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A SURVEY ON THE AUTOMORPHISM GROUPS OF THE COMMUTING GRAPHS AND POWER GRAPHS

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Abstract. Let G be a finite group. The power graph $P(G)$ of a group G is the graph whose vertex set is the set of group elements where two elements are adjacent if one is a power of the other. The commuting graph $\Delta(G)$ of a group G, is the graph whose vertices are the group elements, two of them are joined if they commute. When the vertex set is $G \setminus Z(G)$, this graph is denoted by $\Gamma(G)$. Since the results based on the automorphism groups of these kinds of graphs are so sporadic, in this paper, we give a survey of all results on the automorphism groups of power graphs and commuting graphs obtained in the literature.
Keywords. Finite group; graph; vertex set; commuting graph; automorphism groups.

1. Introduction

There are many connections between graphs and groups. Generating graphs from semigroups and groups has a long history. In 1964, Bosak [6] studied a certain graph over semigroups. In [13], Zelinka studied the intersection graphs of nontrivial subgroups of finite Abelian groups. The well-known study of a directed graphs defined on the elements of a group is the Cayley digraph [7, 22, 40]. The investigation of graphs like these is very important, because they have valuable and numerous applications presented, for example, in the books [27], [28] and [29]. The directed power graph of a group was introduced by Kelarev and Quinn [24]. The definition was formulated so that it applied to semigroups as well. Accordingly, the power graphs of semigroups were first considered in [25], [23] and [26]. It is also explained in the survey [2] that the definition given in [24] covers all undirected graphs as well. This means that the undirected power graphs were also defined in [24] (see [2] for more detailed explanations). All of these papers used only the brief term 'power graph', even though they covered both directed and undirected power graphs. Kelarve and Quinn [23] defined another interesting classes of directed graphs, namely,

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the divisibility graphs of semigroups. Let S be a semigroup, the divisibility graph, $Div(S)$, of a semigroup S is a directed graph with vertex set S and there is an arc from u to v if and only if $u \neq v$ and $u|v$, i.e., the ideal generated by v contains u. On the other hand, the power graph, $\overrightarrow{P}(S)$, of a semigroup S is a directed graph in which the set of vertices is again S and for $a, b \in S$ there is an arc from a to b if and only if $a \neq b$ and $b = a^m$ for some positive integer m.

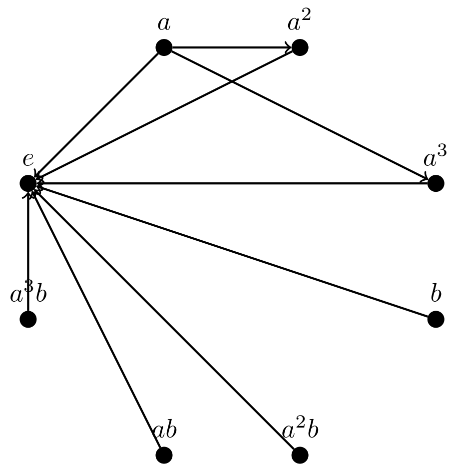

Figure 1. The directed power graph of the dihedral group D_8.

The undirected power graph $P(S)$ was also considered by Chakrabarty, Ghosh and Sen in [11]. Recall that $P(S)$ has vertex set S and two vertices $a, b \in S$ are adjacent if and only if $a \neq b$ and $< a > \subseteq < b >$ or $< b > \subseteq < a >$ (which is equivalent to saying $a \neq b$ and $a^m = b$ or $b^m = a$ for some positive integer m). As a consequence, they proved that $P(G)$ is connected for any finite group G and $P(G)$ is complete if and only if G is a cyclic group of order 1 or p^m [11].

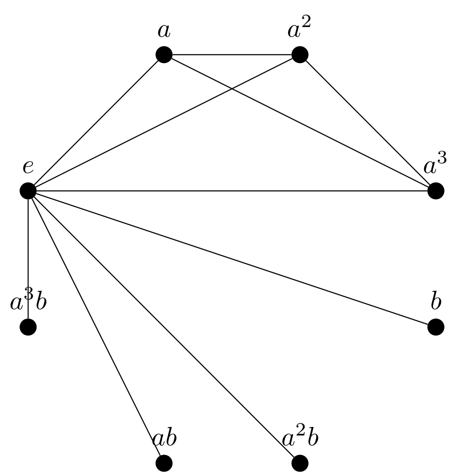

Figure 2. The undirected power graph of the dihedral group D_8.

The undirected power graphs became the main focus of study in [11] and in the subsequent papers by P. J. Cameron et al. [8, 9], which introduced the use of the brief term 'power graph' in the second meaning of an undirected power graph. For a group G, the digraph $\overrightarrow{P}(G)$ was considered in [37] as the main subject of study. The interested readers can be consulted [2, 32, 1] for more information about the power graphs. In this paper, we are also interested in the well-known commuting graphs and their automorphism groups. Let G be a non-abelian group and let $Z(G)$ be the center of G. Associate a graph $\Gamma(G)$ with G as follows: Take $G \setminus Z(G)$ as the vertices of $\Gamma(G)$ and join two distinct vertices x and y, whenever $xy = yx$. The complement of the $\Gamma(G)$ is said to be the noncommuting graph. The noncommuting graph was first considered by Paul Erdos, when he posed the following problem in 1975 [36]: Let G be a group whose noncommuting graph has no infinite complete subgraph. Is it true that there is a finite bound on the cardinalities of complete subgraphs of the noncommuting graph of G? B. H. Neumann [36] answered positively Erdos' question. We refer the readers to [3, 4, 14, 35, 31] for more details about the noncommuting graph. In [1], authors related the power graph to the commuting graph and characterize when they are equal for finite groups. A new graph pops up while considering these graphs, a graph whose vertex set consists of all group elements, in which two vertices x and y are adjacent if they generate a cyclic group. They called this graph as the enhanced power graph of G. The enhanced power graph contains the power graph and is a subgraph of the commuting graph. We consider the commuting graph with vertex set G and denoted it by $\Delta(G)$.

