

SOLVING THE FUZZY INITIAL VALUE PROBLEM WITH NEGATIVE COEFFICIENT BY USING FUZZY LAPLACE TRANSFORM

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Abstract. In this paper, the fuzzy initial value problem with negative coefficient is solved by using fuzzy Laplace transform and generalized differentiability. The solutions are found and the comparison results are given.

Keywords: fuzzy initial value problem, generalized differentiability, fuzzy Laplace transform.

1. Introduction

Fuzzy differential equations have been studied by many researchers. Fuzzy differential equations can be solved by several types. Hukuhara differentiability [10, 17, 23], generalized differentiability [7, 8, 9, 12], extension principle [10, 11], the concept of differential inclusion [16] and the fuzzy problem to be a set of crisp problem [14]. Another types are numeric methods [1, 2, 3, 4, 15] and the fuzzy Laplace transform [5, 21, 22, 24].

This paper is about the solutions of the fuzzy initial value problem with negative coefficient by fuzzy Laplace transform. The aim of this study is to investigate solutions of problem using the properties fuzzy Laplace transform and generalized differentiability.

The paper is organized as follows: Section 2 delas with preliminaries, Section 3 focuses on findings and the main results, and Section 4 refers to conclusions.

2. Preliminaries

Definition 2.1. [20] A fuzzy number is a mapping $u : \mathbb{R} \rightarrow [0, 1]$ satisfying the following properties:

u is normal: $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,

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u is convex fuzzy set: $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$,

u is upper semi-continuous on \mathbb{R} ,

$cl\{x \in \mathbb{R} \mid u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

Let \mathbb{R}_F denote the set of all fuzzy numbers.

Definition 2.2. [18] Let $u \in \mathbb{R}_F$. The α -level set of u , denoted $[u]^\alpha$, $0 < \alpha \leq 1$, is $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$. If $\alpha = 0$, $[u]^0 = cl\{x \in \mathbb{R} \mid u(x) > 0\}$. The notation, $[u]^\alpha = [\underline{u}_\alpha, \bar{u}_\alpha]$ denotes explicitly the α -level set of u , where \underline{u}_α and \bar{u}_α denote the left-hand endpoint and the right-hand endpoint of $[u]^\alpha$, respectively.

The following remark shows when $[\underline{u}_\alpha, \bar{u}_\alpha]$ is a valid α -level set.

Remark 2.1. [13, 18] The sufficient and necessary conditions for $[\underline{u}_\alpha, \bar{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

\underline{u}_α is bounded monotonic increasing (nondecreasing) left-continuous function on $(0, 1]$ and right-continuous for $\alpha = 0$,

\bar{u}_α is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0, 1]$ and right-continuous for $\alpha = 0$,

$$\underline{u}_\alpha \leq \bar{u}_\alpha, 0 \leq \alpha \leq 1.$$

Definition 2.3. [20] If A is a symmetric triangular fuzzy number with support $[\underline{a}, \bar{a}]$, the α -level set of A is $[A]^\alpha = [\underline{A}_\alpha, \bar{A}_\alpha] = \left[\underline{a} + \left(\frac{\bar{a} - \underline{a}}{2} \right) \alpha, \bar{a} - \left(\frac{\bar{a} - \underline{a}}{2} \right) \alpha \right]$, ($\underline{A}_1 = \bar{A}_1, \underline{A}_1 - \underline{A}_\alpha = \bar{A}_\alpha - \bar{A}_1$).

Definition 2.4. [15, 18, 23] Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the Hukuhara difference of fuzzy numbers u and v , and it is denoted by $w = u \ominus v$.

