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# DOMINATION PARAMETERS AND DIAMETER OF ABELIAN CAYLEY GRAPHS

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**Abstract.** Using the domination parameters of Cayley graphs constructed out of $\mathbb{Z}_p \times \mathbb{Z}_m$, where $m \in \{p^\alpha, p^\alpha q^\beta, p^\alpha q^\beta r^\gamma\}, p, q, r$ are distinct prime numbers and $\alpha, \beta, \gamma$ are positive integers, in this paper we have discussed the total and connected domination number and diameter of these Cayley graphs.
**Key words:** Cayley graph, total dominating set, connected dominating set, total domination number, connected domination number

## 1. Introduction and Preliminaries

Let $(G, \cdot)$ be a group and $S = S^{-1}$ be a non empty subset of $G$ not containing the identity element e of $G$. The simple graph $\Gamma$ whose vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{\{v, vs\} | v \in V(\Gamma), s \in S\}$ is called the Cayley graph of $G$ corresponding to the set $S$ and is denoted by $\mathrm{Cay}(G, S)$. By $\mathbb{Z}_n$ we denote the cyclic group of order $n$. For any vertex $v \in V(\Gamma)$, the *open neighborhood* of $v$ is the set $N(v) = \{u \in V(\Gamma) | \{u, v\} \in E(\Gamma)\}$ and the *closed neighborhood* of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(\Gamma)$, the open neighborhood of $X$ is $N(X) = \bigcup_{v \in X} N(v)$ and the closed neighborhood of $X$ is $N[X] = N(X) \cup X$ [6]. A set $D \subseteq V(\Gamma)$ is said to be a *dominating set* if $N[D] = V(\Gamma)$ or equivalently, every vertex in $V(\Gamma) \backslash D$ is adjacent to at least one vertex in $D$. The *domination number* $\gamma(\Gamma)$ is the minimum cardinality of a dominating set in $\Gamma$. A dominating set with cardinality $\gamma(\Gamma)$ is called a $\gamma$*-set*. A set $T \subseteq V(\Gamma)$ is said to be a *total*

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*dominating set* if $N(T) = V(\Gamma)$ or equivalently, every vertex in $V(\Gamma)$ is adjacent to a vertex in $T$. The *total domination number* $\gamma_t(\Gamma)$ is the minimum cardinality of a total dominating set in $\Gamma$. A total dominating set with cardinality $\gamma_t(\Gamma)$ is called a $\gamma_t$*-set*. A graph $\Gamma$ is said to be connected graph if there is at least one path between every pair of vertices in $\Gamma$. The connected components of a graph are its maximal connected subgraphs. A dominating set $D$ of $\Gamma$ is said to be a connected dominating set if the induced subgraph generated by $D$ is connected. The minimum cardinality of a connected dominating set of $\Gamma$ is called the connected domination number of $\Gamma$ and is denoted by $\gamma_c(\Gamma)$, and the corresponding set is denoted by $\gamma_c$*-set* of $\Gamma$. Let $\lambda$ be the length of the longest sequence of consecutive integers in $\mathbb{Z}_m$, each of which shares a prime factor with $m$. Dominating sets were defined by Berge and Ore [1, 16]. The concept of total domination in graphs was initiated by E.J. Cockayne and R.W. Dows and S.T. Hedetniemi [4]. S.T Hedetniemi, R.C. Laskar[7] introduced the connected domination number in graphs. Madhavi [10] present the concept of Euler totient Cayley graphs and their domination parameters studied by Uma Maheswary and B. Maheswary [11]. Also some properties of direct product graphs of Cayley graphs with arithmetic graphs discussed by Uma Maheswary and B. Maheswary [13], and their domination parameters studied by Uma Maheswary and B. Maheswary and M. Manjuri [12, 14, 15].

A walk is a sequence of pairwise adjacent vertices of a graph. A path is a walk in which no vertex is repeated. The distance between two vertices of a graph is the number of edges of the shortest path between them. The diameter of a connected graph is the maximum distance between any two vertices of the graph. According to this definition, the diameter of a disconnected graph is infinite, but if we consider the diameter as the maximum finite shortest path length in the graph, this is the same as the largest of diameters of the graph's connected components. So in this paper by diameter of a disconnected graph we mean the largest diameter of its connected components. Let $v, w \in V(\Gamma)$ then the distance between $v, w$ is denoted by $d(v, w)$ and the diameter of $\Gamma$ is denoted by $\mathrm{diam}(\Gamma)$ [2, 3].

Here we study the total and connected dominating sets and diameter of Cayley graphs constructed out of $\mathbb{Z}_p \times \mathbb{Z}_m$ where $m \in \{p^\alpha, p^\alpha q^\beta, p^\alpha q^\beta r^\gamma\}$, $p, q, r$ are distinct prime numbers and $\alpha, \beta, \gamma$ are positive integers. The domination number of these graphs are presented in [8] and we present some of the results without proofs .

**Theorem 1.1.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}, \Phi)$ *where* $\Phi = \varphi_p \times \varphi_{p^\alpha}$. *Then*

*1)* $\gamma(\Gamma) = 2$ *where* $p = 2$ *and* $\alpha = 1$.

*2)* $\gamma(\Gamma) = 4$ *where* $p = 2$ *and* $\alpha \geq 2$.

*3)* $\gamma(\Gamma) = 3$ *where* $p \geq 3$ *and* $\alpha \geq 1$.

**Theorem 1.2.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$ *where* $\Phi = \varphi_p \times \varphi_{p^\alpha q^\beta}$, $p, q \geq 2$ *and* $\alpha, \beta \geq 1$. *Then* $\gamma(\Gamma)$ *is given by Table 1.1.*

**Theorem 1.3.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$ *where* $\Phi = \varphi_p \times \varphi_{p^\alpha q^\beta r^\gamma}$, $p, q, r \geq 2$ *and* $\alpha, \beta, \gamma \geq 1$. *Then* $\gamma(\Gamma)$ *is given by Table 1.2.*

Table 1.1: $\gamma(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi))$

| $\Gamma$ | $\gamma(\Gamma)$ | Comments |
|---|---|---|
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{pq}, \Phi)$ | 4 | |
| $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta}, \Phi)$ | 8 | $(\alpha, \beta) \neq (1, 1)$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$ | 6 | $(\alpha, \beta) \neq (1, 1)$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$ | 5 | $(\alpha, \beta) \neq (1, 1)$<br>$p = 3, q \geq 5$ *or* $q = 3, p \geq 5$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$ | 4 | $(\alpha, \beta) \neq (1, 1)$<br>$p, q \geq 5$ |

Table 1.2: $\gamma(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi))$

| $\Gamma$ | $\gamma(\Gamma)$ | Comments |
|---|---|---|
| $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2qr}, \Phi)$ | 8 | |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2pr}, \Phi)$ | 8 | |
| $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta r^\gamma}, \Phi)$ | 12 | $\alpha \neq 1$ *or* $\beta \neq 1$ *or* $\gamma \neq 1$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$ | 10 | $\alpha \neq 1$ *or* $\beta \neq 1$ *or* $\gamma \neq 1$<br>$p = 3, r \geq 5$ *or* $r = 3, p \geq 5$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$ | 8 | $\alpha \neq 1$ *or* $\beta \neq 1$ *or* $\gamma \neq 1$<br>$p, r \geq 5$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$ | $6 \leq \gamma(\Gamma) \leq 8$ | $\alpha, \beta, \gamma \geq 1$<br>one of the prime factors is 3 |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$ | 5 | $p, q, r \geq 5$ and $\alpha, \beta, \gamma \geq 1$ |

Let $p_1, p_2, \ldots, p_k$ be consecutive prime numbers, $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers and $\Phi = \varphi_2 \times \varphi_{2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}$.

**Theorem 1.4.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}, \Phi)$*, where* $p_1 = 3$ *and* $\alpha \geq 2$*. Then* $\gamma(\Gamma) \geq 4k + 4$*.*

For $p = 2$, the Cayley graph $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_m, \Phi)$, where $\Phi = \varphi_p \times \varphi_m$ and $m$ is a multiple of 2, is a disconnected graph with two connected components, say $\Gamma_1$ and $\Gamma_2$, where $V(\Gamma_1) = \{(1, v) | v \textit{ is odd}\} \cup \{(0, v) | v \textit{ is even}\}$ and $V(\Gamma_2) = \{(0, v) | v \textit{ is odd}\} \cup \{(1, v) | v \textit{ is even}\}$. Since every Cayley graph $\mathrm{Cay}(G, S)$ is $|S|$-regular (see for example [5]), we find that $\Gamma$ is $|\Phi|$-regular.

Let $X$ be a set of consecutive integers in $\mathbb{Z}_m$ such that for every $x \in X$, we have $\gcd(x, m) > 1$. In this case we call $X_i$ a consecutive set. We use $X_i^k$ to show that the consecutive set $X_i$ has $k$ elements.

Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_m, \Phi)$. In Section 2. we calculate $\gamma_t(\Gamma)$ and $\gamma_c(\Gamma)$ and $\mathrm{diam}(\Gamma)$ where $m = p^\alpha$. We consider the case $m = p^\alpha q^\beta$ in Section 3. and the case $m = p^\alpha q^\beta r^\gamma$ is considered in Section 4.

## 2. Total and connected domination number and diameter of $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}, \Phi)$

Let $p$ be a prime number, $\alpha$ a positive integer and $\Phi = \varphi_p \times \varphi_{p^\alpha}$. In this section, we obtain the total and connected domination number and diameter of $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}, \Phi)$.

**Theorem 2.1.** *Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}, \Phi)$. Then*

*1)* $\mathrm{diam}(\Gamma) = 1$ *where $p = 2$ and $\alpha = 1$.*

*2)* $\mathrm{diam}(\Gamma) = 2$ *where $p = 2, \alpha \geq 2$ or $p \geq 3, \alpha \geq 1$.*

*Proof.* 1) In this case $\Gamma \cong 2K_2$, and clearly the diameter of $\Gamma$ is 1.

2) Let $p = 2$ and $\alpha \geq 2$. Then $\Gamma$ is a disconnected graph with two connected components, say $\Gamma_1$ and $\Gamma_2$, where $V(\Gamma_1) = \{(1, v) | v \text{ is odd}\} \cup \{(0, v) | v \text{ is even}\}$ and $V(\Gamma_2) = \{(0, v) | v \text{ is odd}\} \cup \{(1, v) | v \text{ is even}\}$.

Let $(u, v), (u', v') \in V(\Gamma_1)$. Then we have the following two possibilities:

*i)* $u = u'$ and $v \neq v'$. Obviously $(u, v)$ and $(u, v')$ are not adjacent. This implies that $d((u, v), (u', v')) \geq 2$. On the other hand the vertex $(u - 1, v - 1)$ is adjacent to both vertices. So $d((u, v), (u', v')) = 2$.