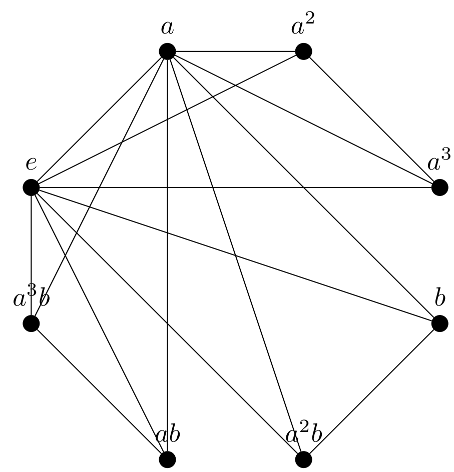

Figure 3. The commuting graph $\Delta(D_8)$.

2. Preliminaries and background information

An action of a group G on a set X is the choice, for each $g \in G$ of a permutation $\pi_g : X \to X$ such that the following two conditions hold:

1. π_e is the identity: $\pi_e(x) = x$ for each $x \in X$,
2. for every g_1, g_2 in G, $\pi_{g_1} \circ \pi_{g_2} = \pi_{g_1 g_2}$.

For example, any group G acts on itself ($X = G$) by left multiplication functions. A group action of G on X is said to be *faithful* if different elements of G act on X in different ways: when $g_1 \neq g_2$ in G, there is an $x \in X$ such that $g_1 \Delta x \neq g_2 \Delta x$. For any graph Γ, we denote the sets of the vertices and the edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. Suppose $v \in V(\Gamma)$ and $V_1(\Gamma) \subseteq V(\Gamma)$, then $N(v)$ is the set of neighbours of v and $\langle V_1(\Gamma) \rangle$ is the subgraph of Γ induced by $V_1(\Gamma)$. The closed neighbourhood of a vertex x, denoted by $N[x]$, is the set of its neighbours and itself. The complement of Γ is the graph $\bar{\Gamma}$ on the same vertices such that two vertices of $\bar{\Gamma}$ are adjacent if and only if they are not adjacent in Γ. For two graphs with disjoint vertex sets V_1 and V_2 their union is the graph H in which $V(H) = V_1 \cup V_2$ and $E(H) = E_1 \cup E_2$. Define nH to be the union of n disjoint copies of G. The automorphism group of a graph Γ is that set of all permutations on $V(\Gamma)$ that fix as a set the edges $E(\Gamma)$. The set of all automorphisms of a graph Γ forms a permutation group, $Aut(\Gamma)$, acting on the object set $V(\Gamma)$. See [10] for the terminology and main results of permutation group theory. Let A and B be permutation groups acting on object sets X and Y, respectively. Define $B \wr A = \{(a, f) \mid a \in A, f : X \to B\}, (a, f)(x, y) = (ax, b_x y)$ where $f(x) = b_x$. $B \wr A$ is said to be *wreath product*. It acts on $X \times Y$ as follows: for each $a \in A$ and any sequence $b_1, b_2, \cdots, b_n$ (where $n = |X|$) in B, there is a unique permutation in $A \wr B$ written $(a; b_1, \cdots, b_n)$, and $(a; b_1, \cdots, b_n)(x_i, y_i) = (ax_i, b_i y_i)$. Suppose S_n denotes the symmetric group on $\{1, 2, \cdots, n\}$, φ is the Euler's totient function . In what follows, we describe some important results relating the automorphism groups of a graph which are crucial in this paper. Frucht [18] described if Γ is a connected graph, then $Aut(n\Gamma) \cong (Aut(\Gamma)) \wr S_n$, if no component of Γ_1 is isomorphic with a component of Γ_2, then $Aut(\Gamma_1 \cup \Gamma_2) \cong Aut(\Gamma_1) \times Aut(\Gamma_2)$ and applying the last two theorems we have the result: Let $\Gamma = n_1\Gamma_1 \cup n_2\Gamma_2 \cup \cdots \cup n_r\Gamma_r$, where n_i is the number of components of Γ isomorphic to Γ_i, then

$$Aut(\Gamma) \cong ((Aut(\Gamma_1)) \wr S_{n_1}) \times ((Aut(\Gamma_2)) \wr S_{n_2}) \times \cdots \times ((Aut(\Gamma_r)) \wr S_{n_r}).$$

An operation $\cdot$ on the set S is associative if it satisfies the following associative law: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in S$. A semigroup is a set S equipped with an associative binary operation $\cdot$. The set of the orders of all elements of G is denoted by $\pi_e(G)$ and is said to be the *spectrum* of G. For $n \in N$, the cyclic group of order n can be defined as the group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of residues modulo n, the set $< g >= \{g^n \mid n \in \mathbb{Z}\}$ is the cyclic group generated by g in G. For a prime p, a group G is said to be an elementary abelian p-group if G is finite, abelian and every nontrivial element of G has order p. A group G is an AC-group, whenever the centralizers of non-central elements are abelian. The dihedral group D_{2n} is an example of an *AC-group*. The group G is said to be an *EPPO-group*, if all elements of G have prime power order.

3. Automorphism groups of power graphs

The first result about the automorphism groups of power graphs was obtained by P. Cameron in [8], where he explained that when the automorphism group and its graph are equal. P. Cameron proved the only finite group G for which $Aut(G) = Aut(P(G))$ is the Klein group $Z_2 \times Z_2$.

In 2013, Doostabadi, Erfanian and Jafarzadeh asserted that the full automorphism group of the power graph of the cyclic group Z_n is isomorphic to the direct product of some symmetry groups.

