}) &= a_{i}(x_{1},...,x_{n})+\varepsilon\left(\sum_{j=1}^{n}x_{j}^{*}\frac{\partial a_{i}}{\partial x_{j}}+\widetilde{a}_{i}(x_{1},...,x_{n})\right)\\
&= a_{i}+\varepsilon a_{i}^{o}
\end{aligned}$$

and, for every $\overline{p}=p+\varepsilon p^{*}\in D^{n}$, we have

$$\overline{a}_{i}(\overline{p})=a_{i}(\widetilde{p})+\varepsilon a_{i}^{o}(\widetilde{p}).$$

The set of dual vector fields is given as follows:

$$\chi(D^{n})=\left\{\overline{X}\mid\overline{X}:D^{n}\to TD^{n},\ \overline{X}(\overline{p})=\overline{X}_{\overline{p}}=\overrightarrow{X}_{\widetilde{p}}+\varepsilon\overrightarrow{X}^{*}_{\widetilde{p}}\right\}.$$

On this set, we are able to define the following operators which make $\chi(D^n)$ a module called D-module. Axioms are as follows:

$$\left(\overline{X}+\overline{Y}\right)(\overline{p})=\overline{X}(\overline{p})+\overline{Y}(\overline{p})=\overrightarrow{X}_{\widetilde{p}}+\overrightarrow{Y}_{\widetilde{p}}+\varepsilon\left(\overrightarrow{X}^{*}_{\widetilde{p}}+\overrightarrow{Y}^{*}_{\widetilde{p}}\right)$$

and

$$\left(\overline{\lambda}\cdot\overline{X}\right)(\overline{p})=\overline{\lambda X}_{\overline{p}}=\lambda\overrightarrow{X}_{\widetilde{p}}+\varepsilon\left(\lambda^{*}\overrightarrow{X}_{\widetilde{p}}+\lambda\overrightarrow{X}^{*}_{\widetilde{p}}\right),$$

where $\overline{X}=\overrightarrow{X}+\varepsilon\overrightarrow{X}^{*}$ and $\overline{Y}=\overrightarrow{Y}+\varepsilon\overrightarrow{Y}^{*}$ are the dual vector fields and $\overline{\lambda}=\lambda+\varepsilon\lambda^{*}$ is the dual number.

Let $\overline{f}=f+\varepsilon f^{o}$ be a dual-valued analytic function on D^n. The function

$$\overline{X}\left[\overline{f}\right]=X[f]+\varepsilon\left(X\left[f^{o}\right]+X^{*}[f]\right) \tag{4.2}$$

is called the derivative of $\overline{f}$ with respect to the dual vector field $\overline{X}$.

Expanding the equality (4.2), we have

$$\overline{X}\left[\overline{f}\right]=\sum_{i=1}^{n}\left[\frac{\partial f}{\partial x_i}a_i+\varepsilon\left(\begin{array}{c}\sum_{j=1}^{n}x_j^{*}\left(\frac{\partial f}{\partial x_i}\frac{\partial a_i}{\partial x_j}+a_i\frac{\partial^2 f}{\partial x_j\partial x_i}\right)\\ +\frac{\partial f}{\partial x_i}\widetilde{a}_i+a_i\frac{\partial\widetilde{f}}{\partial x_i}\end{array}\right)\right]. \tag{4.3}$$

It is clear that the equality (4.3) is a dual analytic function on D^n. In this case, the dual vector field $\overline{X}:C(D^n,D)\to C(D^n,D)$ is able to be defined as follows:

$$\overline{X}\left(\overline{f}\right)=\overline{X}\left[\overline{f}\right]. \tag{4.4}$$

For every $\overline{p}\in D^n$, if the equalities (4.2) and (4.3) are used, the following expressions are obtained, respectively,

$$\left(\overline{X}\left[\overline{f}\right]\right)(\overline{p})=\overline{X}_{\overline{p}}\left[\overline{f}\right]=\overrightarrow{X}_{\widetilde{p}}[f]+\varepsilon\left(\overrightarrow{X}_{\widetilde{p}}\left[f^{o}\right]+\overrightarrow{X}^{*}_{\widetilde{p}}[f]\right)$$

and

$$\overline{X}_{\overline{p}}\left[\overline{f}\right]=\sum_{i=1}^{n}\left[\frac{\partial f}{\partial x_i}(\widetilde{p})\,a_i(\widetilde{p})+\varepsilon\left(\begin{array}{c}\sum_{j=1}^{n}p_j^{*}\left(\frac{\partial f}{\partial x_i}(\widetilde{p})\frac{\partial a_i}{\partial x_j}(\widetilde{p})+a_i(\widetilde{p})\frac{\partial^2 f}{\partial x_i\partial x_j}(\widetilde{p})\right)\\ +\frac{\partial f}{\partial x_i}(\widetilde{p})\,\widetilde{a}_i(\widetilde{p})+a_i(\widetilde{p})\frac{\partial\widetilde{f}}{\partial x_i}(\widetilde{p})\end{array}\right)\right].$$