*ii)* $u \neq u'$ and $v \neq v'$. We know that $u - u' \in \varphi_2$ and $v - v'$ is an odd integer. Since all of the odd integers in $\mathbb{Z}_{2^\alpha}$ to be included into a $\varphi_{2^\alpha}$, hence $v - v' \in \varphi_{2^\alpha}$. Thus $(u, v)$ is adjacent to $(u', v')$. So $d((u, v), (u', v')) = 1$.

Since $(u, v)$ and $(u', v')$ are arbitrary vertices of $\Gamma_1$, hence the diameter of $\Gamma_1$ is 2. Similarly the diameter of $\Gamma_2$ is 2.

Let $p \geq 3$ and $\alpha \geq 1$. Then $\Gamma$ is connected graph where

$$V(\Gamma) = \{(0, 0), \ldots, (0, p^\alpha - 1), \ldots, (p - 1, 0), \ldots, (p - 1, p^\alpha - 1)\}.$$

Assume that $(u, v)$ and $(u', v')$ are arbitrary vertices of $\Gamma$. Now we have the following three possibilities:

*i)* $u = u'$ and $v \neq v'$. Since $(u, v)$ and $(u', v')$ are not adjacent $d((u, v), (u', v')) \geq 2$. Let $v$ and $v'$ be multiple of $p$. Note that 0 is multiple of $p$. Then $(u - 1, p - 1)$ is adjacent to both $(u, v)$ and $(u', v')$. Let $v$ and $v'$ be non-multiple of $p$. Then $(u - 1, p)$ is common neighbor of $(u, v)$ and $(u', v')$. Now let one of either $v$ or $v'$ is multiple of $p$. Without loss of generality let $v$ is multiple of $p$ and $v'$ is non-multiple of $p$. Suppose that $v$ and $v'$ are both even or odd. Then $(u - 1, \frac{v+v'}{2})$ is adjacent to both $(u, v)$ and $(u', v')$. Since $v - v'$ is even so $v - v'$ is divisible by 2. Hence $v - \frac{v+v'}{2} = \frac{2v-v-v'}{2} = \frac{v-v'}{2} \in \varphi_{p^\alpha}$ and also $v' - \frac{v+v'}{2} = \frac{v'-v}{2} \in \varphi_{p^\alpha}$. Now assume that one of either $v$ or $v'$ is even. Then $(u - 1, 2v')$ is common neighbor of $(u, v)$ and $(u', v')$. Therefore $d((u, v), (u', v')) = 2$.

*ii)* $u \neq u'$ and $v = v'$. In this case vertex $(u'', v - 1)$ where $u'' \neq u, u'$ is adjacent to both $(u, v)$ and $(u', v')$. Thus $d((u, v), (u', v')) = 2$.

*iii*) $u \neq u'$ and $v \neq v'$. If $(u, v)$ and $(u', v')$ be adjacent then $d((u, v), (u', v')) = 1$. If $(u, v)$ and $(u', v')$ be non-adjacent then similar to *i*) and *ii*), $d((u, v), (u', v')) = 2$. Therefore in this case $\mathrm{diam}(\Gamma) = 2$. $\square$

**Theorem 2.2.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}, \Phi)$. *Then*

*1)* $\gamma_t(\Gamma) = 4$ *and* $\gamma_c(\Gamma)$ *does not exist where* $p = 2$ *and* $\alpha \geq 1$.

*2)* $\gamma_t(\Gamma) = \gamma_c(\Gamma) = 3$ *where* $p \geq 3$ *and* $\alpha \geq 1$.

*Proof.* 1) Let $p = 2$ and $\alpha = 1$. Then $\Gamma \cong 2K_2$, and obviously $\gamma_t(\Gamma) = 4$.

Assume that $p = 2$ and $\alpha \geq 2$. Then by [8, Theorem 2.1], $\gamma(\Gamma) = 4$ and $D = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ is a $\gamma$-set for $\Gamma$. Since $(0, 0)$ and $(0, 1)$ are adjacent to $(1, 1)$ and $(1, 0)$, respectively. Hence $D$ is a $\gamma_t$-set for $\Gamma$. Thus $\gamma_t(\Gamma) = 4$.

In this case $\Gamma$ is a disconnected graph. Hence by the definition of connected dominating set, $\gamma_c$-*set* does not exist for $\Gamma$

2) Let $p \geq 3$ and $\alpha \geq 1$. By [8, Theorem 2.1], we find that $\gamma(\Gamma) = 3$ and $D = \{(0, 1), (1, 0), (2, 2)\}$ is a $\gamma$-set for $\Gamma$. Vertices of $D$ dominate among themselves. Therefore $\gamma_t(\Gamma) = \gamma_c(\Gamma) = 3$. $\square$

**Example 2.1.** Let $\Gamma_1 = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^4}, \Phi)$ and $\Gamma_2 = \mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \Phi)$, which are shown in Figures 2.1 and 2.2, respectively. Clearly $\Gamma_1$ is a disconnected graph with two connected components. Thus $\gamma_c$-*set* does not exist for $\Gamma_1$. Also, total dominating set of $\Gamma_1$, is $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Note that total and connected dominating set of $\Gamma_2$ is $\{(0, 1), (1, 0), (2, 2)\}$.

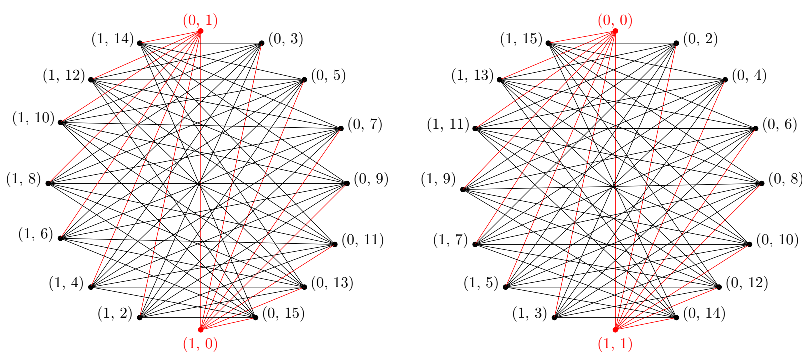


FIG. 2.1: The graph $\Gamma_1 = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^4}, \Phi)$ and its total dominating set.

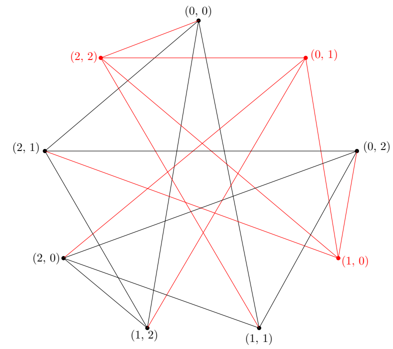


FIG. 2.2: The graph $\Gamma_2 = \mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \Phi)$ and its total and connected dominating set.

## 3. Total and connected domination number and diameter of $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$

Let $p, q$ be prime numbers, $\alpha, \beta$ positive integers and $\Phi = \varphi_p \times \varphi_{p^\alpha q^\beta}$. In this section, we find the total and connected domination number and diameter of $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$.

**Lemma 3.1.** *Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta}, \Phi)$, where $\alpha, \beta \geq 1$. Then* $\mathrm{diam}(\Gamma) = 3$.

*Proof.* $\Gamma$ is a disconnected graph with two connected components, say $\Gamma_1$ and $\Gamma_2$, where $V(\Gamma_1) = \{(1, v) | v \ is \ odd\} \cup \{(0, v) | v \ is \ even\}$ and $V(\Gamma_2) = \{(0, v) | v \ is \ odd\} \cup \{(1, v) | v \ is \ even\}$.

Let $(u, v), (u', v') \in V(\Gamma_1)$. Then we have the following two possibilities:

*i)* $u = u'$ and $v \neq v'$. Clearly $d((u, v), (u', v')) \geq 2$. Let $v$ and $v'$ be multiple of $2q$. Then $(u-1, 2q-1)$ is common neighbor of $(u, v)$ and $(u', v')$. Also if $v$ and $v'$ be non-multiple of $2q$, then $(u-1, 2q)$ is adjacent to both $(u, v)$ and $(u', v')$. Note that a trivial observation shows that $v$ and $v'$ have the same parity. Let $v$ and $v'$ be both multiple of one of the prime factors 2 or $q$. Then the other prime factor is adjacent to both $v$ and $v'$. Now let one of either $v$ or $v'$ is odd and is multiple of $q$. Then $(u, v), (u', v') \in \{(1, v) | v \ is \ odd\}$. If $\frac{v+v'}{2}$ be even, then $(u-1, \frac{v+v'}{2})$ is common neighbor of $(u, v)$ and $(u', v')$. Also if $\frac{v+v'}{2}$ be odd, then $(u-1, \frac{v+v'}{2}+q)$ is adjacent to both $(u, v)$ and $(u', v')$. Let one of either $v$ or $v'$ is multiple of $2q$. So $(u, v), (u', v') \in \{(0, v) | v \ is \ even\}$. If $\frac{v+v'}{2} \in \varphi_{2^\alpha q^\beta}$, then $(u-1, \frac{v+v'}{2})$ is common neighbor of $(u, v)$ and $(u', v')$. Moreover if $\frac{v+v'}{2}$ be even, then $(u-1, \frac{v+v'}{2}+q)$ is adjacent to both $(u, v)$ and $(u', v')$. Thus in this case $d((u, v), (u', v')) = 2$.

*ii)* $u \neq u'$ and $v \neq v'$. If $v - v' \in \varphi_{2^\alpha q^\beta}$, then $d((u, v), (u', v')) = 1$. Suppose that $v - v' \notin \varphi_{2^\alpha q^\beta}$, since $u \neq u'$ and $u, u' \in \mathbb{Z}_2$, we have no common neighbor between $(u, v)$ and $(u', v')$. This implies that $d((u, v), (u', v')) \geq 3$. Without loss of generality assume that $u = 0$ and $u' = 1$. Since $v - v'$ is an odd integer, we find that $v - v' + 2 \in \varphi_{2^\alpha q^\beta}$. Thus $(0, v)(1, v+1)(0, v+2)(1, v')$ is a path of length 3 between $(0, v)$ and $(1, v')$. So $\mathrm{diam}(\Gamma_1) = 3$ and similarly $\mathrm{diam}(\Gamma_2) = 3$. Therefore $\mathrm{diam}(\Gamma) = 3$. □

**Lemma 3.2.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta}, \Phi)$, *where* $\alpha, \beta \geq 1$. *Then* $\gamma_c(\Gamma)$ *does not exist and* $\gamma_t(\Gamma) = 8$.