Definition 2.5. [6, 15, 18] Let $f : [a, b] \rightarrow \mathbb{R}_F$ and $t_0 \in [a, b]$. We say that f is Hukuhara differentiable at t_0 , if there exists an element $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits hold

$$\lim_{h \rightarrow 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \rightarrow 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0).$$

Definition 2.6. [18] Let $f : [a, b] \rightarrow \mathbb{R}_F$ and $t_0 \in [a, b]$. We say that f is (1)-differentiable at t_0 , if there exists an element $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small near to 0, exist $f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits

$$\lim_{h \rightarrow 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \rightarrow 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0),$$

and f is (2)-differentiable if for all $h > 0$ sufficiently small near to 0, exist $f(t_0) \ominus f(t_0 + h)$, $f(t_0 - h) \ominus f(t_0)$ and the limits

$$\lim_{h \rightarrow 0} \frac{f(t_0) \ominus f(t_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{f(t_0 - h) \ominus f(t_0)}{-h} = f'(t_0).$$

Theorem 2.1. [19] Let $f : [a, b] \rightarrow \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = [\underline{f}_\alpha(t), \bar{f}_\alpha(t)]$, for each $\alpha \in [0, 1]$.

(i) If f is (1)-differentiable, then \underline{f}_α and \bar{f}_α are differentiable functions and $[f'(t)]^\alpha = [\underline{f}'_\alpha(t), \bar{f}'_\alpha(t)]$,

(ii) If f is (2)-differentiable, then \underline{f}_α and \bar{f}_α are differentiable functions and $[f'(t)]^\alpha = [\bar{f}'_\alpha(t), \underline{f}'_\alpha(t)]$.

Theorem 2.2. [19] Let $f' : [a, b] \rightarrow \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = [\underline{f}_\alpha(t), \bar{f}_\alpha(t)]$, for each $\alpha \in [0, 1]$, f is (1)-differentiable or (2)-differentiable.

(i) If f and f' are (1)-differentiable, then \underline{f}'_α and \bar{f}'_α are differentiable functions and $[f''(t)]^\alpha = [\underline{f}''_\alpha(t), \bar{f}''_\alpha(t)]$,

(ii) If f is (1)-differentiable and f' is (2)-differentiable, then \underline{f}'_α and \bar{f}'_α are differentiable functions and $[f''(t)]^\alpha = [\bar{f}''_\alpha(t), \underline{f}''_\alpha(t)]$,

(iii) If f is (2)-differentiable and f' is (1)-differentiable, then \underline{f}'_α and \bar{f}'_α are differentiable functions and $[f''(t)]^\alpha = [\bar{f}''_\alpha(t), \underline{f}''_\alpha(t)]$,

(iv) If f and f' are (2)-differentiable, then \underline{f}'_α and \bar{f}'_α are differentiable functions and $[f''(t)]^\alpha = [\underline{f}''_\alpha(t), \bar{f}''_\alpha(t)]$.

Definition 2.7. [22, 24] The fuzzy Laplace transform of fuzzy-valued function f is defined as follows:

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t) dt = \lim_{\tau \rightarrow \infty} \int_0^\tau e^{-st} f(t) dt,$$

$$F(s) = L(f(t)) = \left[\lim_{\tau \rightarrow \infty} \int_0^\tau e^{-st} \underline{f}(t) dt, \lim_{\tau \rightarrow \infty} \int_0^\tau e^{-st} \bar{f}(t) dt \right].$$

$$F(s, \alpha) = L(f(t, \alpha)) = [L(\underline{f}(t, \alpha)), L(\bar{f}(t, \alpha))]$$

where,

$$L(\underline{f}(t, \alpha)) = \int_0^{\infty} e^{-st} \underline{f}(t, \alpha) dt = \lim_{\tau \rightarrow \infty} \int_0^{\tau} e^{-st} \underline{f}(t, \alpha) dt,$$

$$L(\overline{f}(t, \alpha)) = \int_0^{\infty} e^{-st} \overline{f}(t, \alpha) dt = \lim_{\tau \rightarrow \infty} \int_0^{\tau} e^{-st} \overline{f}(t, \alpha) dt.$$

Theorem 2.3. [5, 22, 24] Suppose that f is continuous fuzzy-valued function on $[0, \infty)$ and exponential order α and that f' is piecewise continuous fuzzy-valued function on $[0, \infty)$, then

$$L(f'(t)) = sL(f(t)) \ominus f(0),$$

if f is (1) differentiable,

$$L(f'(t)) = (-f(0)) \ominus (-sL(f(t))),$$

if f is (2) differentiable.