Conjecture 3.1. [16] *For every positive integer n,*

$$Aut(P(Z_n)) \cong S_{\varphi(n)+1} \times \prod_{d \in D(n) \setminus \{1,n\}} S_{\varphi(d)}$$

where $D(n)$ is the set of positive divisors of n, and φ is the Euler's totient function.

In fact, if n is a prime power, then $P(Z_n)$ is a complete graph by [11] which implies that $Aut(P(Z_n) \cong S_n$. Hence, the conjecture does not hold if $n = p^m$ for any prime p and integer $m > 2$. In [17], proved that this conjecture holds for the remaining case. Feng, Ma and Wang [17], describe the full automorphism group of the power (di)graph of an arbitrary finite group. As an application, this conjecture is valid if n is not a prime power. Denote by $C(G)$ the set of all cyclic subgroups of G. For $C \in C(G)$, let $[C]$ denote the set of all generators of C. Write

$$C(G) = \{C_1, \cdots C_k\} \textit{ and } [C_i] = \{[C_i]_1, \cdots [C_i]_{s_i}\}.$$

Define $\mathbf{P}(G)$ as the set of permutations σ on $C(G)$ preserving order, inclusion and noninclusion, i.e., $|C_i^\sigma| = |C_i|$ for each $i \in \{1, \cdots, k\}$ and $C_i \subseteq C_j$ if and only if $C_i^\sigma \subseteq C_j^\sigma$. Note that $\mathbf{P}(G)$ is a permutation group on C(G). This group induces the faithful action on the set G:

$$G \times \mathbf{P}(G) \longrightarrow G, \qquad ([C_i]_j, \sigma) \longmapsto [C_i^\sigma]_j. \tag{3.1}$$

For $\Omega \subseteq G$, let S_Ω denote the symmetric group on Ω. Since G is the disjoint union of $[C_1], \cdots, [C_k]$, we get the faithful group action on the set G:

$$G \times \prod_{i=1}^{k} S_{[C_i]} \longrightarrow G, \quad ([C_i]_j, (\xi_1, \cdots, \xi_k)) \longmapsto ([C_i]_j)^{\xi_i}. \tag{3.2}$$

By using the above-mentioned symbols we have:

Theorem 3.1. [17] *Let G be a finite group. Then*

$$Aut(\overrightarrow{P}(G)) = (\prod_{i=1}^{k} S_{[C_i]}) \times \boldsymbol{P}(G),$$

where $\boldsymbol{P}(G)$ and $\prod_{i=1}^{k} S_{[C_i]}$ act on G as in (3.1) and (3.2), respectively.

In the power graph $P(G)$, for $x, y \in G$, define $x \equiv y$ if $N[x] = N[y]$. Observe that $\equiv$ is an equivalence relation. Let $\bar{x}$ denote the equivalence class containing x. Write

$$\mathcal{U}(G) = \{\bar{x} | x \in G\} = \{\bar{u_1}, \cdots, \bar{u_l}\}.$$

Since G is the disjoint union of $u_1, \cdots, u_l$, the following is a faithful group action on the set G:

$$G \times \prod_{i=1}^{l} S_{\bar{u}_i} \longrightarrow G, \quad (x, (\tau_1, \tau_2, \cdots, \tau_l)) \longmapsto x^{\tau_i}, \quad where \quad x \in \bar{u}_i. \tag{3.3}$$

Similar to the last theorem, for the automorphism groups of undirected power graphs we have:

Theorem 3.2. [17] *Let G be a finite group. Then*

$$Aut(P(G)) = (\prod_{i=1}^{l} S_{\bar{u}_i)} \times \boldsymbol{P}(G),$$

where $\boldsymbol{P}(G)$ and $\prod_{i=1}^{l} S_{\bar{u}_i}$ act on G as in (3.1) and (3.3), respectively.

By combining Theorems 3.1 and 3.2, the authors in [17], obtained that $Aut(P(G)) = Aut(\overrightarrow{P}(G))$ if and only if $x = [x]$ for each $x \in G$. Indeed, this result demonstrates relationship between power graphs and directed power graphs.

A graph Γ is said to be a *subgraph* of another graph Δ (or Δ is a supergraph of Γ), if $V(\Gamma) \subset V(\Delta)$ and $E(\Gamma) \subset E(\Delta)$. Hamzeh and Ashrafi [19] defined the main supergraph $\mathcal{S}(G)$ of $P(G)$ with the vertex set G and two elements $x, y \in G$ are adjacent if and only if $o(x)|o(y)$ or $o(y)|o(x)$ and proved that there is not a group G, such that $Aut(\mathcal{S}(G)) = Aut(G)$. In what follows, $\Omega_{a_i}(G) = |\{y|o(y) = a_i\}|$. Authors in [19] also define the graph Δ with vertex set $V(\delta) = \pi_e(G)$ and two vertices a_i and a_j are adjacent if and only if $a_i|a_j$ or $a_j|a_i$.

Theorem 3.3. [19] *Let G be a finite group with spectrum $\pi_e(G) = \{a_1, \cdots, a_k\}$ and choose a representative set $\{t_1, t_2, \cdots, t_k\}$, where for each i, $1 \leq i \leq k, ti \in K_{\Omega_{a_i}}(G)$. Then,*

1. *If $deg(t_i)$'s are distinct then $Aut(\mathcal{S}(G)) = S_{\Omega_{a_1}}(G) \times \cdots \times S_{\Omega_{a_k}}(G)$.*

2. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, any two distinct vertices of $K_{\Omega_{a_{i_1}}}(G), \cdots, K_{\Omega_{a_{i_r}}}(G)$ are adjacent and $N_\Delta[a_{i_1}] = \cdots = N_\Delta[a_{i_r}]$ then $Aut(\mathcal{S}(G))$ has a subgroup isomorphic to $S_{\Omega_{a_{i_1}}(G)+\cdots+\Omega_{a_{i_r}}(G)}$.*

3. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, all vertices of $K_{\Omega_{a_{i_1}}}(G), \cdots, K_{\Omega_{a_{i_r}}}(G)$ are adjacent and $N_\Delta[a_{i_l}]$'s are distinct then $Aut(\mathcal{S}(G))$ has a subgroup isomorphic to $S_{\Omega_{a_{i_1}}}(G) \times \cdots \times S_{\Omega_{a_{i_r}}}(G)$.*

4. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, $N_\Delta[a_{i_1}] = \cdots = N_\Delta[a_{i_r}]$ and for each two $m, n, 1 \leq m, n \leq r$, $K_{\Omega_{a_{i_m}}}(G)$ and $K_{\Omega_{a_{i_n}}}(G)$ are disjoint then $Aut(\mathcal{S}(G))$ has a subgroup isomorphic to $S_{\Omega_{a_{i_1}}}(G) \wr S_r$.*

5. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$,$N_\Delta[a_{i_l}]$'s are distinct and for each $m, n, 1 \leq m, n \leq r$, $K_{\Omega_{a_{i_m}}}(G)$ and $K_{\Omega_{a_{i_n}}}(G)$ are disjoint then $Aut(\mathcal{S}(G))$ has a subgroup isomorphic to $S_{\Omega_{a_{i_1}}}(G) \times \cdots \times S_{\Omega_{a_{i_r}}}(G)$.*

6. *$Aut(\mathcal{S}(G)) = A_1 \times \cdots \times A_q$, where $A_i, 1 \leq i \leq q$, are subgroups appeared in Cases (2–5).*

In [[20], Theorem 2.8], it is proved that if G is an EPPO-group of order $p_1^{n_1} \cdots p_k^{n_k}$ and $V_i = \{1 \neq g \in G \mid o(g)|p_i^{ni}\}$ then $\mathcal{S}(G) = K_1 + (\bigcup_{i=1}^k K_{|Vi|})$. The authors applied the structure of $\mathcal{S}(G)$ to determine its automorphism.

Theorem 3.4. [19] *Let G be a finite group and $e_1, \cdots, e_t$ are distinct values of $|V_1|, \cdots, |V_k|$. Define $B_i = |\{|V_j| \mid |V_j| = e_i\}|$. Then,*

$$Aut(\mathcal{S}(G)) = (S_{|V_1|} \wr S_{B_1}) \times \cdots \times (S_{|V_k|} \wr S_{B_k}).$$

Suppose G is a finite group and $C(G) = \{C_1, \cdots, C_k\}$ is the set of all cyclic subgroups of G. Define L_G to be the graph with vertex set $C(G)$ in which two cyclic subgroups C_i and C_j are adjacent if one is contained in the other or there is a cyclic subgroup C_k such that $C_i \subseteq C_k$ and $C_j \subseteq C_k$. It is clear that the subgraphs of $P(G)$ induced by a cyclic subgroup are complete. So, $P(G) = W_G[K_{b_1}, K_{b_2}, \cdots, K_{b_k}]$ with $b_i = \varphi(|C_i|)$.

Theorem 3.5. [19] *Let G be a finite group with $C(G) = \{C_1, \cdots, C_k\}$ and choose a representative set $\{t_1, t_2, \cdots, t_k\}$, where for each i, $1 \leq i \leq k, ti \in K_{b_i}$. Then,*

1. *If $deg(t_i)$'s are distinct then $Aut(P(G)) = S_{b_1} \times \cdots \times S_{b_k}$.*

2. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, any two distinct vertices of $K_{b_{i_1}}, \cdots, K_{b_{i_r}}$ are adjacent and $N_{W_G}[C_{i_1}] = \cdots = N_{W_G}[C_{i_r}]$ then $Aut(P(G))$ has a subgroup isomorphic to $S_{b_{a_{i_1}}+\cdots+b_{a_{i_r}}}$.*

3. *If* $deg(t_{i_1}) = \cdots = deg(t_{i_r})$*, all vertices of* $K_{b_{i_1}}, \cdots, K_{b_{i_r}}$ *are adjacent and* $N_{W_G}[C_{i_l}]$*'s are distinct then* $Aut(P(G))$ *has a subgroup isomorphic to* $S_{b_{i_1}} \times \cdots \times S_{b_{i_r}}$.
4. *If* $deg(t_{i_1}) = \cdots = deg(t_{i_r})$*,* $N_{W_G}[C_{i_1}] = \cdots = N_{W_G}[C_{i_r}]$ *and for each two* $m, n, 1 \leq m, n \leq r$*,* $K_{b_{i_m}}$ *and* $K_{b_{i_n}}$ *are disjoint then* $Aut(P(G))$ *has a subgroup isomorphic to* $S_{b_{i_1}} \wr S_r$.
5. *If* $deg(t_{i_1}) = \cdots = deg(t_{i_r})$*,* $N_{W_G}[C_{i_l}]$*'s are distinct and for each* $m, n, 1 \leq m, n \leq r$*,* $K_{b_{i_m}}$ *and* $K_{b_{i_n}}$ *are disjoint then* $Aut(P(G))$ *has a subgroup isomorphic to* $S_{b_{i_1}} \times \cdots \times S_{b_{i_r}}$.
6. $Aut(P(G)) = A_1 \times \cdots \times A_q$*, where* $A_i, 1 \leq i \leq q$*, are subgroups appeared in Cases (2–5).*

3.1. Examples

In this section, we present $Aut(P(G))$ and $Aut(\overrightarrow{P}(G))$ for some families of finite groups such as $Z_n, Z_n^p, D_{2n}, Q_{4n}, U_{6n}, V_{8n}$ and so on. These results obtained in several papers in different ways. In [5], the authors used the graph structure from [30] and computed the automorphism groups of $P(G)$ for the above groups. In [17], the authors by using Theorem 3.1 and Theorem 3.2, computed the automorphism groups of $P(G)$ and $\overrightarrow{P}(G)$ for these groups. In [19], authors obtained these results from Theorem 3.3.