*Proof.* $\Gamma$ is a disconnected graph with exactly two connected components $\Gamma_1$ and $\Gamma_2$ where $V(\Gamma_1) = \{(1, v) | v \textit{ is odd}\} \cup \{(0, v) | v \textit{ is even}\}$ and $V(\Gamma_2) = \{(0, v) | v \textit{ is odd}\} \cup \{(1, v) | v \textit{ is even}\}$. Hence by the definition of connected dominating set, $\gamma_c$-*set* does not exist for $\Gamma$.

Assume first that $(\alpha, \beta) = (1, 1)$. Then by [8, Proposition 3.1], $A = \{(0, 0), (1, q)\}$ and $B = \{(0, 1), (1, q+1)\}$ dominate $V(\Gamma_1) \backslash A$ and $V(\Gamma_2) \backslash B$, respectively. Hence $\gamma(\Gamma) = 4$. Vertices of $A$ are not adjacent to each other and $A$ is not dominated by one vertex. Note that $(1, 1)$ and $(0, q+1)$ are adjacent to $(0, 0)$ and $(1, q)$, respectively. Hence $T_1 = \{(0, 0), (1, 1), (1, q), (0, q+1)\}$ is a $\gamma_t$-set for $\Gamma_1$. Similarly $T_2 = \{(0, 1), (1, 0), (0, q), (1, q+1)\}$ is a $\gamma_t$-set for $\Gamma_2$. Therefore $\gamma_t(\Gamma) = 8$.

Next consider the case where $(\alpha, \beta) \neq (1, 1)$. By [8, Lemma 3.2], $\gamma(\Gamma) = 8$ and $D = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3)\}$ is a $\gamma$-set for $\Gamma$. Vertices $(0, 1), (0, 0), (0, 3)$,
$(0, 2)$ are adjacent to vertices $(1, 0), (1, 1), (1, 2), (1, 3)$ respectively. Thus $D$ becomes a $\gamma_t$-set for $\Gamma$. Hence $\gamma_t(\Gamma) = 8$. □

**Proposition 3.1.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$, *where* $\alpha, \beta \geq 1$. *Then* $\mathrm{diam}(\Gamma) = 3$.

*Proof.* Let $(u, v), (u', v') \in V(\Gamma)$. Then we have the following three possibilities:

*i)* $u = u'$ and $v \neq v'$. In this case $d((u, v), (u', v')) \geq 2$. Suppose that $v$ and $v'$ are both even or odd. Hence by case *i)* of Lemma 3.1, $d((u, v), (u', v')) = 2$. Since in $\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha p^\beta}$ we have two connected components, where in each of them, if $u = u'$ then $v$ and $v'$ are both even or odd.

Assume that one of either $v$ or $v'$ is even. Without loss of generality let $v$ is even and $v'$ is odd. Also let $(u'', v'')$ where $u'' \neq u$, is common neighbor between $(u, v), (u', v')$. If $v''$ be even then $v - v'' \notin \varphi_{2^\alpha p^\beta}$ and if $v''$ be odd then $v' - v'' \notin \varphi_{2^\alpha p^\beta}$. Thus we have no common neighbor between $(u, v)$ and $(u', v')$. Hence $d((u, v), (u', v')) \geq 3$. We consider $u'', u''' \neq u$, if $v$ and $v'$ be multiple of $p$, then $(u, v)(u'', p-2)(u''', p-1)(u, v')$ is a path of length 3 between $(u, v)$ and $(u', v')$. If $v$ and $v'$ be non-multiple of $p$, then the path $(u, v)(u'', p)(u''', 2v')(u', v')$ is connected. If $v$ be multiple of $p$ and $v'$ be non-multiple of $p$, since $v - v' \in \varphi_{2^\alpha p^\beta}$ then $(u, v)(u'', v')(u''', v)(u', v')$ is a path of length 3 between $(u, v)$ and $(u', v')$.

*ii)* $u \neq u'$ and $v = v'$. In this case $(u'', v-1)$ where $u'' \neq u, u'$ is common neighbor of $(u, v)$ and $(u', v')$. Hence $d((u, v), (u', v')) = 2$.

*iii)* $u \neq u^{'}$ and $v \neq v^{'}$. If $(u, v)$ and $(u^{'}, v^{'})$ be adjacent then $d((u, v), (u^{'}, v^{'})) = 1$. Now assume that $(u, v)$ and $(u^{'}, v^{'})$ are not adjacent. Let $v$ and $v^{'}$ be both even or odd. Then by case *i)* of Lemma 3.1, we know that there is a vertex $(u^{''}, v^{''})$, where $u^{''} \neq u, u^{'}$ and $v^{''}$ is adjacent to $v$ and $v^{'}$, that is adjacent to both $(u, v)$ and $(u^{'}, v^{'})$. Thus $d((u, v), (u^{'}, v^{'})) = 2$.

Now let one of either $v$ or $v^{'}$ is even. Then by second paragraph of case *i)* and also by using of case *ii)* of Lemma 3.1, we see that $d((u, v), (u^{'}, v^{'})) = 3$. Therefore $\mathrm{diam}(\Gamma) = 3$. □

**Proposition 3.2.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$, *where* $\alpha, \beta \geq 1$. *Then*

*i)* $\gamma_t(\Gamma) = 6$.

*ii)* $\gamma_c(\Gamma)$ *is given by Table* 3.1.

Table 3.1: $\gamma_c(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi))$ where $\alpha, \beta \geq 1$.

| $\Gamma$ | $\gamma_c(\Gamma)$ | Comments |
|---|---|---|
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$ | 7 | $p = 3$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$ | 6 | $p \geq 5$ |

*Proof. i)* Let $(\alpha, \beta) = (1, 1)$. By [8, Proposition 3.1], we see that $\gamma(\Gamma) = 4$ and $D = \{(0, 0), (0, 1), (1, p), (1, p+1)\}$ is a $\gamma$-set for $\Gamma$. Vertices of $D$ are not adjacent to each other. Hence $\gamma_t(\Gamma) > 4$. Let a vertex say $(u, v)$ dominates all vertices of $D$. Then $(u, v)$ is adjacent to $(0, 0)$ hence $(u, v) \in \Phi$. On the other hand $(u, v)$ is adjacent to $(0, 1)$ thus $(u, v) \notin \Phi$, which is impossible. We conclude that $\gamma_t(\Gamma) > 5$. Since vertex $(p-1, p-1)$ is adjacent to vertices $(0, 1), (1, p)$ and also vertex $(p-1, 2p-1)$ is adjacent to vertices $(0, 0), (1, p+1)$. Hence $T = \{(0, 0), (0, 1), (1, p), (1, p+1), (p-1, p-1), (p-1, 2p-1)\}$ is a $\gamma_t$-set for $\Gamma$.

Finally $(\alpha, \beta) \neq (1, 1)$. In this case by [8, Proposition 3.3], $\gamma(\Gamma) = 6$ and $D' = \{(0, 0), (0, 1), (1, 2), (1, 3), (2, 4), (2, 5)\}$ is a $\gamma$-set for $\Gamma$. If $p = 3$, then we find that vertices $(0, 0), (0, 1), (1, 3)$ are adjacent to vertices $(2, 5), (1, 2), (2, 4)$, respectively and if $p \geq 5$ then vertices $(0, 0), (1, 3), (2, 4), (0, 1), (1, 2)$ are adjacent to vertices $(1, 3), (2, 4), (0, 1), (1, 2), (2, 5)$, respectively. Thus $D'$ becomes a $\gamma_t$-set for $\Gamma$.

Note that both $T$ and $D^{'}$ are two $\gamma_t$-sets for $\Gamma$, where $\alpha, \beta \geq 1$. Therefore $\gamma_t(\Gamma) = 6$.

*ii)* By using a similar argument given in the proof of case *i)*, we have $\gamma_c(\Gamma) \geq 6$.

Assume first that $p = 3$. Then the subgraphs generated by $T$ and $D^{'}$ are disconnected. Since the subgraph generated by $D^{'}$ has exactly three connected components which are induced subgraphs generated by sets $\{(0, 0), (2, 5)\}$, $\{(0, 1), (1, 2)\}$ and $\{(1, 3), (2, 4)\}$, also the subgraph generated by $T$ has exactly two connected components which are induced subgraphs generated by sets $\{(0, 1), (1, p), (p-1, p-1)\}$ and $\{(0, 0), (1, p+1), (p-1, 2p-1)\}$. We conclude that $\gamma_c(\Gamma) \geq 7$.

Note that vertex $(0,p)$ is adjacent to vertices $(p-1,p-1)$ and $(1,p+1)$. Therefore $C=\{(0,0),(0,1),(1,p),(1,p+1),(p-1,p-1),(p-1,2p-1),(0,p)\}$ is a connected dominating set for $\Gamma$ with minimum cardinality. Therefore $\gamma_c(\Gamma)=7$.

Now suppose that $p\geq 5$. According to the proof of final part of case $i$), we see that $D^{'}$ becomes a connected dominating set for $\Gamma$ with minimum cardinality. Therefore in this case $\gamma_c(\Gamma)=6$. □

**Proposition 3.3.** *Let* $\Gamma=\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{p^\alpha q^\beta},\Phi)$*, where* $p,q\geq 3$ *and* $\alpha,\beta\geq 1$*. Then* $\mathrm{diam}(\Gamma)=2$.