Theorem 2.4. [22, 24] Suppose that f and f' are continuous fuzzy-valued functions on $[0, \infty)$ and of exponential order α and that f'' is piecewise continuous fuzzy-valued function on $[0, \infty)$, then

$$L(f''(t)) = s^2L(f(t)) \ominus sf(0) \ominus f'(0),$$

if f and f' are (1) differentiable,

$$L(f''(t)) = -f'(0) \ominus (-s^2)L(f(t)) - sf(0),$$

if f is (1) differentiable and f' is (2) differentiable,

$$L(f''(t)) = -sf(0) \ominus (-s^2)L(f(t)) \ominus f'(0),$$

if f is (2) differentiable and f' is (1) differentiable,

$$L(f''(t)) = s^2L(f(t)) \ominus sf(0) - f'(0),$$

if f and f' are (2) differentiable.

Theorem 2.5. [5, 22] Let $f(x)$, $g(x)$ be continuous fuzzy-valued functions suppose that c_1 and c_2 are constant, then

$$L(c_1f(x) + c_2g(x)) = (c_1L(f(x))) + (c_2L(g(x))).$$

Theorem 2.6. [5] Let $f(x)$ be continuous fuzzy-valued function on $[0, \infty)$ and $\lambda \geq 0$, then

$$L(\lambda f(x)) = \lambda(L(f(x))).$$

3. Findings and Main Results

In this section, we consider solutions of the fuzzy initial value problem

$$(3.1) \quad y''(t) = -\lambda y(t), \quad t > 0,$$

$$(3.2) \quad y(0) = [A]^\alpha, \quad y'(0) = [B]^\alpha,$$

by Laplace transform, where $\lambda > 0$, A and B are symmetric triangular fuzzy numbers with supports $[\underline{a}, \bar{a}]$ and $[\underline{b}, \bar{b}]$, respectively,

$$[A]^\alpha = [\underline{A}_\alpha, \bar{A}_\alpha] = \left[\underline{a} + \left(\frac{\bar{a} - \underline{a}}{2} \right) \alpha, \bar{a} - \left(\frac{\bar{a} - \underline{a}}{2} \right) \alpha \right],$$

$$[B]^\alpha = [\underline{B}_\alpha, \bar{B}_\alpha] = \left[\underline{b} + \left(\frac{\bar{b} - \underline{b}}{2} \right) \alpha, \bar{b} - \left(\frac{\bar{b} - \underline{b}}{2} \right) \alpha \right],$$

(i,j) solution means that y is (i) differentiable, y' is (j) differentiable.

Case 1) If y and y' are (1) differentiable, since

$$s^2 L(y(t, \alpha)) \ominus sy(0, \alpha) \ominus y'(0, \alpha) = -\lambda L(y(t, \alpha)),$$

we have the equations

$$s^2 L(\underline{y}(t, \alpha)) - s\underline{y}(0, \alpha) - \underline{y}'(0, \alpha) = -\lambda \underline{y}(t, \alpha),$$

$$s^2 L(\bar{y}(t, \alpha)) - s\bar{y}(0, \alpha) - \bar{y}'(0, \alpha) = -\lambda \bar{y}(t, \alpha).$$

Using the initial values and taking the necessary operations,

$$L(\underline{y}(t, \alpha)) = \frac{s^2}{s^4 - \lambda^2} \underline{B}_\alpha + \frac{s^3}{s^4 - \lambda^2} \underline{A}_\alpha - \frac{\lambda s}{s^4 - \lambda^2} \bar{A}_\alpha - \frac{\lambda}{s^4 - \lambda^2} \bar{B}_\alpha,$$

$$L(\bar{y}(t, \alpha)) = \frac{s^2}{s^4 - \lambda^2} \bar{B}_\alpha + \frac{s^3}{s^4 - \lambda^2} \bar{A}_\alpha - \frac{\lambda s}{s^4 - \lambda^2} \underline{A}_\alpha - \frac{\lambda}{s^4 - \lambda^2} \underline{B}_\alpha.$$