Corollary 4.1. *If $\overline{X}=\overrightarrow{X}+\varepsilon\overrightarrow{X}^{*}$ is a dual vector field on D^n and $\overline{f}=f+\varepsilon f^{o}$ and $\overline{g}=g+\varepsilon g^{o}$ are dual-valued analytic functions on D^n, then*

(1) $\overline{X}\left[\overline{f}+\overline{g}\right]=\overline{X}\left[\overline{f}\right]+\overline{X}\left[\overline{g}\right]$.

(2) $\overline{X}\left[\overline{\lambda}\overline{f}\right]=\overline{\lambda X}\left[\overline{f}\right]$, *for all dual numbers* $\overline{\lambda}=\lambda+\varepsilon\lambda^{*}$.

(3) $\overline{X}\left[\overline{f}\overline{g}\right]=\overline{X}\left[\overline{f}\right]\overline{g}+\overline{fX}\left[\overline{g}\right]$.

Proof. For $\overline{p} = p + \varepsilon p^* \in D^n$, in the section III, the equalities (1) and (3) were calculated in detail. Now, we know that

$$\left(\overline{X}\left[\overline{\lambda f}\right]\right)(\overline{p}) = \overline{X}_{\overline{p}}\left[\overline{\lambda f}\right].$$

In this case, we have

$$\begin{aligned}
\left(\overline{X}\left[\overline{\lambda f}\right]\right)(\overline{p}) &= \overline{X}_{\overline{p}}\left[\overline{\lambda f}\right] = \overrightarrow{X}_{\widetilde{p}}[\lambda f] + \varepsilon\left(\overrightarrow{X}_{\widetilde{p}}[\lambda^* f + \lambda f^o] + \overrightarrow{X}^*_{\widetilde{p}}[\lambda f]\right) \\
&= \lambda\overrightarrow{X}_{\widetilde{p}}[f] + \varepsilon\left(\lambda^*\overrightarrow{X}_{\widetilde{p}}[f] + \lambda\overrightarrow{X}_{\widetilde{p}}[f^o] + \lambda\overrightarrow{X}^*_{\widetilde{p}}[f]\right) \\
&= (\lambda + \varepsilon\lambda^*)\left(\overrightarrow{X}_{\widetilde{p}}[f] + \varepsilon\left(\overrightarrow{X}_{\widetilde{p}}[f^o] + \overrightarrow{X}^*_{\widetilde{p}}[f]\right)\right) \qquad (4.5) \\
&= \overline{\lambda X}_{\overline{p}}\left[\overline{f}\right] \\
&= \left(\overline{\lambda X}\left[\overline{f}\right]\right)(\overline{p}).
\end{aligned}$$

For every $\overline{p} = p + \varepsilon p^* \in D^n$, since the equality (4.5) is correct,

$$\overline{X}\left[\overline{\lambda f}\right] = \overline{\lambda X}\left[\overline{f}\right]$$

is obtained. □

5. Tangent Maps on Dual Space

Let $\overline{f} \in C(D^n, D^m)$ be a dual analytic function. For every $\overline{p} = p + \varepsilon p^* \in D^n$, the dual function

$$\overline{f}_{*\overline{p}} : T_{\overline{p}}D^n \to T_{\overline{f}(\overline{p})}D^m$$

is called as dual tangent map of $\overline{f}$ at dual point $\overline{p}$, and is defined by