*Proof.* Let $(u,v),(u^{'},v^{'})\in V(\Gamma)$. Then we have the following three possibilities:

*i*) $u=u^{'}$ and $v\neq v^{'}$. Hence $d((u,v),\,(u^{'},v^{'}))\geq 2$. Let $v$ and $v^{'}$ be multiple of $pq$, then $(u-1,pq-1)$ is common neighbor of $(u,v)$ and $(u^{'},v^{'})$. Let $v,v^{'}\in\varphi_{p^\alpha q^\beta}$, then $(u-1,pq)$ is adjacent to both $(u,v)$ and $(u^{'},v^{'})$. Let $v$ and $v^{'}$ be multiple of $p$, then $q$ is adjacent to both $v$ and $v^{'}$. Also let $v$ and $v^{'}$ be multiple of $q$, then $p$ is adjacent to both $v$ and $v^{'}$. So $d((u,v),\,(u^{'},v^{'}))=2$. Let $v$ is multiple of $p$ and $v^{'}$ is multiple of $q$. If $v$ and $v^{'}$ be both even or odd, then we show that $(u-1,\frac{v+v^{'}}{2})$ is a common neighbor of $(u,v)$ and $(u^{'},v^{'})$. Assume that $v=kp$ and $v^{'}=k^{'}q$; $k,k^{'}\in\mathbb{Z}$. Then $v-\frac{v+v^{'}}{2}=\frac{v-v^{'}}{2}=\frac{kp-k^{'}q}{2}$. Suppose that $\frac{kp-k^{'}q}{2}\notin\varphi_{p^\alpha q^\beta}$ and without loss of generality assume $\frac{kp-k^{'}q}{2}=k^{''}p$; $k^{''}\in\mathbb{Z}$. Then $kp-k^{'}q=2k^{''}p$ which implies $kp-2k^{''}p=k^{'}q$. Hence $(\frac{k-2k^{''}}{k^{'}})p=q$, which is impossible, since $q$ is not a multiple of $p$. Hence $\frac{kp-k^{'}q}{2}\in\varphi_{p^\alpha q^\beta}$, and $v$ is adjacent to $\frac{v+v^{'}}{2}$. Similarly $v^{'}$ is adjacent to $\frac{v+v^{'}}{2}$. If one of either $v$ or $v^{'}$ be odd, then $2(v+v^{'})$ is adjacent to both $v$ and $v^{'}$. Assume that $v=kp$ is even and $v^{'}=k^{'}q$ is odd. Without loss of generality let $2(v+v^{'})-v=v+2v^{'}=k^{''}p$. Then $kp+2k^{'}q=k^{''}p$. This implies $(\frac{k^{''}-k}{2k^{'}})p=q$, which is impossible. Thus $v$ is adjacent to $2(v+v^{'})$. Similarly $v^{'}$ is adjacent to $2(v+v^{'})$. Hence $d((u,v),\,(u^{'},v^{'}))=2$. Let $v$ be multiple of $p$ or $q$ and $v^{'}\in\varphi_{p^\alpha q^\beta}$. Assume that $v$ and $v^{'}$ be both even or odd. If $v-v^{'}\in\varphi_{p^\alpha q^\beta}$ then it is easy to see that $\frac{v+v^{'}}{2}$ is adjacent to both $v$ and $v^{'}$ and if $v-v^{'}\notin\varphi_{p^\alpha q^\beta}$ then $v-v^{'}$ is adjacent to both $v$ and $v^{'}$. Now suppose that one of either $v$ or $v^{'}$ is odd. If $v$ be multiple of $p$ then $v^{'}q$ is adjacent to both $v$ and $v^{'}$. If $v$ be multiple of $q$ then $v^{'}p$ is adjacent to both $v$ and $v^{'}$. Moreover if $v$ be multiple of $pq$ then $v^{'}(p+q)$ is adjacent to both $v$ and $v^{'}$. Thus $d((u,v),\,(u^{'},v^{'}))=2$. Let one of either $v$ or $v^{'}$ is multiple of $p$ or $q$ and other is multiple of $pq$. We know that $+2$ and $-2$ is adjacent to all of the multiple of $pq$. Since by proof of [8, Proposition 3.1], $\lambda=2$, hence $v$ is adjacent to $+2$ or $-2$ or both of them. So we have a common neighbor between $(u,v)$ and $(u^{'},v^{'})$. Therefore $d((u,v),\,(u^{'},v^{'}))=2$.

*ii*) $u\neq u^{'}$ and $v=v^{'}$. In this case the vertex $(u^{''},v-1)$ where $u^{''}\neq u,u^{'}$, is a common neighbor of $(u,v)$ and $(u^{'},v^{'})$. Thus $d((u,v),\,(u^{'},v^{'}))=2$.

*iii*) $u\neq u^{'}$ and $v\neq v^{'}$. Hence by *i*) and *ii*), $d((u,v),\,(u^{'},v^{'}))=2$.

Therefore $\mathrm{diam}(\Gamma)=2$. □

**Proposition 3.4.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$*, where* $p, q \geq 3$ *and* $\alpha, \beta \geq 1$*. Then* $\gamma_t(\Gamma)$ *and* $\gamma_c(\Gamma)$ *is given by Table* 3.2*.*

Table 3.2: $\gamma_t(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)) = \gamma_c(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi))$ where $p, q \geq 3$ and $\alpha, \beta \geq 1$.

| $\Gamma$ | $\gamma_t(\Gamma), \gamma_c(\Gamma)$ | Comments |
|---|---|---|
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$ | 5 | one of the prime factors is 3 |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$ | 4 | $p, q \geq 5$ |

*Proof.* Assume first that one of the prime factors is 3. Let $(\alpha, \beta) = (1, 1)$. Then by [8, Proposition 3.1], $\gamma(\Gamma) = 4$ and $D = \{(0,0), (0,1), (1,x'), (1,y')\}$ is a $\gamma$-set for $\Gamma$, where $x, x'$ and $y, y'$ are consecutive integers in $\mathbb{Z}_{pq}$, each of which shares a prime factor with $pq$ where $x'$ is a multiple of $p$ and $y'$ is a multiple of $q$. Note that vertices of $D$ are not adjacent to each other. Hence $\gamma_t(\Gamma) > 4$. Also $D$ is dominated by $\{(2,2)\}$. Thus $T = \{(0,0), (0,1), (1,x'), (1,y'), (2,2)\}$ is a $\gamma_t$-set and $\gamma_c$-set for $\Gamma$.

The next case is where $(\alpha, \beta) \neq (1, 1)$. By [8, Table 1], $\gamma(\Gamma) = 5$ and $D = \{(0,0), (0,1), (1,2), (2,3), (2,4)\}$ is a $\gamma$-set for $\Gamma$. Vertices $(0,0), (2,4), (1,2), (0,1)$ are adjacent to vertices $(2,4), (1,2), (0,1), (2,3)$, respectively. Hence $D$ dominates all vertices of $\Gamma$ and the subgraph generated by $D$ is connected. Thus $D$ becomes a $\gamma_t$-set and $\gamma_c$-set for $\Gamma$. Therefore $\gamma_t(\Gamma) = \gamma_c(\Gamma) = 5$.

Finally assume that $p, q \geq 5$. Then by [8, Proposition 3.1, Table 1], $\gamma(\Gamma) = 4$ and $D = \{(0,0), (1,1), (2,2), (3,3)\}$ is a $\gamma$-set for $\Gamma$. Since $p, q \geq 5$ then vertices of $D$ dominate among themselves. Therefore $\gamma_t(\Gamma) = \gamma_c(\Gamma) = 4$. $\square$

As an immediate consequence of Lemma 3.2 and Propositions 3.2, 3.4, we have the following theorem.

**Theorem 3.1.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$*, where* $p, q \geq 2$ *and* $\alpha, \beta \geq 1$*. Then* $\gamma_t(\Gamma)$ *and* $\gamma_c(\Gamma)$ *is given by Table 3.3.*

Table 3.3: $\gamma_t(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi))$, $\gamma_c(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi))$ where $\alpha, \beta \geq 1$.

| $\Gamma$ | $\gamma_t(\Gamma)$ | $\gamma_c(\Gamma)$ | Comments |
|---|---|---|---|
| $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta}, \Phi)$ | 8 | does not exist | |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$ | 6 | 7 | $p = 3$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$ | 6 | 6 | $p \geq 5$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$ | 5 | 5 | one of the prime factors is 3 |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$ | 4 | 4 | $p, q \geq 5$ |

**Example 3.1.** The graph $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2\times 3^2}, \Phi)$, which is shown in Figure 3.1, is a disconnected graph with two connected components, say $\Gamma_1$ and $\Gamma_2$. Thus $\gamma_c$-*set* does not exist for $\Gamma$. In this graph two sets $T_1 = \{(0,0),(0,4),(1,1),(1,3)\}$ and $T_2 = \{(0,1),(0,3),(1,0),(1,4)\}$ are $\gamma_t$-*sets* sets for $\Gamma_1$ and $\Gamma_2$, respectively. Hence $\gamma_t(\Gamma) = 8$.

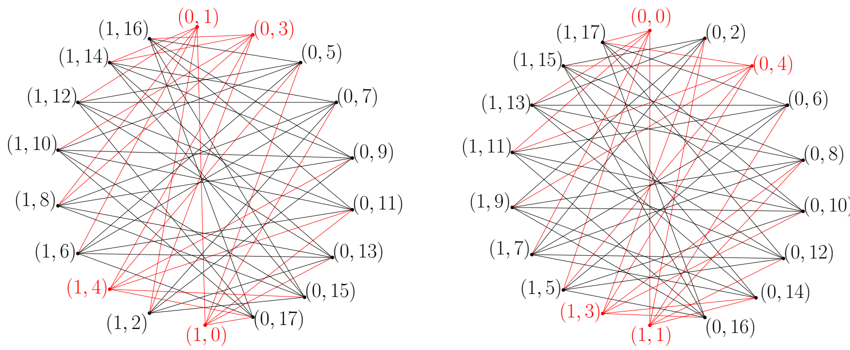


FIG. 3.1: Two connected components of $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2\times 3^2}, \Phi)$, left $\Gamma_1$, right $\Gamma_2$

**Example 3.2.** Let $p = 3$, $q = 5$. Then total and connected dominating set of $\Gamma = \mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_{15}, \Phi)$, which is shown in Figure 3.2, is $\{(0,0),(0,1),(1,6),(1,10),(2,2)\}$.

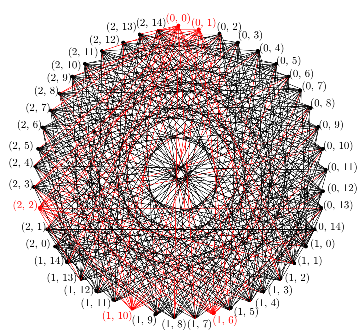

FIG. 3.2: The graph $\Gamma = \mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_{15}, \Phi)$ and its total dominating set.

## 4. Total and connected domination number and diameter of $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$

Let $p, q, r$ be three prime numbers, $\alpha, \beta, \gamma$ positive integers and $\Phi = \varphi_p \times \varphi_{p^\alpha q^\beta r^\gamma}$. In this section, we obtain the total and connected domination number of $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$ and we extend the results in the previous section for diameter of this graph.

**Lemma 4.1.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta r^\gamma}, \Phi)$*, where* $p, q, r$ *are distinct prime numbers and* $\alpha, \beta, \gamma \geq 1$*. Then* $\mathrm{diam}(\Gamma) = 3$.

*Proof.* $\Gamma$ is a disconnected graph with two connected components, say $\Gamma_1$ and $\Gamma_2$, where $V(\Gamma_1) = \{(1,v)|v\ is\ odd\} \cup \{(0,v)|v\ is\ even\}$ and $V(\Gamma_2) = \{(0,v)|v\ is\ odd\} \cup \{(1,v)|v\ is\ even\}$.