Example 3.1. [17] If n be a positive integer then,

$$\begin{aligned} Aut(\overrightarrow{P}(Z_n)) &\cong \prod_{d \in D(n)} S_{\varphi(d)}, \\ Aut(P(Z_n)) &\cong \begin{cases} S_n & n \text{ is a prime power} \\ S_{\varphi(n)+1} \times \prod_{d \in D(n) \setminus \{1,n\}} S_{\varphi(d)} & \text{otherwise} \end{cases}, \end{aligned}$$

and if $n \geq 2$ then,

$$Aut(P(Z_p^n)) = Aut(\overrightarrow{P}(Z_p^n) \cong S_{p-1} \wr S_m,$$

where $m = \frac{p^n - 1}{p-1}$ and Z_p^n denote the elementary abelian $p-$group.

In the [21, 15], the dihedral group D_{2n}, semi-dihedral group SD_{2^n}, generalized quaternion group of Q_{4n}, semidihedral groups SD_{8n} are defined by the following presentations:

$$\begin{aligned} D_{2n} &= \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\ SD_{2^n} &= \langle a, b \mid a^{2^n} = b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\ Q_{4n} &= \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, \\ U_{6n} &= \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle, \\ V_{8n} &= \langle a, b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle. \end{aligned}$$

Now, we are ready to state next example.

Example 3.2. [17] For $n \geq 3$,

$$Aut(\overrightarrow{P}(D_{2n})) \cong \prod_{d\in D(n)} S_{\varphi(d)} \times S_n,$$

$$Aut(P(D_{2n})) \cong \begin{cases} S_{n-1} \times S_n, & n \text{ is a prime power} \\ S_n \times \prod_{d\in D(n)} S_{\varphi(d)} & otherwise \end{cases},$$

and let $n \geq 3$ then,

$$Aut(\overrightarrow{P}(Q_{4n})) \cong \prod_{d\in D(2n)} S_{\varphi(d)} \times (S_2 \wr S_n),$$

$$Aut(P(Q_{4n})) \cong \begin{cases} S_2 \times S_{2n-2} \times (S_2 \wr S_n), & n \text{ is a power of } 2 \\ \prod_{d\in D(2n)} S_{\varphi(d)} \times (S_2 \wr S_n) & otherwise \end{cases}.$$

Example 3.3. [5] If k is nonnegative integer and satisfies $n = 3^k t$ for some positive integer t such that $3 \nmid t$ then,

$$Aut(P(U_{6n})) \cong \begin{cases} \prod_{d|3n} S_{\varphi(d)} \times \prod_{d|2n, d\nmid n} S_{\varphi(d)} \wr S_3 & k=0 \\ \prod_{d|2n, d\nmid n} S_{\varphi(d)} \wr S_3 \times \prod_{d|n} S_{\varphi(d)} \times \prod_{d|n, d\nmid t} S_{\varphi(d)} \wr S_3 & k=1 \\ \prod_{d|2n, d\nmid n} S_{\varphi(d)} \wr S_3 \times \prod_{d|n} S_{\varphi(d)} \times \prod_{d|3t, d\nmid t} S_{\varphi(d)} \wr S_3 & \\ \times \prod_{d|n, d\nmid 3t} S_{\varphi(d)} \wr S_2 & k \geq 2 \end{cases},$$

if $n = 2^k t$ for a nonnegative k and some positive odd integer t then,

$$Aut(P(V_{8n})) \cong \begin{cases} S_{2n} \times S_2 \wr S_n \times \prod_{d|2n, d\nmid n} S_{\varphi(d)} \wr S_2 \times \prod_{d|2n} S_{\varphi(d)} & k=0 \\ S_{2n+1} \times S_2 \wr S_n \times \prod_{l=1}^{k-1} S_{2^l}^2 \times S_{2^k} \wr S_2 & t=1, k \geq 1 \\ S_{2n} \times S_2 \wr S_n \times \prod_{d|t} S_{\varphi(d)}^4 \times \prod_{s=2}^{k} \prod_{d|2^s t, d\nmid 2^{s-1} t} S_{\varphi(d)}^2 & \\ \times \prod_{d|2^{k+1} t, d\nmid 2^k t} S_{\varphi(d)} \wr S_2 & t>1, k \geq 1 \end{cases},$$

also,

$$Aut(P(SD_{8n})) \cong \begin{cases} S_{4n-2} \times S_{2n} \times (S_2 \wr S_n), & n \text{ is a power of } 2 \\ \prod_{d|4n} S_{\varphi(d)} \times S_{2n} \times (S_2 \wr S_n) & otherwise \end{cases}.$$

The smallest sporadic group is the first Mathieu group M_{11}, it has order 7920. There are many presentations for the group M_{11}, we give two of its known presentation, [39].

$$M_{11} \cong < a, b, c | a^{11} = b^5 = c^4 - (ac)^3 = 1, b^4 ab = a^4, c^3 bc = b^2 >,$$
$$\cong < a, b, c, d | a^2 = b^2 = c^2 = d^2 = (ab)^5 = (bc)^3 = (bd)^4 = (cd)^3 = (abdbd)^3 = 1 > .$$

The paper by Around (1960) increased the interest to finite simple groups, as Janko in Australia found (1965) the first new sporadic group J_1 a century later after Mathieu's. It turns out that J_1 had order 175560. A presentation for J_1 in terms of its standard generators is given below [12]:

$$J_1 \cong < a, b | a^2 = b^3 = (ab)^7 = (ab(abab^{-1})^3)^5 = (ab(abab^{-1})^6 abab(ab^{-1})^2)^2 = 1 > .$$

The automorphism groups of M_{11} and J_1 are determined as follows:

Example 3.4. [5] Let M_{11} be the first Mathieu group and J_1 be the first Janko group, then,

$$\begin{aligned} Aut(P(M_{11})) &\cong (S_{10} \wr S_{144}) \times (S_4 \wr S_{396}) \times (S_2 \wr S_{55}) \times ((S_6 \wr S_3) \times (S_2 \wr S_4) \times S_2) \wr S_{165}, \\ Aut(P(J_1)) &\cong (S_{10} \wr S_{596}) \times (S_6 \wr S_{4180}) \times (S_{18} \wr S_{1540}) \\ &\times ((S_2 \times S_8 \times S_4 \times (S_4 \wr S_3) \times (S_2 \wr S_5)) \wr S_2) \wr S_{1463}. \end{aligned}$$

Moreover, in [30] the automorphism groups of $P(Z_{pq})$, $P(Z_{pqr})$ and $P(Z_{p^2q^2})$ are calculated as follows:

$$\begin{aligned} Aut(P(Z_{pq})) &\cong S_{\varphi(pq)+1} \times S_{p-1} \times S_{q-1}, \\ Aut(P(Z_{pqr})) &\cong S_{\varphi(pqr)} \times S_{p-1} \times S_{q-1} \times S_{r-1} \times S_{\varphi(pq)} \times S_{\varphi(pr)} \times S_{\varphi(qr)}, \\ Aut(P(Z_{p^2q^2})) &\cong S_{\varphi(p^2q^2)+1} \times S_{p-1} \times S_{\varphi(p^2)} \times S_{q-1} \times S_{\varphi(q^2)} \times S_{\varphi(pq)} \times S_{\varphi(pq^2)} \times S_{\varphi(p^2q)}. \end{aligned}$$

As we mentioned in above Theorem 3.4 is playing a main role in finding automorphism group of power graphs. In [19], the authors obtained the following results from Theorem 3.3.

Example 3.5. [19] If n is odd, then

$$Aut(\mathcal{S}(D_{2n})) = \begin{cases} S_{n-1} \times S_n & n \textit{ is a prime power} \\ S_n \times \prod_{d|n} S_{\varphi(d)} & \textit{otherwise} \end{cases},$$

and if n is even then

$$Aut(\mathcal{S}(D_{2n})) = \begin{cases} S_{2n} & n \textit{ is a power of } 2 \\ S_{\varphi(n)+1} \times S_{n+1} \prod_{\{1,n,2\} \neq d|n} S_{\varphi(d)} & \textit{otherwise} \end{cases},$$

if n is odd, then

$$Aut(\mathcal{S}(T_{4n})) = S_{2n} \times \prod_{d|2n} S_{\varphi(d)},$$

and if n is even then

$$Aut(\mathcal{S}(T_{4n})) = \begin{cases} S_{4n} & n \textit{ is a power of } 2 \\ S_{\varphi(2n)+1} \times S_{2n+2} \prod_{\{1,2n,4\} \neq d|2n} S_{\varphi(d)} & \textit{otherwise} \end{cases},$$

for arbitrary n,

$$Aut(\mathcal{S}(SD_{8n})) = \begin{cases} S_{8n} & n \textit{ is a power of } 2 \\ S_{\varphi(4n)+1} \times S_{2n+1} \times S_{2n+2} \prod_{\{1,4n,2,4\} \neq d|4n} S_{\varphi(d)} & \textit{otherwise} \end{cases},$$

if $n = 2^k$ then $Aut(\mathcal{S}(V_{8n})) \cong S_{8n}$, and if n is an odd prime then $Aut(\mathcal{S}(V_{8n})) = S_{2n+3} \times S_{2n} \times S_{3\varphi(n)} \times \prod_{\{1,2n,2\} \neq d|2n} S_{\varphi(d)}$.

4. Automorphism groups of commuting graphs

The commuting graphs $\Delta(G)$ and $\Gamma(G)$ of a group G are defined in the introduction. The following theorem established the relation between $Aut(G)$, $Aut(\Delta(G))$ and $Aut(\Gamma(G))$.

Theorem 4.1. [33] *Let G be a finite group, then*

1. $Aut(G) = Aut(\Delta(G))$ *if and only if* $|G| = 1$.
2. $Aut(\Delta(G)) \cong Aut(\Gamma(G)) \times S_{Z(G)}$.

Mirzargar, Pach and Ashrafi studied the subgroups of $Aut(\Delta(G))$ in [33, 34]. The first subgroups are $Aut(\Gamma(G))$ and $Aut(G)$, then they added some automorphisms of graph to $Aut(G)$ and constructed bigger subgroups. Define two permutations $\Phi_{x,y}, \phi : G \to G$ as follows: $\Phi_{x,y}$ fixed each element $a \in G \setminus \{x, y\}$ and maps x into y and vice-versa; and, the permutation ϕ is defined by $x \to x^{-1}$ for each element $x \in G$. They also defined $Aut^*(G) = \langle Aut(G), \phi\rangle$ and considered to the equality of the subgroups and the main group.

Theorem 4.2. [33] $Aut^*(G) = Aut(\Delta(G))$ *if and only if* $G \cong S_3$.

Let the cosets $Z(G)x_1, Z(G)x_2, \cdots, Z(G)x_{m-1}$ of the group $G/Z(G)$ and define a new graph $\Delta^u(G)$ with $V(\Delta^u(G)) = \{x_0 = 1, x_1, \cdots, x_{m-1}\}$ and $E(\Delta^u(G)) = \{x_ix_j | x_ix_j = x_jx_i, 0 \le i < j \le m-1\}$. Notice when $|Z(G)| = 1$ then $\Delta(G) \cong \Delta^u(G)$. It is clear that every two elements in one of these cosets commute. Hence we have a complete graph in any of these cosets. On the other hand, if there exists $x_i \in Z(G)x_i, x_j \in Z(G)x_j$ satisfying $x_ix_j = x_jx_i$, then for every $y_i \in Z(G)x_i, y_j \in Z(G)x_j$ we have $y_iy_j = y_jy_i$. Finally, the set of all $\phi \in Aut(\Delta(G))$ such that for $a, b \in G$ if $ab^{-1} \in Z(G)$, then $\phi(a)\phi(b)^{-1} \in Z(G)$ is denoted by T. These notations are applied in [33] to prove two following theorems.