$$\begin{aligned}
\overline{f}_{*\overline{p}}(\overline{v}_{\overline{p}}) &= \left(\left(v_{\widetilde{p}}[f_1], ..., v_{\widetilde{p}}[f_m]\right) + \varepsilon\left(v_{\widetilde{p}}[f_1^o] + v^*_{\widetilde{p}}[f_1], ..., v_{\widetilde{p}}[f_m^o] + v^*_{\widetilde{p}}[f_m]\right)\right)|_{q+\varepsilon q^*} \\
&= f_{*\widetilde{p}}(v_{\widetilde{p}}) + \varepsilon\left(f^o_{*\widetilde{p}}(v_{\widetilde{p}}) + f_{*\widetilde{p}}\left(v^*_{\widetilde{p}}\right)\right) \qquad (5.1) \\
&= w_{\widetilde{q}} + \varepsilon w^*_{\widetilde{q}},
\end{aligned}$$

where $q + \varepsilon q^* = f(\widetilde{p}) + \varepsilon f^o(\widetilde{p})$ is the dual point of D^n. It is seen from the above formula that $\overline{f}_{*\overline{p}}$ sends dual tangent vectors at $\overline{p} = p + \varepsilon p^*$ to dual tangent vectors at $\overline{f}(\overline{p}) = f(\widetilde{p}) + \varepsilon f^o(\widetilde{p})$. On the other hand, the function $\overline{f}_* : \chi(D^n) \to \chi(D^m)$ is named as dual tangent map of $\overline{f}$ and is given as

$$\begin{aligned}
\overline{f}_*\left(\overline{X}\right) &= \left(\overline{X}\left[\overline{f}_1\right], ..., \overline{X}\left[\overline{f}_m\right]\right) \\
&= f_*\left(\overrightarrow{X}\right) + \varepsilon\left(f^o_*\left(\overrightarrow{X}\right) + f_*\left(\overrightarrow{X}^*\right)\right).
\end{aligned}$$

Theorem 5.1. *If the function $\overline{f} : D^n \to D^m$ is a dual analytic function, then the dual tangent map $\overline{f}_{*\overline{p}} : T_{\overline{p}}D^n \to T_{\overline{f}(\overline{p})}D^m$ is a linear transformation.*

Proof. Let $\overline{v}_{\overline{p}}$ and $\overline{w}_{\overline{p}}$ be dual tangent vectors and $\overline{\lambda} = \lambda + \varepsilon\lambda^*$ be dual number. We must show that

(1) $\overline{f}_{*\overline{p}}\left(\overline{v}_{\overline{p}} + \overline{w}_{\overline{p}}\right) = \overline{f}_{*\overline{p}}\left(\overline{v}_{\overline{p}}\right) + \overline{f}_{*\overline{p}}\left(\overline{w}_{\overline{p}}\right)$

(2) $\overline{f}_{*\overline{p}}\left(\overline{\lambda}\overline{v}_{\overline{p}}\right) = \overline{\lambda f}_{*\overline{p}}\left(\overline{v}_{\overline{p}}\right).$

Since the dual tangent vectors are shown as $\overline{v}_{\overline{p}} = \overrightarrow{v}_{\widetilde{p}} + \varepsilon\overrightarrow{v}^*_{\widetilde{p}}$ and $\overline{w}_{\overline{p}} = \overrightarrow{w}_{\widetilde{p}} + \varepsilon\overrightarrow{w}^*_{\widetilde{p}}$, the addition of these vectors is

$$\overline{v}_{\overline{p}} + \overline{w}_{\overline{p}} = \overrightarrow{v}_{\widetilde{p}} + \overrightarrow{w}_{\widetilde{p}} + \varepsilon\left(\overrightarrow{v}^*_{\widetilde{p}} + \overrightarrow{w}^*_{\widetilde{p}}\right).$$

Considering the equality (5.1), we get

$$\begin{aligned}
\overline{f}_{*\overline{p}}\left(\overline{v}_{\overline{p}} + \overline{w}_{\overline{p}}\right) &= f_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}} + \overrightarrow{w}_{\widetilde{p}}\right) + \varepsilon\left(f^o_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}} + \overrightarrow{w}_{\widetilde{p}}\right) + f_{*\widetilde{p}}\left(\overrightarrow{v}^*_{\widetilde{p}} + \overrightarrow{w}^*_{\widetilde{p}}\right)\right) \\
&= f_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}}\right) + \varepsilon\left(f^o_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}}\right) + f_{*\widetilde{p}}\left(\overrightarrow{v}^*_{\widetilde{p}}\right)\right) \\
&\quad + f_{*\widetilde{p}}\left(\overrightarrow{w}_{\widetilde{p}}\right) + \varepsilon\left(f^o_{*\widetilde{p}}\left(\overrightarrow{w}_{\widetilde{p}}\right) + f_{*\widetilde{p}}\left(\overrightarrow{w}^*_{\widetilde{p}}\right)\right) \\
&= \overline{f}_{*\overline{p}}\left(\overline{v}_{\overline{p}}\right) + \overline{f}_{*\overline{p}}\left(\overline{w}_{\overline{p}}\right).
\end{aligned}$$