Let $(u,v),(u^{'},v^{'}) \in V(\Gamma_1)$. Then we have the following two possibilities:

*i)* $u = u^{'}, v \neq v^{'}$. Since $u = u^{'}$ hence $d((u,v),(u^{'},v^{'})) \geq 2$. Now by Table 4.1 we show that $d((u,v),(u^{'},v^{'})) = 2$. In this table, when $v, v^{'}$ are odd we have $u = u^{'} = 1,\ u^{''} = 0$ and when $v, v^{'}$ are even we have $u = u^{'} = 0,\ u^{''} = 1$. We prove the rows $6, 8$ of the table and the rest is similarly proven.

Let $v, v^{'}$ are odd and $v = kq,\ v^{'} = k^{'}qr,\ k, k^{'} \in \mathbb{Z}$. If $\frac{v+v^{'}}{q}$ be non-multiple of $q$ then we show that $\frac{v+v^{'}}{q}$ is adjacent to both $v$ and $v^{'}$.

Let $k^{''} \in \mathbb{Z}$. If $v - \frac{v+v^{'}}{q} = 2k^{''}$, then $k(q-1) - k^{'}r = 2k^{''}$. This implies $k = \frac{2k^{''}+k^{'}r}{q-1}$. Since $2k^{''} + k^{'}r$ is odd and $q-1$ is even hence $k$ is non-integer, which is impossible. If $v - \frac{v+v^{'}}{q} = k^{''}q$, then $\frac{v+v^{'}}{q} = (k - k^{''})q$, which is inaccurate because $\frac{v+v^{'}}{q}$ is non-multiple of $q$. Moreover if $v - \frac{v+v^{'}}{q} = k^{''}r$, then $k = (\frac{k^{'}+k^{''}}{q-1})r$. But we know that $k$ is non-multiple of $r$. So $v - \frac{v+v^{'}}{q} \in \varphi_{2^\alpha q^\beta r^\gamma}$ and similarly $v^{'}$ is adjacent to $\frac{v+v^{'}}{q}$. Since $u^{''}$ is adjacent to $u, u^{'}$ thus $(u^{''}, \frac{v+v^{'}}{q})$ is common neighbor between $(u,v),(u^{'},v^{'})$. Similarly it is easy to see that if $\frac{v+v^{'}}{q}$ be multiple of $q$ then $(u^{''}, \frac{v+v^{'}}{q} + 2r)$ is adjacent to both $(u,v)$ and $(u^{'},v^{'})$.

Let $v \in \varphi_{2^\alpha q^\beta r^\gamma}, v^{'} = kq$ is odd and $k, k^{''} \in \mathbb{Z}$. If $v - (v+v^{'})r = 2k^{''}$, then $v = 2k^{''} + (v+v^{'})r$. Hence $v$ is even, which is inaccurate. Also if $v - (v+v^{'})r = k^{''}q$, then $v = (\frac{k^{''}+kr}{1-r})q$ and if $v - (v+v^{'})r = k^{''}r$, then $v = (k^{''} + v + v^{'})r$, which are impossible. Hence $v$ is adjacent to $(v+v^{'})r$. Similarly it is easy to see that $v^{'}$ is adjacent to $(v+v^{'})r$. Therefore $(u^{''},(v+v^{'})r)$ is adjacent to both $(u,v)$ and $(u^{'},v^{'})$.

*ii)* $u \neq u^{'}, v \neq v^{'}$. If $v$ be adjacent to $v^{'}$, then $d((u,v),(u^{'},v^{'})) = 1$. Suppose that $v$ be non-adjacent to $v^{'}$, since $u \neq u^{'}$ and $u, u^{'} \in \mathbb{Z}_2$, hence we have no common neighbor between $(u,v)$ and $(u^{'},v^{'})$. This implies that $d((u,v),(u^{'},v^{'})) \geq 3$. Without loss of generality assume that $u = 0$ and $u^{'} = 1$. Now by Table 4.2 we show that $d((u,v),(u^{'},v^{'})) = 3$. In this table $u, u^{'''} = 0$ and also $u^{'}, u^{''} = 1$. Now we prove the fifth row and the rest is similarly proven. Let $v = 2kr;\ k \in \mathbb{Z}$ and $v^{'} \in \varphi_{2^\alpha q^\beta r^\gamma}$. Clearly $u, u^{'''}$ are adjacent to $u^{'}, u^{''}$.

First we show that $v$ is adjacent to $q$. Let $k^{''} \in \mathbb{Z}$.

$$\text{If } v - q = 2k^{''}, \text{ then } q = 2(kr - k^{''}).$$

$$\text{If } v - q = k^{''}q, \text{ then } r = (\frac{k^{''}+1}{2k})q.$$

$$\text{If } v-q=k^{''}r, \text{ then } q=(2k-k^{''})r.$$

In all three cases, we came across a contradiction. So $v-q \in \varphi_{2^\alpha q^\beta r^\gamma}$.

Next we prove that $q$ is adjacent to $(q+v^{'})r$.

$$\text{If } (q+v^{'})r-q=2k^{''}, \text{ then } k^{''}=\frac{(q+v^{'})r-q}{2}.$$

$$\text{If } (q+v^{'})r-q=k^{''}q, \text{ then } v^{'}=(\frac{k^{''}-r+1}{r})q.$$

$$\text{If } (q+v^{'})r-q=k^{''}r, \text{ then } q=(q+v^{'}-k^{''})r.$$

which is impossible, since $k^{''}$ is integer and $v^{'} \in \varphi_{2^\alpha q^\beta r^\gamma}$ and also $q$ is non-integer of $r$.

Finally we show that $(q+v^{'})r$ is adjacent to $v^{'}$.

$$\text{If } (q+v^{'})r-v^{'}=2k^{''}, \text{ then } k^{''}=\frac{(q+v^{'})r-v^{'}}{2}.$$

$$\text{If } (q+v^{'})r-v^{'}=k^{''}q, \text{ then } v^{'}=(\frac{k^{''}-r}{r-1})q.$$

$$\text{If } (q+v^{'})r-v^{'}=k^{''}r, \text{ then } v^{'}=(q+v^{'}-k^{''})r.$$

Again which are impossible. This implies that $(u,v)(u^{''},q)(u^{'''},(q+v^{'})r)(u^{'},v^{'})$ is shortest path between $(u,v)$ and $(u^{'},v^{'})$. Thus $\mathrm{diam}(\Gamma_1)=3$ and similarly $\mathrm{diam}(\Gamma_2)=3$. Therefore $\mathrm{diam}(\Gamma)=3$. $\square$

**Lemma 4.2.** *Let $\Gamma=\mathrm{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{2^\alpha q^\beta r^\gamma},\Phi)$, where $\alpha,\beta,\gamma\geq 1$. Then $\gamma_c(\Gamma)$ does not exist and $\gamma_t(\Gamma)=12$.*

*Proof.* Clearly $\Gamma$ is a disconnected graph with two connected components say $\Gamma_1$ and $\Gamma_2$. Let $V_1=V(\Gamma_1)$ and $V_2=V(\Gamma_2)$. Then $V_1=\{(1,v)|v\ is\ odd\}\cup\{(0,v)|v\ is\ even\}$ and $V_2=\{(0,v)|v\ is\ odd\}\cup\{(1,v)|v\ is\ even\}$. Hence by the definition of connected dominating set, $\gamma_c$-*set* does not exist for $\Gamma$.

Let $(\alpha,\beta,\gamma)=(1,1,1)$. Then we find by [8, Lemma 4.1], that $\gamma(\Gamma)=8$ and $D_1=\{(0,0),(0,2),(1,x_4),(1,x_4')\}$ and $D_2=\{(0,1),(0,3),(1,x_5),(1,x_5')\}$ are minimal dominating sets for $\Gamma_1$ and $\Gamma_2$ respectively, where $X_i^5=\{x_1,x_2,x_3,x_4,x_5\}$ and $X_j^5=\{x_1',x_2',x_3',x_4',x_5'\}$ are consecutive integers in $\mathbb{Z}_{2qr}$, each of which shares a prime factor with $2qr$. Since vertices of $D_1$ are not adjacent to each other, we conclude that $\gamma_t(\Gamma_1)>4$. On the other hand it is clear that $D_1$ is not dominated by one vertex. Hence $\gamma_t(\Gamma_1)>5$. Vertex $(1,1)$ is adjacent to vertices $(0,0),(0,2)$ and vertex $(0,4)$ is adjacent to vertices $(1,x_4),(1,x_4')$. Thus $T_1=\{(0,0),(0,2),(0,4),(1,1),(1,x_4),(1,x_4')\}$ dominates all vertices of $\Gamma_1$. Similarly $T_2=\{(0,1),(0,3),(0,5),(1,2),(1,x_5),(1,x_5')\}$ dominates all vertices of $\Gamma_2$. Hence $\gamma_t(\Gamma)=12$.

Table 4.1: Common neighbor between $(u,v),(u^{'},v^{'})$ in $\Gamma=\mathrm{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{2^\alpha q^\beta r^\gamma},\Phi)$