Theorem 4.3. [33] *Let G be a group. Then,*

1. $Aut(\Delta^u(G))$ *is a subgroup of* $Aut(\Delta(G))$. *Moreover,* $Aut(\Delta^u(G)) = Aut(\Delta(G))$ *if and only if* $|Z(G)| = 1$.
2. *If G is not centerless then T is a subgroup of $Aut(\Delta(G))$, and $Aut(\Delta(G)) = T$ if and only if for each pair a, b of elements of G with $C_G(a) = C_G(b)$, we have $ab^{-1} \in Z(G)$.*

Theorem 4.4. [33] *Let $|Z(G)| \ge 2$, where G be a nonabelian group. If $T = Aut(\Delta(G))$ then $G/Z(G)$ is an elementary abelian $2-$group.*

For a finite group G define a labelled graph $\Delta^v(G)$ as follows. For $a, b \in G$ let $a \sim b$ if $C_G(a) = C_G(b)$. Clearly, $\sim$ is an equivalence relation, the equivalence class of $a \in G$ is $A(a) = \{x | C_G(x) = C_G(a)\}$. Let us denote the equivalence classes by $A_1, \ldots, A_k$, these are the vertices of $\Delta^v(G)$. Two vertices A_i and A_j are connected if and only if $a_i a_j = a_j a_i$, for some $a_i \in A_i, a_j \in A_j$. At first, we note that if there exists $a_i \in A_i, a_j \in A_j$ satisfying $a_i a_j = a_j a_i$, then for every $b_i \in A_i, b_j \in A_j$ we have $a_j \in C_G(a_i) = C_G(b_i)$. So, $b_i \in C_G(a_j) = C_G(b_j)$ implies that $b_i b_j = b_j b_i$. Each equivalence class is the union of some sets of the form $tZ(G)$, hence there exists a positive integers c_i such that $|A_i| = c_i|Z(G)|$. Let $\alpha(A_i) = c_i$ be the label of the vertex A_i in $\Delta^v(G)$. One can see $\phi : V(\Delta^v(G)) \to V(\Delta^v(G))$ is an automorphism of the labelled graph $\Delta^v(G)$ if ϕ is a bijection, it preserves the edges (and the non-edges) and it preserves the labels. The automorphism group formed by these automorphisms is denoted by $Aut(\Delta^v(G))$. Define $S_{A_i} = \{f_\sigma \mid \sigma \in S_{|A_i|}, \forall x \in A_i, f_\sigma(x) = \sigma(x), \forall x \notin A_i, f_\sigma(x) = x\}$, $1 \leq i \leq k$. Clearly, S_{A_i} is a subgroup of $Aut(\Delta(G))$. The connection between $Aut(\Delta(G))$ and $Aut(\Delta^v(G))$ is described by the following theorem:

Theorem 4.5. [33] *There is a subgroup A of $Aut(\Delta(G))$ such that $A \cong Aut(\Delta^v(G))$ and $Aut(\Delta(G)) = \langle S_{A_1}, \cdots, S_{A_k}\rangle \times A$.*

In [38], Rocke proved that the following are equivalent:

1. G has abelian centralizers;
2. If $xy = yx$, then $C_G(x) = C_G(y)$ whenever $x, y \notin Z(G)$;
3. If $xy = yx$ and $xz = zx$, then $yz = zy$ whenever $x \notin Z(G)$;
4. If U and B are subgroups of G and $Z(G) < C_G(U) \leq C_G(B) < G$ then $C_G(U) = C_G(B)$.

Therefore, the intersection of two proper element centralizers of an AC-group is the center of G. If G is an AC-group, then $\Delta(G)$ is a union of some complete graphs with all vertices adjacent to the elements of $Z(G)$. So, $\Delta(G)$ is $n_1(C_G(x_1) \setminus Z(G)) \cup n_2(C_G(x_2) \setminus Z(G)) \cup \cdots \cup (n_r C_G(x_r) \setminus Z(G))$ and also every element of $Z(G)$ is adjacent to all elements of G, such that for each $i, 1 \leq i \leq r$, we have n_i isomorphic components with complete graph of size $|C_G(x_i) \setminus Z(G)|$. In [33], the authors proved that if G is an AC-group with the above notations then,

$$\begin{aligned} Aut(\Delta(G)) &\cong ((S_{|C_G(x_1)|-|Z(G)|}) \wr S_{n_1}) \times ((S_{|C_G(x_2)|-|Z(G)|}) \wr S_{n_2}) \times \cdots \\ &\times ((S_{|C_G(x_n)|-|Z(G)|}) \wr S_{n_r}) \times S_{Z(G)}. \end{aligned}$$

Finally, from [33], $|Aut(\Delta(G))|$ can not be a prime power or a square-free number. Moreover, $|Aut(\Delta(G))| = 1$ if and only if G is trivial, $Aut(\Gamma(G))$ is abelian if and only if G is a group of order 1 or 2. Also if $|G| > 2$ then $Aut(\Delta(G))$ is a nonabelian group.