On the other hand, the multiplication of dual tangent vector with dual number is

$$\overline{\lambda}\overline{v}_{\overline{p}} = \lambda\overrightarrow{v}_{\widetilde{p}} + \varepsilon\left(\lambda^*\overrightarrow{v}_{\widetilde{p}} + \lambda\overrightarrow{v}^*_{\widetilde{p}}\right).$$

In this case, we have

$$\begin{aligned}
\overline{f}_{*\overline{p}}\left(\overline{\lambda}\overline{v}_{\overline{p}}\right) &= \left(\left(\lambda\overrightarrow{v}_{\widetilde{p}}\right)[f_1], ..., \left(\lambda\overrightarrow{v}_{\widetilde{p}}\right)[f_m]\right) \\
&\quad + \varepsilon\left(\left(\left(\lambda^*\overrightarrow{v}_{\widetilde{p}}\right)[f_1]\right)\left(\lambda\overrightarrow{v}^*_{\widetilde{p}}\right)[f_1] +, ..., \left(\left(\lambda^*\overrightarrow{v}_{\widetilde{p}}\right)[f_m]\right)\left(\lambda\overrightarrow{v}^*_{\widetilde{p}}\right)[f_m]\right).
\end{aligned}$$

When the above mentioned equality is taken into consideration, it is easily seen that

$$\begin{aligned}
\overline{f}_{*\overline{p}}\left(\overline{\lambda}\overline{v}_{\overline{p}}\right) &= \left(\lambda + \varepsilon\lambda^*\right)\left(f_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}}\right) + \varepsilon\left(f^o_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}}\right) + f_{*\widetilde{p}}\left(\overrightarrow{v}^*_{\widetilde{p}}\right)\right)\right) \\
&= \overline{\lambda f}_{*\overline{p}}\left(\overline{v}_{\overline{p}}\right).
\end{aligned}$$

The equalities (1) and (2) show that the map $\overline{f}_{*\overline{p}}$ is a linear transformation. □

According to given bases, each linear transformation corresponds to a matrix. Now, let's find the matrix which is called dual Jacobian corresponding to this linear transformation. Let us consider that the bases of $T_{\overline{p}}D^n$ and $T_{\overline{q}}D^n$ are defined as follows:

$$\left\{e_{1\overline{p}}, ..., e_{n\overline{p}}\right\} \text{ and } \left\{e_{1\overline{q}}, ..., e_{m\overline{q}}\right\},$$

respectively, where $\overline{q} = f\left(\widetilde{p}\right) + \varepsilon f^o\left(\widetilde{p}\right)$. Thus, for $1 \leq j \leq n$, the following expression can be written:

$$\overline{f}_{*\overline{p}}\left(e_{j\overline{p}}\right) = \left(\left(e_{j\widetilde{p}}\left[f_1\right], ..., e_{j\widetilde{p}}\left[f_m\right]\right) + \varepsilon\left(e_{j\widetilde{p}}\left[f^o_1\right], ..., e_{j\widetilde{p}}\left[f^o_m\right]\right)\right)|_{q+\varepsilon q^*}$$

$$
\begin{aligned}
&= \left(\frac{\partial f_1}{\partial x_j}(\widetilde{p})\, e_{1\widetilde{q}} + ... + \frac{\partial f_m}{\partial x_j}(\widetilde{p})\, e_{m\widetilde{q}} + \varepsilon\left(\frac{\partial f_1^o}{\partial x_j}(\widetilde{p})\, e_{1\widetilde{q}} + ... + \frac{\partial f_m^o}{\partial x_j}(\widetilde{p})\, e_{m\widetilde{q}}\right)\right) \\
&= \left(\frac{\partial f_1}{\partial x_j}(\widetilde{p}), ..., \frac{\partial f_m}{\partial x_j}(\widetilde{p})\right) + \varepsilon\left(\frac{\partial f_1^o}{\partial x_j}(\widetilde{p}), ..., \frac{\partial f_m^o}{\partial x_j}(\widetilde{p})\right) \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial x_j}(\widetilde{p}) \\ . \\ . \\ . \\ \frac{\partial f_m}{\partial x_j}(\widetilde{p}) \end{bmatrix} + \varepsilon \begin{bmatrix} \frac{\partial f_1^o}{\partial x_j}(\widetilde{p}) \\ . \\ . \\ . \\ \frac{\partial f_m^o}{\partial x_j}(\widetilde{p}) \end{bmatrix},
\end{aligned}
$$