| $u=u^{'},v\neq v^{'}$ | common neighbor | Comments |
|---|---|---|
| $v,v^{'}\in\varphi_{2^\alpha q^\beta r^\gamma}$ | $(u^{''},2qr)$ | |
| $v,v^{'}$ are odd and multiple of $q$ | $(u^{''},2r)$ | |
| $v,v^{'}$ are odd and multiple of $r$ | $(u^{''},2q)$ | |
| $v,v^{'}$ are odd and multiple of $qr$ | $(u^{''},2)$ | |
| $v$ is odd and multiple of $q$ and $v^{'}$ is odd and multiple of $r$ | $(u^{''},\frac{v+v^{'}}{2})$<br>$(u^{''},\frac{v+v^{'}}{2}+qr)$ | if $\frac{v+v^{'}}{2}$ be even<br>if $\frac{v+v^{'}}{2}$ be odd |
| $v$ is odd and multiple of $q$ and $v^{'}$ is odd and multiple of $qr$ | $(u^{''},\frac{v+v^{'}}{q})$<br>$(u^{''},\frac{v+v^{'}}{q}+2r)$ | if $\frac{v+v^{'}}{q}\neq kq,\ k\in\mathbb{Z}$<br>if $\frac{v+v^{'}}{q}=k^{'}q,\ k^{'}\in\mathbb{Z}$ |
| $v$ is odd and multiple of $r$ and $v^{'}$ is odd and multiple of $qr$ | $(u^{''},\frac{v+v^{'}}{r})$<br>$(u^{''},\frac{v+v^{'}}{r}+2q)$ | if $\frac{v+v^{'}}{r}\neq kr,\ k\in\mathbb{Z}$<br>if $\frac{v+v^{'}}{r}=k^{'}r,\ k^{'}\in\mathbb{Z}$ |
| $v\in\varphi_{2^\alpha q^\beta r^\gamma}$ and $v^{'}$ is odd and multiple of $q$ | $(u^{''},(v+v^{'})r)$ | |
| $v\in\varphi_{2^\alpha q^\beta r^\gamma}$ and $v^{'}$ is odd and multiple of $r$ | $(u^{''},(v+v^{'})q)$ | |
| $v\in\varphi_{2^\alpha q^\beta r^\gamma}$ and $v^{'}$ is odd multiple of $qr$ | $(u^{''},(v+v^{'})2)$ | |
| $v,v^{'}$ are even and multiple of $r$ | $(u^{''},q)$ | |
| $v,v^{'}$ are even and multiple of $q$ | $(u^{''},r)$ | |
| $v,v^{'}$ are even and non-multiple of $q$ and $r$ | $(u^{''},qr)$ | |
| $v,v^{'}$ are even and multiple of $2qr$ | $(u^{''},2qr-1)$ | |
| $v$ is even and multiple of $q$ and $v^{'}$ is even and multiple of $r$ | $(u^{''},\frac{v+v^{'}}{2}+qr)$<br>$(u^{''},\frac{v+v^{'}}{2})$ | if $\frac{v+v^{'}}{2}$ be even<br>if $\frac{v+v^{'}}{2}$ be odd |
| $v$ is even and multiple of $qr$ and $v^{'}$ is even and non-multiple of $q$ and $r$ | $(u^{''},\frac{v^{'}}{2})$<br>$(u^{''},\frac{v^{'}}{2}+qr)$ | if $\frac{v^{'}}{2}\in\varphi_{2^\alpha q^\beta r^\gamma}$<br>if $\frac{v^{'}}{2}\notin\varphi_{2^\alpha q^\beta r^\gamma}$ |
| $v$ is even and multiple of $qr$ and $v^{'}$ is even and multiple of $q$ | $(u^{''},\frac{v^{'}}{q}+qr)$<br>$(u^{''},\frac{v^{'}}{q}+r)$ | if $\frac{v^{'}}{q}\neq kq,\ k\in\mathbb{Z}$<br>if $\frac{v^{'}}{q}=k^{'}q,\ k^{'}\in\mathbb{Z}$ |
| $v$ is even and multiple of $qr$ and $v^{'}$ is even and multiple of $r$ | $(u^{''},\frac{v^{'}}{r}+qr)$<br>$(u^{''},\frac{v^{'}}{r}+q)$ | if $\frac{v^{'}}{r}\neq kr,\ k\in\mathbb{Z}$<br>if $\frac{v^{'}}{r}=k^{'}r,\ k^{'}\in\mathbb{Z}$ |

Let $(\alpha,\beta,\gamma)\neq(1,1,1)$. Then by [8, Lemma 4.3], $\gamma(\Gamma)=12$. Indeed $D_1=\{(0,0),(0,2),(0,4),(1,1),(1,3),(1,5)\}$ and $D_2=\{(0,1),(0,3),(0,5),(1,0),(1,2),(1,4)\}$ are minimal dominating sets for $\Gamma_1,\ \Gamma_2$, respectively. Vertex $(1,1)$ is adjacent to vertices $(0,0),(0,2)$ and vertex $(0,4)$ is adjacent to vertices $(1,3),(1,5)$. Thus $D_1$ becomes a $\gamma_t$-set for $\Gamma_1$. Similarly $D_2$ becomes a $\gamma_t$-set for $\Gamma_2$. Therefore $\gamma_t(\Gamma)=12$. □

**Proposition 4.1.** *Let* $\Gamma=\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^\alpha p^\beta r^\gamma},\Phi)$, *where* $\alpha,\beta,\gamma\geq 1$. *Then* $\mathrm{diam}(\Gamma)=3$.

Table 4.2: Shortest path between $(u,v),(u^{'},v^{'})$ in $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^{\alpha}q^{\beta}r^{\gamma}}, \Phi)$

| $u \neq u^{'}, v \neq v^{'}$ | shortest path between $(u,v),(u^{'},v^{'})$ | Comments |
|---|---|---|
| $v = 2kqr, k \in \mathbb{Z}, v^{'} \in \varphi_{2^{\alpha}q^{\beta}r^{\gamma}}$ | $(u,v)(u^{'},v^{'})$ | $d((u,v),(u^{'},v^{'})) = 1$ |
| $v$ is multiple of $2qr$ and $v^{'}$ is odd and multiple of $q$ | $(u,v)(u^{''},1)(u^{'''},(1+v^{'})r)(u^{'},v^{'})$ | |
| $v$ is multiple of $2qr$ and $v^{'}$ is odd and multiple of $r$ | $(u,v)(u^{''},1)(u^{'''},(1+v^{'})q)(u^{'},v^{'})$ | |
| $v$ is multiple of $2qr$ and $v^{'}$ is odd and multiple of $qr$ | $(u,v)(u^{''},1)(u^{'''},(1+v^{'})2)(u^{'},v^{'})$ | |
| $v = 2kr, k \in \mathbb{Z}, v^{'} \in \varphi_{2^{\alpha}q^{\beta}r^{\gamma}}$ | $(u,v)(u^{''},q)(u^{'''},(q+v^{'})r)(u^{'},v^{'})$ | |
| $v$ is multiple of $2r$ and $v^{'}$ is odd and multiple of $r$ | $(u,v)(u^{''},q)(u^{'''},\frac{q+v^{'}}{2})(u^{'},v^{'})$<br>$(u,v)(u^{''},q)(u^{'''},\frac{q+v^{'}}{2}+qr)(u^{'},v^{'})$ | if $\frac{q+v^{'}}{2}$ be even<br>if $\frac{q+v^{'}}{2}$ be odd |
| $v$ is multiple of $2r$ and $v^{'}$ is odd and multiple of $q$ | $(u,v)(u^{''},q)(u^{'''},2r)(u^{'},v^{'})$ | |
| $v$ is multiple of $2r$ and $v^{'}$ is odd and multiple of $qr$ | $(u,v)(u^{''},q)(u^{'''},\frac{q+v^{'}}{q})(u^{'},v^{'})$<br>$(u,v)(u^{''},q)(u^{'''},\frac{q+v^{'}}{q}+2r)(u^{'},v^{'})$ | if $\frac{q+v^{'}}{q} \neq kq, k \in \mathbb{Z}$<br>if $\frac{q+v^{'}}{q} = k^{'}q, k^{'} \in \mathbb{Z}$ |
| $v = 2kq, k \in \mathbb{Z}, v^{'} \in \varphi_{2^{\alpha}q^{\beta}r^{\gamma}}$ | $(u,v)(u^{''},r)(u^{'''},(r+v^{'})q)(u^{'},v^{'})$ | |
| $v$ is multiple of $2q$ and $v^{'}$ is odd and multiple of $q$ | $(u,v)(u^{''},r)(u^{'''},\frac{r+v^{'}}{2})(u^{'},v^{'})$<br>$(u,v)(u^{''},r)(u^{'''},\frac{r+v^{'}}{2}+qr)(u^{'},v^{'})$ | if $\frac{r+v^{'}}{2}$ be even<br>if $\frac{r+v^{'}}{2}$ be odd |
| $v$ is multiple of $2q$ and $v^{'}$ is odd and multiple of $r$ | $(u,v)(u^{''},r)(u^{'''},2q)(u^{'},v^{'})$ | |
| $v$ is multiple of $2q$ and $v^{'}$ is odd and multiple of $qr$ | $(u,v)(u^{''},r)(u^{'''},\frac{r+v^{'}}{r})(u^{'},v^{'})$<br>$(u,v)(u^{''},r)(u^{'''},\frac{r+v^{'}}{r}+2q)(u^{'},v^{'})$ | if $\frac{r+v^{'}}{r} \neq kr, k \in \mathbb{Z}$<br>if $\frac{r+v^{'}}{r} = k^{'}r, k^{'} \in \mathbb{Z}$ |
| $v = 2k, k \in \mathbb{Z}, v^{'} \in \varphi_{2^{\alpha}q^{\beta}r^{\gamma}}$ | $(u,v)(u^{''},qr)(u^{'''},(qr+v^{'})2)(u^{'},v^{'})$ | |
| $v$ is multiple of 2 and $v^{'}$ is odd and multiple of $q$ | $(u,v)(u^{''},qr)(u^{'''},\frac{qr+v^{'}}{q})(u^{'},v^{'})$<br>$(u,v)(u^{''},qr)(u^{'''},\frac{qr+v^{'}}{q}+2r)(u^{'},v^{'})$ | if $\frac{qr+v^{'}}{q} \neq kq, k \in \mathbb{Z}$<br>if $\frac{qr+v^{'}}{q} = k^{'}q, k^{'} \in \mathbb{Z}$ |
| $v$ is multiple of 2 and $v^{'}$ is odd and multiple of $r$ | $(u,v)(u^{''},qr)(u^{'''},\frac{qr+v^{'}}{r})(u^{'},v^{'})$<br>$(u,v)(u^{''},qr)(u^{'''},\frac{qr+v^{'}}{r}+2q)(u^{'},v^{'})$ | if $\frac{qr+v^{'}}{r} \neq kr, k \in \mathbb{Z}$<br>if $\frac{qr+v^{'}}{r} = k^{'}r, k^{'} \in \mathbb{Z}$ |
| $v$ is multiple of 2 and $v^{'}$ is odd and multiple of $qr$ | $(u,v)(u^{''},qr)(u^{'''},2)(u^{'},v^{'})$ | |

*Proof.* We proceed along the lines of Theorem 4.1, and $q := p$. Let $(u,v),(u^{'},v^{'})$ are arbitrary vertices of $\Gamma$. Then we have following three possibilities:

*i)* $u = u^{'}, v \neq v^{'}$. We know that $d((u,v),(u^{'},v^{'})) \geq 2$. Assume that $v$ and $v^{'}$ are both even or odd. Thus by case *i)* of Theorem 4.1, we have $d((u,v),(u^{'},v^{'})) = 2$. Suppose that one of either $v$ or $v^{'}$ is odd. Hence we have no path of length 2 between

$(u,v),(u^{'},v^{'})$. Now we show that $d((u,v),(u^{'},v^{'}))=3$. Without loss of generality assume that $v$ is even and $v^{'}$ is odd. If $v$ be multiple of $2pr$ and $v^{'}$ be multiple of $pr$, then $(u,v)(u^{''},pr-2)(u^{'''},pr-1)(u^{'},v^{'})$ is a path of length 3 between $(u,v)$ and $(u^{'},v^{'})$, where $u=u^{'}\neq u^{''}\neq u^{'''}$. If $v$ be multiple of $2pr$ and $v^{'}\in\varphi_{2^{\alpha}p^{\beta}r^{\gamma}}$, note that $v$ and $v^{'}$ are adjacent, then $(u,v)(u^{''},v^{'})(u^{'''},v)(u^{'},v^{'})$ is a shortest path. For other cases of $v$ and $v^{'}$ we are using of Table 4.2, where $u=u^{'}\neq u^{''}\neq u^{'''}$.