From here,

$$\begin{aligned} \underline{y}(t, \alpha) &= L^{-1} \left(\frac{s^2}{s^4 - \lambda^2} \right) \underline{B}_\alpha + L^{-1} \left(\frac{s^3}{s^4 - \lambda^2} \right) \underline{A}_\alpha \\ &\quad - L^{-1} \left(\frac{\lambda s}{s^4 - \lambda^2} \right) \bar{A}_\alpha - L^{-1} \left(\frac{\lambda}{s^4 - \lambda^2} \right) \bar{B}_\alpha, \end{aligned}$$

$$\begin{aligned}\bar{y}(t, \alpha) = & L^{-1} \left(\frac{s^2}{s^4 - \lambda^2} \right) \bar{B}_\alpha + L^{-1} \left(\frac{s^3}{s^4 - \lambda^2} \right) \bar{A}_\alpha \\ & - L^{-1} \left(\frac{\lambda s}{s^4 - \lambda^2} \right) \underline{A}_\alpha - L^{-1} \left(\frac{\lambda}{s^4 - \lambda^2} \right) \underline{B}_\alpha.\end{aligned}$$

Thus, the lower solution and the upper solution are obtained as

$$\begin{aligned}\underline{y}(t, \alpha) = & \frac{e^{\sqrt{\lambda}t}}{4} \left(\frac{\underline{B}_\alpha - \bar{B}_\alpha}{\sqrt{\lambda}} + \underline{A}_\alpha - \bar{A}_\alpha \right) \\ & + \frac{e^{-\sqrt{\lambda}t}}{4} \left(\frac{\bar{B}_\alpha - \underline{B}_\alpha}{\sqrt{\lambda}} + \underline{A}_\alpha - \bar{A}_\alpha \right) \\ & + \frac{\sin(\sqrt{\lambda}t)}{2\sqrt{\lambda}} (\underline{B}_\alpha + \bar{B}_\alpha) + \frac{\cos(\sqrt{\lambda}t)}{2} (\underline{A}_\alpha + \bar{A}_\alpha),\end{aligned}$$

$$\begin{aligned}\bar{y}(t, \alpha) = & \frac{e^{\sqrt{\lambda}t}}{4} \left(\frac{\bar{B}_\alpha - \underline{B}_\alpha}{\sqrt{\lambda}} + \bar{A}_\alpha - \underline{A}_\alpha \right) \\ & + \frac{e^{-\sqrt{\lambda}t}}{4} \left(\frac{\underline{B}_\alpha - \bar{B}_\alpha}{\sqrt{\lambda}} + \bar{A}_\alpha - \underline{A}_\alpha \right) \\ & + \frac{\sin(\sqrt{\lambda}t)}{2\sqrt{\lambda}} (\underline{B}_\alpha + \bar{B}_\alpha) + \frac{\cos(\sqrt{\lambda}t)}{2} (\underline{A}_\alpha + \bar{A}_\alpha).\end{aligned}$$

Case 2) If y is (1) differentiable and y' is (2) differentiable, since

$$-y'(0, \alpha) \ominus (-s^2) L(y(t, \alpha)) - sy(0, \alpha) = -\lambda L(y(t, \alpha)),$$

we have the equations

$$-\bar{y}'(0, \alpha) - (-s^2 L(\bar{y}(t, \alpha))) - s\bar{y}(0, \alpha) = -\lambda L(\bar{y}(t, \alpha)),$$

$$-\underline{y}'(0, \alpha) - (-s^2 L(\underline{y}(t, \alpha))) - s\underline{y}(0, \alpha) = -\lambda L(\underline{y}(t, \alpha)).$$