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REFERENCES

1. G. Aalipour, S. Akbari, P. J. Cameron, R. Nikandish and F. Shaveisi: *On the structure of the power graph and the enhanced power graph of a group*. Electron. J. Comb. **24** (2017), 3–16.
2. J. Abawajy, A. Kelarev and M. Chowdhury: *Power graphs: a survey*. Electron. J. Graph Theory Appl. (EJGTA) **1** (2013), 125-–147.
3. A. Abdollahi, S. Akbari and H. R. Maimani :*Non-commuting graph of a group*. J. Algebra **298** (2006), 468–492.
4. A. Abdollahi and H. Shahverdi :*Non-commuting graphs of nilpotent groups*. Commun. Algebra **42** (2014), 3944-–3949.
5. A. R. Ashrafi, A. Gholami and Z. Mehranian :*Automorphism group of certain power graphs of finite groups*. Electron. J. Graph Theory Appl. (EJGTA) **5** (2017), 70-–82.
6. J. Bosak :*The graphs of semigroups, Theory of Graphs and Application*. Academic Press, New York, 1964.
7. F. Budden: *Cayley graphs for some well-known groups*. Math. Gaz. **69** (1985), 271–278.
8. P. J. Cameron and S. Ghosh: *The power graph of a finite group*. Discrete Math. **311** (2011), 1220–1222.
9. P. J. Cameron: *The power graph of a finite group II*. J. Group Theory, **13** (2010), 779–783.
10. P. J. Cameron: *Permutation Groups*. Cambridge Univ. Press, Cambridge, 1999.
11. I. Chakrabarty, S. Ghosh and M. K. Sen:*Undirected power graphs of semigroups*. Semigroup Forum **78** (2009), 410–426.
12. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson: *Atlas of finite simple groups, Maximal subgroups and ordinary characters for simple groups*. Oxford University Press, Eynsham, 1985.
13. B. Zelinka:*Intersection graphs of finite Abelian groups*. Czech. Math. J. **25** (1975), 171–174.
14. M. R. Darafsheh: *Groups with the same non-commuting graph*. Discrete Applied Math. **157** (2009), 833–837.
15. M. R. Darafsheh and N. S. Poursalavati: *On the existence of the orthogonal basis of the symmetry classes of tensors associated with certain groups*. Sut J. Math. **37** (2001), 1–17.
16. A. Doostabadi, A. Erfanian and A. Jafarzadeh: *Some results on the power graph of groups*.In: The Extended Abstracts of the 44th Annual Iranian Mathematics Conference, Ferdowsi University of Mashhad, Iran, 2013, pp. 27–30.

17. M. Feng, X. Ma and K. Wang: *The full automorphism group of the power (di)graph of a finite group*. European J. Combin. **52** (2016), 197–206.

18. R. Frucht: *On the groups of repeated graphs*. Bull. Amer. Math. Soc. **55** (1949), 418–420.

19. A. Hamzeh and A. R. Ashrafi: *Automorphism groups of supergraphs of the power graph of a finite group*. Eur. J. Comb. **60** (2017), 82–88.

20. A. Hamzeh and A. R. Ashrafi: *The order supergraph of the power graph of a finite group*. Turk. J. Math. **42** (2018), 1978–1989.

21. G. James and M. Liebeck: *Representations and Characters of Groups*. 2nd ed., Cambridge University Press, New York, 2001.

22. J. A. Gallian: *Contemporary Abstract Algebra*. Narosa Publishing House, London, 1999.

23. A. V. Kelarev and S. J. Quinn: *Directed graph and combinatorial properties of semigroups*. J. Algebra **251** (2002), 16–26.

24. A. V. Kelarev and S. J. Quinn: *A combinatorial property and power graphs of groups*. The Vienna Conference, Contrib. General Algebra. **12** (2000), 229–235.

25. A. V. Kelarev, S. J. Quinn and R. Smolikova: *Power graphs and semigroups of matrices*. Bull. Austral. Math. Soc. **63** (2001), 341–344.

26. A. V. Kelarev and S. J. Quinn: *A combinatorial property and power graphs of semigroups*. Comment. Math. Univ. Carolinae. **45** (2004), 1–7.

27. A. V. Kelarev: *Graph Algebras and Automata*. Marcel Dekker, New York, 2003.

28. A. V. Kelarev: *Ring Constructions and Applications*. World Scientifc, River Edge, NJ, 2002.

29. A. V. Kelarev, J. Ryan and J. Yearwood: *Cayley graphs as for data mining: The infuence of asymmetries*. Discrete Mathematics. **309** (2009), 5360–5369.

30. Z. Mehranian, A. Gholami and A. R. Ashrafi: *A note on the power graph of a finite group*. Int. J. Group Theory **5** (2016), 1–10.

31. M. Mirzargar and A. R. Ashrafi: *Some distance-based topological indices of the non-commuting graph*. Hacet. J. Math. Stat. **41** (2012), 515–526.

32. M. Mirzargar, A. R. Ashrafi and M. J. Nadjafi-Arani: *On the power graph of a finite group*. Filomat **26** (2012), 1201–1208.

33. M. Mirzargar, P. P. Pach and A. R. Ashrafi: *The automorphism graph of commuting graph of a finite group*. Bull. Korean Math. Soc. **51** (2014), 1145–1153.

34. M. Mirzargar, P. P. Pach and A. R. Ashrafi: *Remarks On Commuting Graph of a Finite Group*. Electron. Notes Discrete Math. **45** (2014), 103–106.

35. A. R. Moghaaddamfar: *About noncommuting graphs*, Siberian Math. J. **47** (2006), 911–914.

36. B. H. Neumann, *A problem of Paul Erdos on groups*. J. Aust. Math. Soc. Ser. **21** (1976), 467–472.

37. G. R. Pourgholi and H. Yousefi-Azari: *On the 2-connected power graphs of finite groups*. Australiasian J. Combinatorics **62** (2015), 1–7.

38. D. M. Rocke: *p-groups with abelian centralizers*. Proc. London Math. Soc. **30** (1975), 55–75.

39. H. E. Rose: *A Course on Finite Groups*. Cambridge University press, Cambridge, 1978.

40. D. Witte, G. Letzter and J. A. Gallian: *On Hamiltonian circuits in Cartesian products of Cayley digraphs*. Discrete Math. **43** (1983), 297–307.

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