*ii)* $u\neq u^{'}, v=v^{'}$. In this case vertex $(u^{''},v-1)$ is a common neighbor between $(u,v)$ and $(u^{'},v^{'})$, where $u^{''}\neq u,u^{'}$. Thus $d((u,v),(u^{'},v^{'}))=2$.

*iii)* $u\neq u^{'}, v\neq v^{'}$. Let $v$ and $v^{'}$ be both even or odd. Then by Table 4.1, where $u^{''}\neq u,u^{'}$, we see that $d((u,v),(u^{'},v^{'}))=2$. Let one of either $v$ or $v^{'}$ be even and other be odd. Then by Table 4.2, where $u=u^{'''}$ and $u^{'}=u^{''}$, we see that $d((u,v),(u^{'},v^{'}))=3$. Therefore $\mathrm{diam}(\Gamma)=3$. □

**Proposition 4.2.** *Let* $\Gamma=\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^{\alpha}p^{\beta}r^{\gamma}},\Phi)$, *where* $\alpha,\beta,\gamma\geq 1$. *Then* $\gamma_t(\Gamma)$ *and* $\gamma_c(\Gamma)$ *is given by Table* 4.3.

Table 4.3: $\gamma_t(\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^{\alpha}p^{\beta}r^{\gamma}},\Phi))$ and $\gamma_c(\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^{\alpha}p^{\beta}r^{\gamma}},\Phi))$

| $\Gamma$ | $\gamma_t(\Gamma)$ | $\gamma_c(\Gamma)$ | Comments |
|---|---|---|---|
| $\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^{\alpha}p^{\beta}r^{\gamma}},\Phi)$ | 10 | 12 | one of the prime factors is 3 $p=3$ |
| $\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^{\alpha}p^{\beta}r^{\gamma}},\Phi)$ | 10 | 10 | one of the prime factors is 3 $p\geq 5$ |
| $\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^{\alpha}p^{\beta}r^{\gamma}},\Phi)$ | 8 | 8 | $p,r\geq 5$ |

*Proof.* Assume first that one of the prime factors is 3. In this case if $(\alpha,\beta,\gamma)=(1,1,1)$ then by [8, Lemma 4.2], $\gamma(\Gamma)=8$ and

$$D=\{(0,0),(0,1),(0,2),(0,3),(1,x_4),(1,x_4^{'}),(1,x_5),(1,x_5^{'})\}$$

is a $\gamma$-set for $\Gamma$. Vertices of $D$ are not adjacent to each other. Hence $\gamma_t(\Gamma)>8$. Note that $D$ is not dominated by one vertex, since every vertex $(u,v)\in V$, where $v$ is an odd (even) integer, is not adjacent to the vertex $(u^{'},v^{'})$, where $v^{'}$ is an odd (even) integer. This implies that $\gamma_t(\Gamma)>9$. Now we take another dominating set with cardinality 10. By [8, Proposition 4.4], we have $D^{'}=\{(0,0),(0,1),(0,2),(0,3),(1,4),(1,5),(2,6),(2,7),(2,8),(2,9)\}$ is a dominating set of $\Gamma$. If the other prime factor is 5, then vertices $(0,0),(0,1),(0,2),(0,3),(1,5)$ are adjacent to vertices $(2,7),(2,8),(2,9),(1,4),(2,6)$, respectively. Also let other prime factor be $\geq 7$ then vertices $(0,0),(0,1),(0,2),(0,3),(1,4)$ are adjacent to vertices $(1,5),(2,6),(2,7),(2,8),(2,9)$, respectively. Hence $D^{'}$ becomes a $\gamma_t$-set for $\Gamma$. Therefore $\gamma_t(\Gamma)=10$.

Let $(\alpha,\beta,\gamma)\neq(1,1,1)$. By [8, Proposition 4.4], $\gamma(\Gamma)=10$. By previous paragraph, $\gamma_t(\Gamma)=10$.

Now we find the connected domination number of $\Gamma$ where one of the prime factors is 3. By above discussion $\gamma_c(\Gamma) > 9$. We use again from $D^{'}$.

Let $p = 3$. Without loss of generality assume that $r = 5$. Then the subgraph generated by $D^{'}$ has exactly five connected components which are induced the subgraphs generated by sets $\{(0,0),(2,7)\}$, $\{(0,1),(2,8)\}$, $\{(0,2),(2,9)\}$, $\{(0,3),(1,4)\}$ and $\{(1,5),(2,6)\}$. Hence $\gamma_c(\Gamma) > 10$. Let a vertex say $(u,v) \in V(\Gamma)$, where $v$ is an odd integer, dominates all vertices $(0,0),(2,8),(0,2),(1,4),(2,6)$. Since $u \in \mathbb{Z}_3$, it is impossible. This implies that $\gamma_c(\Gamma) > 11$. Next consider another dominating with cardinality 12.

Let

$$A = \{(1,1),(2,2),(1,4),(2,5),(1,7),(2,8),(1,10),(2,11)\},$$

$$B = \{(0,0),(2,2),(0,3),(2,5),(0,6),(2,8),(0,9),(2,11)\}$$

and

$$C = \{(0,0),(1,1),(0,3),(1,4),(0,6),(1,7),(0,9),(1,10)\}.$$

Then $A$, $B$ and $C$ dominate $\{(0,v)|v \in \mathbb{Z}_{2^\alpha 3^\beta r^\gamma}\}$, $\{(1,v)|v \in \mathbb{Z}_{2^\alpha 3^\beta r^\gamma}\}$ and $\{(2,v)|v \in \mathbb{Z}_{2^\alpha 3^\beta r^\gamma}\}$ respectively. Thus

$$D^{''} = \{(0,0),(1,1),(2,2),(0,3),(1,4),(2,5),(0,6),(1,7),(2,8),(0,9),(1,10),(2,11)\}$$

is a dominating set for $\Gamma$. Both vertices next to each other in $D^{''}$ are adjacent. Hence the subgraph generated by $D^{''}$ is connected. Therefore $\gamma_c(\Gamma) = 12$.

Let $p \geq 5$. Then $D^{'''} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(0,5),(1,6),(2,7),(3,8),(4,9)\}$ is a dominating set for $\Gamma$. Both vertices next to each other in $D^{'''}$ are adjacent. Thus the subgraph generated by $D^{'''}$ is connected. Therefore $\gamma_c(\Gamma) = 10$.

Finally assume that $p, r \geq 5$. By [8, Lemma 4.2, Proposition 4.4], $\gamma(\Gamma) = 8$ and by using a proof of proposition 4.4, we know that $D^{''''} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(2,5),(1,6),(0,7)\}$ is a $\gamma$-set for $\Gamma$, where $\alpha,\beta,\gamma \geq 1$. Both vertices next to each other in $D^{''''}$ are adjacent. Hence $D^{''''}$ is a $\gamma_t$-set and $\gamma_c$-set for $\Gamma$. Therefore $\gamma_t(\Gamma) = \gamma_c(\Gamma) = 8$. □

**Proposition 4.3.** *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$*, where* $p, q, r \geq 3$ *and* $\alpha, \beta, \gamma \geq 1$. *Then* $\mathrm{diam}(\Gamma) = 2$.

*Proof.* Let $(u,v),(u^{'},v^{'})$ are arbitrary vertices of $\Gamma$. Then we have following three possibilities:

*i)* $u = u^{'}, v \neq v^{'}$. By Table 4.5, we show that $d((u,v),(u^{'},v^{'})) = 2$. In this table $u^{''} \neq u$.

*ii)* $u \neq u^{'}$ and $v = v^{'}$. In this case the vertex $(u^{''}, v-1)$ where $u^{''} \neq u, u^{'}$, is a common neighbor of $(u,v)$ and $(u^{'},v^{'})$. Thus $d((u,v),\,(u^{'},v^{'})) = 2$.

*iii)* $u \neq u^{'}$ and $v \neq v^{'}$. Hence by $(i)$ and $(ii)$, $d((u,v),\,(u^{'},v^{'})) = 2$.

Therefore $\mathrm{diam}(\Gamma) = 2$. □

Table 4.4: Common neighbor between $(u, v), (u^{'}, v^{'})$ in $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$