where $\widetilde{q} = (q_1, ..., q_n, q_1^*, ..., q_n^*)$ is the point of $\mathbb{R}^{2n}$. Thus, the dual Jacobian matrix is shown by $\overline{J}\left(\overline{f}\right)(\overline{p})$ and is defined as follow equality

$$
\begin{aligned}
\overline{J}\left(\overline{f}\right)(\overline{p}) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\widetilde{p}) & . & . & . & \frac{\partial f_1}{\partial x_n}(\widetilde{p}) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{\partial f_m}{\partial x_1}(\widetilde{p}) & . & . & . & \frac{\partial f_m}{\partial x_n}(\widetilde{p}) \end{bmatrix} + \varepsilon \begin{bmatrix} \frac{\partial f_1^o}{\partial x_1}(\widetilde{p}) & . & . & . & \frac{\partial f_1^o}{\partial x_n}(\widetilde{p}) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{\partial f_m^o}{\partial x_1}(\widetilde{p}) & . & . & . & \frac{\partial f_m^o}{\partial x_n}(\widetilde{p}) \end{bmatrix} \\
&= J(f)(\widetilde{p}) + \varepsilon J(f^o)(\widetilde{p}),
\end{aligned}
$$

where

$$
\frac{\partial f_k^o}{\partial x_j}(\widetilde{p}) = \sum_{i=1}^{n} p_i^* \frac{\partial^2 f_k}{\partial x_i \partial x_j}(\widetilde{p}) + \frac{\partial f_k}{\partial x_j}(\widetilde{p})
$$

for $1 \leq k \leq m$ and $1 \leq j \leq n$. In this case, the dual analytic tangent map $\overline{f}_{*\overline{p}} : T_{\overline{p}} D^n \to T_{\overline{f}(\overline{p})} D^n$ can be rewritten as follows

$$
\begin{aligned}
\overline{f}_{*\overline{p}}(\overline{v}_{\overline{p}}) &= \left(f_{*\widetilde{p}}(\overrightarrow{v}_{\widetilde{p}}) + \varepsilon\left(f^o_{*\widetilde{p}}(\overrightarrow{v}_{\widetilde{p}}) + f_{*\widetilde{p}}\left(\overrightarrow{v}^*_{\widetilde{p}}\right)\right)\right) \\
&= J(f)(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}} + \varepsilon\left(J(f^o)(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}} + J(f)(\widetilde{p})\,\overrightarrow{v}^*_{\widetilde{p}}\right). \qquad (5.2)
\end{aligned}
$$

Example 5.1. Let

$$
\begin{aligned}
\overline{f} &: D^2 \to D^3, \\
\overline{f}(\overline{x}) &= (\cos x_1, \sin x_1, x_2) + \varepsilon\left(-x_1^* \sin x_1 + \cos x_1, x_1^* \cos x_1 + \sin x_1, x_2^*\right)
\end{aligned}
$$

be a dual analytic function with $\overline{p} = \left(\frac{\pi}{4}, 0\right) + \varepsilon\left(1, \frac{\pi}{2}\right)$ and $\overline{v}_{\overline{p}} = (2, -3)_{\widetilde{p}} + \varepsilon(1, 2)_{\widetilde{p}}$. The dual tangent map of $\overline{f}$ is obtained from (5.1) as

$$
\begin{aligned}
\overline{f}_{*\overline{p}}(\overline{v}_{\overline{p}}) &= \left(\begin{array}{c} (v_{\widetilde{p}}[f_1], v_{\widetilde{p}}[f_2], v_{\widetilde{p}}[f_3]) \\ +\varepsilon\left(v_{\widetilde{p}}[f_1^o] + v^*_{\widetilde{p}}[f_1], v_{\widetilde{p}}[f_2^o] + v^*_{\widetilde{p}}[f_2], v_{\widetilde{p}}[f_3^o] + v^*_{\widetilde{p}}[f_3]\right) \end{array}\right)_{|_{q+\varepsilon q^*}} \\
&= \left(-\sqrt{2}, \sqrt{2}, -3\right)_{\widetilde{q}} + \varepsilon\left(-\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\right)_{\widetilde{q}},
\end{aligned}
$$