Using the initial values, we get

$$L(\underline{y}(t, \alpha)) = \frac{1}{s^2 + \lambda} \underline{B}_\alpha + \frac{s}{s^2 + \lambda} \underline{A}_\alpha.$$

$$L(\bar{y}(t, \alpha)) = \frac{1}{s^2 + \lambda} \bar{B}_\alpha + \frac{s}{s^2 + \lambda} \bar{A}_\alpha.$$

Taking inverse Laplace transforms of the equations, we have

$$\underline{y}(t, \alpha) = L^{-1} \left(\frac{1}{s^2 + \lambda} \right) \underline{B}_\alpha + L^{-1} \left(\frac{s}{s^2 + \lambda} \right) \underline{A}_\alpha,$$

$$\bar{y}(t, \alpha) = L^{-1} \left(\frac{1}{s^2 + \lambda} \right) \bar{B}_\alpha + L^{-1} \left(\frac{s}{s^2 + \lambda} \right) \bar{A}_\alpha.$$

From this, the lower and the upper solutions are obtained as

$$\underline{y}(t, \alpha) = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t) \underline{B}_\alpha + \cos(\sqrt{\lambda}t) \underline{A}_\alpha,$$

$$\bar{y}(t, \alpha) = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t) \bar{B}_\alpha + \cos(\sqrt{\lambda}t) \bar{A}_\alpha.$$

Case 3) If y is (2) differentiable and y' is (1) differentiable, since

$$-sy(0, \alpha) \ominus (-s^2)L(y(t, \alpha)) \ominus y'(0, \alpha) = -\lambda L(y(t, \alpha))$$

we have the equations

$$-s\bar{y}(0, \alpha) - (-s^2L(\bar{y}(t, \alpha))) - \bar{y}'(0, \alpha) = -\lambda L(\bar{y}(t, \alpha))$$

$$-s\underline{y}(0, \alpha) - (-s^2L(\underline{y}(t, \alpha))) - \underline{y}'(0, \alpha) = -\lambda L(\underline{y}(t, \alpha)).$$

Using the initial values and taking inverse Laplace transforms of the equations, we have

$$\underline{y}(t, \alpha) = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t) \underline{B}_\alpha + \cos(\sqrt{\lambda}t) \underline{A}_\alpha.$$

$$\bar{y}(t, \alpha) = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t) \bar{B}_\alpha + \cos(\sqrt{\lambda}t) \bar{A}_\alpha.$$

Case 4) If y is (2) differentiable and y' is (2) differentiable, since

$$s^2L(y(t, \alpha)) \ominus sy(0, \alpha) - y'(0, \alpha) = -\lambda L(y(t, \alpha))$$

we have the equations

$$s^2L(\underline{y}(t, \alpha)) - s\underline{y}(0, \alpha) - \underline{y}'(0, \alpha) = -\lambda L(\underline{y}(t, \alpha))$$

$$s^2L(\bar{y}(t, \alpha)) - s\bar{y}(0, \alpha) - \bar{y}'(0, \alpha) = -\lambda L(\bar{y}(t, \alpha))$$

Using the initial values and taking the necessary operations,

$$L(\underline{y}(t, \alpha)) = \frac{s^3}{s^4 - \lambda^2} \underline{A}_\alpha + \frac{s^2}{s^4 - \lambda^2} \underline{B}_\alpha - \frac{\lambda s}{s^4 - \lambda^2} \bar{A}_\alpha - \frac{\lambda}{s^4 - \lambda^2} \underline{B}_\alpha,$$

$$L(\bar{y}(t, \alpha)) = \frac{s^3}{s^4 - \lambda^2} \bar{A}_\alpha + \frac{s^2}{s^4 - \lambda^2} \underline{B}_\alpha - \frac{\lambda s}{s^4 - \lambda^2} \underline{A}_\alpha - \frac{\lambda}{s^4 - \lambda^2} \bar{B}_\alpha.$$