| $u = u^{'}, v \neq v^{'}$ | common neighbor | Comments |
|---|---|---|
| $v, v^{'} \in \varphi_{p^\alpha q^\beta r^\gamma}$ | $(u^{''}, pqr)$ | |
| $v, v^{'}$ are multiples of $pqr$ | $(u^{''}, pqr-1)$ | |
| $v, v^{'}$ are multiples of $pq$ | $(u^{''}, r)$ | |
| $v, v^{'}$ are multiples of $pr$ | $(u^{''}, q)$ | |
| $v, v^{'}$ are multiples of $qr$ | $(u^{''}, p)$ | |
| $v, v^{'}$ are multiples of $p$ | $(u^{''}, qr)$ | |
| $v \neq v^{'}$ and each of them is multiple of one of the prime factor and both of them are even or odd | $(u^{''}, \frac{v+v^{'}}{2})$<br>$(u^{''}, 2(v+v^{'}))$ | if $v - v^{'} \in \varphi_{p^\alpha q^\beta r^\gamma}$<br>if $v - v^{'} \notin \varphi_{p^\alpha q^\beta r^\gamma}$ |
| $v$ is multiple of $p$ and $v^{'}$ is multiple of $pq$ | $(u^{''}, (\frac{v}{p})r + pqr)$<br>$(u^{''}, (\frac{v}{p})r + qr)$ | if $\frac{v}{p} \in \varphi_{p^\alpha q^\beta r^\gamma}$<br>if $\frac{v}{p}$ be multiple of $p$ |
| $v$ is multiple of $p$ and $v^{'}$ is multiple of $pr$ | $(u^{''}, (\frac{v}{p})q + pqr)$<br>$(u^{''}, (\frac{v}{p})q + qr)$ | if $\frac{v}{p} \in \varphi_{p^\alpha q^\beta r^\gamma}$<br>if $\frac{v}{p}$ be multiple of $p$ |
| $v$ is multiple of $p$ and $v^{'}$ is multiple of $qr$ | $(u^{''}, \frac{v+v^{'}}{2})$<br>$(u^{''}, 2(v+v^{'}))$ | if $v, v^{'}$ be both even or odd<br>if one of them be odd and other be even |
| $v$ is multiple of $p$ and $v^{'}$ is multiple of $pqr$ | $(u^{''}, \frac{v}{p} + pqr)$<br>$(u^{''}, \frac{v}{p} + qr)$ | if $\frac{v}{p}$ be non-multiple of $p$<br>if $\frac{v}{p}$ be multiple of $p$ |
| $v$ is multiple of $p$ and $v^{'} \in \varphi_{p^\alpha q^\beta r^\gamma}$ | $(u^{''}, (v+v^{'})qr)$<br>$(u^{''}, v^{'}qr)$ | if $v, v^{'}$ be both even or odd<br>if one of them be odd and other be even |
| $v, v^{'}$ are multiples of $q$ | $(u^{''}, pr)$ | |
| $v$ is multiple of $q$ and $v^{'}$ is multiple of $pq$ | $(u^{''}, (\frac{v}{q})r + pqr)$<br>$(u^{''}, (\frac{v}{q})r + pr)$ | if $\frac{v}{q} \in \varphi_{p^\alpha q^\beta r^\gamma}$<br>if $\frac{v}{q}$ be multiole of $q$ |
| $v$ is multiple of $q$ and $v^{'}$ is multiple of $qr$ | $(u^{''}, (\frac{v}{q})p + pqr)$<br>$(u^{''}, (\frac{v}{q})p + pr)$ | if $\frac{v}{q} \in \varphi_{p^\alpha q^\beta r^\gamma}$<br>if $\frac{v}{q}$ be multiole of $q$ |
| $v$ is multiple of $q$ and $v^{'}$ is multiple of $pr$ | $(u^{''}, \frac{v+v^{'}}{2})$<br>$(u^{''}, 2(v+v^{'}))$ | if $v, v^{'}$ be both even or odd<br>if one of them be odd and other be even |
| $v$ is multiple of $q$ and $v^{'}$ is multiple of $pqr$ | $(u^{''}, \frac{v}{q} + pqr)$<br>$(u^{''}, \frac{v}{q} + pr)$ | if $\frac{v}{q}$ be non-multiple of $q$<br>if $\frac{v}{q}$ be multiple of $q$ |
| $v$ is multiple of $q$ and $v^{'} \in \varphi_{p^\alpha q^\beta r^\gamma}$ | $(u^{''}, (v+v^{'})pr)$<br>$(u^{''}, v^{'}pr)$ | if $v, v^{'}$ be both even or odd<br>if one of them be odd and other be even |

Table 4.5: Shortest path between $(u,v),(u^{'},v^{'})$ in $\Gamma=\mathrm{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{2^{\alpha}q^{\beta}r^{\gamma}},\Phi)$

| $u=u^{'},v\neq v^{'}$ | common neighbor | Comments |
|---|---|---|
| $v,v^{'}$ are multiple of $r$ | $(u^{''},pq)$ | |
| $v$ is multiple of $r$ and $v^{'}$ is multiple of $pr$ | $(u^{''},(\frac{v}{r})q+pqr)$<br>$(u^{''},(\frac{v}{r})q+pq)$ | if $\frac{v}{r}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $\frac{v}{r}$ be multiole of $r$ |
| $v$ is multiple of $r$ and $v^{'}$ is multiple of $qr$ | $(u^{''},(\frac{v}{r})p+pqr)$<br>$(u^{''},(\frac{v}{r})p+pq)$ | if $\frac{v}{r}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $\frac{v}{r}$ be multiole of $r$ |
| $v$ is multiple of $r$ and $v^{'}$ is multiple of $pq$ | $(u^{''},\frac{v+v^{'}}{2})$<br>$(u^{''},2(v+v^{'}))$ | if $v,v^{'}$ be both even or odd<br>if one of them be odd |
| $v$ is multiple of $r$ and $v^{'}$ is multiple of $pqr$ | $(u^{''},\frac{v}{r}+pqr)$<br>$(u^{''},\frac{v}{r}+pq)$ | if $\frac{v}{r}$ be non-multiple of $r$<br>if $\frac{v}{r}$ be multiple of $r$ |
| $v$ is multiple of $r$ and $v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ | $(u^{''},(v+v^{'})pq)$<br>$(u^{''},v^{'}pq)$ | if $v,v^{'}$ be both even or odd<br>if one of them be odd |
| $v$ is multiple of $pq$ and $v^{'}$ is multiple of $pr$ | $(u^{''},\frac{v+v^{'}}{p})$<br>$(u^{''},\frac{v+v^{'}}{p}+qr)$ | if $\frac{v+v^{'}}{p}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $\frac{v+v^{'}}{p}$ be multiple of $p$ |
| $v$ is multiple of $pq$ and $v^{'}$ is multiple of $rq$ | $(u^{''},\frac{v+v^{'}}{q})$<br>$(u^{''},\frac{v+v^{'}}{q}+pr)$ | if $\frac{v+v^{'}}{q}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $\frac{v+v^{'}}{q}$ be multiple of $q$ |
| $v$ is multiple of $pr$ and $v^{'}$ is multiple of $rq$ | $(u^{''},\frac{v+v^{'}}{r})$<br>$(u^{''},\frac{v+v^{'}}{r}+pq)$ | if $\frac{v+v^{'}}{r}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $\frac{v+v^{'}}{r}$ be multiple of $r$ |
| $v=kpq,k\in\mathbb{Z},v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ and<br>$v,v^{'}$ are both even or odd<br>one of the $v$ or $v^{'}$ is odd | $(u^{''},\frac{v+v^{'}}{2})$<br>$(u^{''},v-v^{'})$<br>$(u^{''},v^{'}r)$ | if $v-v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ |
| $v=kpr,k\in\mathbb{Z},v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$and<br>$v,v^{'}$ are both even or odd<br>one of the $v$ or $v^{'}$ is odd | $(u^{''},\frac{v+v^{'}}{2})$<br>$(u^{''},v-v^{'})$<br>$(u^{''},v^{'}q)$ | if $v-v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ |
| $v=kqr,k\in\mathbb{Z},v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ and<br>$v,v^{'}$ are both even or odd<br>one of the $v$ or $v^{'}$ is odd | $(u^{''},\frac{v+v^{'}}{2})$<br>$(u^{''},v-v^{'})$<br>$(u^{''},v^{'}p)$ | if $v-v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$<br>if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ |
| $v=kpqr,k\in\mathbb{Z}$ and $v^{'}$<br>is multiple of $pq$ or $pr$ or $qr$ | $(u^{''},+4)$ or $(u^{''},-4)$ | by Proposition 4.5[8], $\lambda=4$<br>then $pqr$ is adjacent by $\pm4$<br>and $pq,pr,qr$ are<br>adjacent by $+4$ or $-4$ |
| $v$ is multiple of $pqr$ and $v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ | $(u^{''},\frac{v+v^{'}}{2})$<br>$(u^{''},2(v+v^{'}))$ | if $v,v^{'}$ be both even or odd<br>if one of them be odd |

**Proposition 4.4.** *Let* $\Gamma=\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{p^{\alpha}q^{\beta}r^{\gamma}},\Phi)$, *where* $p,q,r\geq 3$ *and* $\alpha,\beta,\gamma\geq 1$. *Then* $\gamma_t(\Gamma)$ *and* $\gamma_c(\Gamma)$ *is given by Table 4.6.*

Table 4.6: $\gamma_t(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi))$ and $\gamma_c(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi))$ where $p, q, r \geq 3$

| $\Gamma$ | $\gamma_t(\Gamma), \gamma_c(\Gamma)$ | Comments |
|---|---|---|
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$ | $6 \leq \gamma_t(\Gamma), \gamma_c(\Gamma) \leq 8$ | one of the prime factors is 3 |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$ | 5 | $p, q, r \geq 5$ |

*Proof.* By using the [8, Proposition 4.5], $D, D^{'}, D^{''}, D^{'''}$ are minimal dominating sets for various cases in this graph. Clearly the subgraphs generated by $D$, $D^{'}$, $D^{''}$ and $D^{'''}$ are all connected. Therefore $\gamma(\Gamma) = \gamma_t(\Gamma) = \gamma_c(\Gamma)$. □

As an immediate consequence of Lemma 4.2 and Propositions 4.2, 4.4, we have the following theorem.

**Theorem 4.1.** *Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$, where $p, q, r \geq 2$ and $\alpha, \beta, \gamma \geq 1$. Then $\gamma_t(\Gamma)$and $\gamma_c(\Gamma)$ is given by Table 4.7.*

Table 4.7: $\gamma_t(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi))$ and $\gamma_c(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi))$ where $\alpha, \beta, \gamma \geq 1$

| $\Gamma$ | $\gamma_t(\Gamma)$ | $\gamma_c(\Gamma)$ | Comments |
|---|---|---|---|
| $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta r^\gamma}, \Phi)$ | 12 | does not exist | |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$ | 10 | 12 | one of the prime factors is 3<br>$p = 3$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$ | 10 | 10 | one of the prime factors is 3<br>$p \geq 5$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$ | 8 | 8 | $p, r \geq 5$ |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$ | $6 \leq \gamma_t(\Gamma) \leq 8$ | $6 \leq \gamma_c(\Gamma) \leq 8$ | one of the prime factors is 3 |
| $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$ | 5 | 5 | $p, q, r \geq 5$ |

As an immediate consequence of Lemmas 3.1, 4.1 and Propositions 3.1, 3.3, 4.1, 4.3, we have the following theorem.

**Theorem 4.2.** *Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_m, \Phi)$, where $m \in \{p^\alpha q^\beta, p^\alpha q^\beta r^\gamma\}$. Then*

*1)* $\mathrm{diam}(\Gamma) = 3$ *where one of the prime factors is* 2.

*2)* $\mathrm{diam}(\Gamma) = 2$ *where* $p, q, r \geq 3$.

**Remark 4.1.** Let $p_1, p_2, \ldots, p_k$ be consecutive prime numbers, $p_1 = 3$, $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers, $\alpha \geq 2$, and $\Phi = \varphi_2 \times \varphi_{2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}$. Then by [8, Theorem 4.7], we have $\gamma(\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}, \Phi)) \geq 4k+4$. Therefore $\gamma_t(\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}, \Phi)) \geq 4k + 4$. Since $\Gamma$ is a disconnected graph, the $\gamma_c$*-set* does not exist for $\Gamma$.

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