where $q+\varepsilon q^{*}=f\left(\widetilde{p}\right)+\varepsilon f^{o}\left(\widetilde{p}\right)=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right)+\varepsilon\left(0,\sqrt{2},\frac{\pi}{2}\right)$ is a dual point of D^{3}. On the other hand, if we use the dual Jacobian matrix of $\overline{f}$ at dual point $\overline{p}$, we have

$$\begin{aligned}\overline{J}\left(\overline{f}\right)\left(\overline{p}\right) &= J\left(f\right)\left(\widetilde{p}\right)+\varepsilon J\left(f^{o}\right)\left(\widetilde{p}\right)\\ &= \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0\\ \frac{\sqrt{2}}{2} & 0\\ 0 & 1\end{bmatrix}+\varepsilon\begin{bmatrix} -\sqrt{2} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}.\end{aligned}$$

Considering the equality (5.2) , we get the following equality

$$\begin{aligned}\overline{f}_{*\overline{p}}\left(\overline{v}_{\overline{p}}\right) &= J\left(f\right)\left(\widetilde{p}\right)\overrightarrow{v}_{\widetilde{p}}+\varepsilon\left(J\left(f^{o}\right)\left(\widetilde{p}\right)\overrightarrow{v}_{\widetilde{p}}+J\left(f\right)\left(\widetilde{p}\right)\overrightarrow{v}^{*}_{\widetilde{p}}\right)\\ &= \begin{bmatrix} -\sqrt{2}\\ \sqrt{2}\\ -3\end{bmatrix}+\varepsilon\begin{bmatrix} -\frac{5\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\\ 2\end{bmatrix}\\ &= \left(-\sqrt{2},\sqrt{2},-3\right)_{\widetilde{q}}+\varepsilon\left(-\frac{5\sqrt{2}}{2},\frac{\sqrt{2}}{2},2\right)_{\widetilde{q}}.\end{aligned}$$

6. Acknowledgements

The first and the second authors would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial supports during their Ph.D. studies.

REFERENCES

1. A. ALASTUEY, M. CLUSEL, M. MAGRO and P. PUJOL: *Physics and Mathematical Tools*, World Scientific Co. Pte. Ltd., Singapore, 2016.
2. V. BRODSKY and M. SHOHA: *Dual Numbers Representations of Rigid Body Dynamics,* Mechanism and Machine Theory, **34**(1999), 693-718.
3. W. K. CLIFFORD: *Preliminary Sketch of bi-quaternions*, Proc. London Mathematical Society, **4**(1873), 361-395.
4. F. M. DIMENTBERG: *The Screw Calculus and its Applications in Mechanisms*, Izdat. "Nauka", Moskow, USSR, 1965.
5. J. R. DOOLEY and J. M. McCARTHY: *Spatial Rigid Body Dynamics Using Dual Quaternion Components*, Proc. of IEEE International Conf. on Robotics and Automation, Sacremento, CA, **1(**1991), 90-95.
6. N. A. GROMOV: *Contractions and Analytical Continuations of Classical Groups*, Komi Science Center, Syktyvkar, Russia, 1990.
7. N. A. GROMOV: *The matrix Quantum Unitary Cayley-Klein Groups*, J. Phys. A: Math. Gen., **26**(1993), L5-L8.
8. N. A. GROMOV, I. V. KOSTYAKOV and V. V. KURATOV: *Quantum Orthogonal Cayley-Klein Groups and Algebras*, P.Kasperkovitz and D. Grau, World Scientific, Singapore, (1998), 19–21.

9. A. P. KOTELNIKOV: *Screw calculus and Some of Its Applications in Geometry and Mechanics*, Annals of the Imperial University, Kazan, 1895.
10. J. M. McCARTHY: *An Introduction to Theoretical Kinematics*, The MIT Press, London, 1990.
11. B. O'NEILL: *Elementary Differential Geometry*, Elsevier, USA, 2006.
12. G. R. PENNOCH and A. T. YANG: *Dynamics Analysis of Multi-Rigid-Body Open-Chain System*, J. Mech., Trans., and Automation, **105**(1983), 28-34.
13. E. STUDY: *Geometrie der Dynamen*, Druck und Verlag von B.G. Teubner, Leipzig, 1903.

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