Taking inverse Laplace transforms of the equations, the lower and the upper solutions are obtained as

$$\begin{aligned} \underline{y}(t, \alpha) &= \frac{e^{\sqrt{\lambda}t}}{4} \left(\frac{\bar{B}_\alpha - \underline{B}_\alpha}{\sqrt{\lambda}} + \underline{A}_\alpha - \bar{A}_\alpha \right) \\ &+ \frac{e^{-\sqrt{\lambda}t}}{4} \left(\frac{\underline{B}_\alpha - \bar{B}_\alpha}{\sqrt{\lambda}} + \underline{A}_\alpha - \bar{A}_\alpha \right) \\ &+ \frac{\sin(\sqrt{\lambda}t)}{2\sqrt{\lambda}} (\underline{B}_\alpha + \bar{B}_\alpha) + \frac{\cos(\sqrt{\lambda}t)}{2} (\underline{A}_\alpha + \bar{A}_\alpha), \end{aligned}$$

$$\begin{aligned} \bar{y}(t, \alpha) &= \frac{e^{\sqrt{\lambda}t}}{4} \left(\frac{\underline{B}_\alpha - \bar{B}_\alpha}{\sqrt{\lambda}} + \bar{A}_\alpha - \underline{A}_\alpha \right) \\ &+ \frac{e^{-\sqrt{\lambda}t}}{4} \left(\frac{\bar{B}_\alpha - \underline{B}_\alpha}{\sqrt{\lambda}} + \bar{A}_\alpha - \underline{A}_\alpha \right) \\ &+ \frac{\sin(\sqrt{\lambda}t)}{2\sqrt{\lambda}} (\underline{B}_\alpha + \bar{B}_\alpha) + \frac{\cos(\sqrt{\lambda}t)}{2} (\underline{A}_\alpha + \bar{A}_\alpha). \end{aligned}$$

Theorem 3.1. (1,1) solution of the initial value problem (3.1)-(3.2) is a valid α -level set for $t > 0$ satisfying the inequality

$$e^{2\sqrt{\lambda}t} \geq \left(\frac{(\bar{b} - \underline{b}) - \sqrt{\lambda}(\bar{a} - \underline{a})}{(\bar{b} - \underline{b}) + \sqrt{\lambda}(\bar{a} - \underline{a})} \right).$$

Proof. If

$$\frac{\partial \underline{y}(t, \alpha)}{\partial \alpha} \geq 0, \quad \frac{\partial \bar{y}(t, \alpha)}{\partial \alpha} \leq 0, \quad \underline{y}(t, \alpha) \leq \bar{y}(t, \alpha)$$

(1,1) solution of the initial value problem (3.1)-(3.2) is valid α -level set. Thus, it must be

$$e^{\sqrt{\lambda}t} \left(\frac{\bar{b} - \underline{b}}{\sqrt{\lambda}} + \bar{a} - \underline{a} \right) - e^{-\sqrt{\lambda}t} \left(\frac{\bar{b} - \underline{b}}{\sqrt{\lambda}} - \bar{a} - \underline{a} \right) \geq 0.$$

Since

$$(\bar{b} - \underline{b}) + \sqrt{\lambda}(\bar{a} - \underline{a}) \geq 0,$$

we have

$$e^{\sqrt{\lambda}t} \geq e^{-\sqrt{\lambda}t} \left(\frac{(\bar{b} - \underline{b}) - \sqrt{\lambda}(\bar{a} - \underline{a})}{(\bar{b} - \underline{b}) + \sqrt{\lambda}(\bar{a} - \underline{a})} \right).$$

Consequently, (1,1) solution of the initial value problem (3.1)-(3.2) is a valid α -level set for $t > 0$ satisfying the inequality

$$e^{2\sqrt{\lambda}t} \geq \left(\frac{(\bar{b} - \underline{b}) - \sqrt{\lambda}(\bar{a} - \underline{a})}{(\bar{b} - \underline{b}) + \sqrt{\lambda}(\bar{a} - \underline{a})} \right).$$

□

Theorem 3.2. (1,2) solution of the initial value problem (3.1)-(3.2) is valid α -level set, for $t \in \left(0, \frac{\pi}{2\sqrt{\lambda}}\right)$ satisfying the inequality

$$t \geq \frac{1}{\sqrt{\lambda}} \tan^{-1} \left(-\sqrt{\lambda} \left(\frac{\bar{a} - \underline{a}}{\bar{b} - \underline{b}} \right) \right).$$

Proof. If

$$\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t) (\bar{b} - \underline{b}) + \cos(\sqrt{\lambda}t) (\bar{a} - \underline{a}) \geq 0,$$

(1,2) solution of the initial value problem (3.1)-(3.2) is valid α -level set. For $\sqrt{\lambda}t \in \left(0, \frac{\pi}{2}\right) \Rightarrow t \in \left(0, \frac{\pi}{2\sqrt{\lambda}}\right)$, we have

$$\tan(\sqrt{\lambda}t) \geq -\sqrt{\lambda} \left(\frac{\bar{a} - \underline{a}}{\bar{b} - \underline{b}} \right) \Rightarrow t \geq \frac{1}{\sqrt{\lambda}} \tan^{-1} \left(-\sqrt{\lambda} \left(\frac{\bar{a} - \underline{a}}{\bar{b} - \underline{b}} \right) \right).$$

This completes the proof. □

Theorem 3.3. (2,1) solution of the initial value problem (3.1)-(3.2) is a valid α -level set for $t \in \left(0, \frac{\pi}{2\sqrt{\lambda}}\right)$ satisfying the inequality

$$t \leq \frac{1}{\sqrt{\lambda}} \tan^{-1} \left(\sqrt{\lambda} \left(\frac{\bar{a} - \underline{a}}{\bar{b} - \underline{b}} \right) \right).$$

Proof. The proof is similar. □

Theorem 3.4. (2,2) solution of the initial value problem (3.1)-(3.2) is a valid α -level set for $t > 0$ satisfying the inequality

$$e^{-2\sqrt{\lambda}t} \geq \left(\frac{(\bar{b} - \underline{b}) - \sqrt{\lambda}(\bar{a} - \underline{a})}{(\bar{b} - \underline{b}) + \sqrt{\lambda}(\bar{a} - \underline{a})} \right).$$

Proof. The proof is similar. □

Theorem 3.5. All of the solutions are symmetric triangular fuzzy numbers for any $t > 0$.

Proof. For (1,1) solution, since

$$\underline{y}(t, 1) = \frac{\sin(\sqrt{\lambda t})}{2\sqrt{\lambda}} (\bar{b} + \underline{b}) + \frac{\cos(\sqrt{\lambda t})}{2} (\bar{a} + \underline{a}) = \bar{y}(t, 1),$$

and

$$\begin{aligned} \underline{y}(t, 1) - \underline{y}(t, \alpha) &= \frac{e^{\sqrt{\lambda t}}}{4} (1 - \alpha) \left(\frac{\bar{b} - \underline{b}}{\sqrt{\lambda}} + (\bar{a} - \underline{a}) \right) \\ &\quad - \frac{e^{-\sqrt{\lambda t}}}{4} (\alpha - 1) \left(\frac{\bar{b} - \underline{b}}{\sqrt{\lambda}} - (\bar{a} - \underline{a}) \right) \\ &= \bar{y}(t, \alpha) - \bar{y}(t, 1), \end{aligned}$$

(1,1) solution of the initial value problem (3.1)-(3.2) is a symmetric triangular fuzzy number for any $t > 0$. For (1,2) solution, since

$$\underline{y}(t, 1) = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda t}) \left(\frac{\bar{b} + \underline{b}}{2} \right) + \cos(\sqrt{\lambda t}) \left(\frac{\bar{a} + \underline{a}}{2} \right) = \bar{y}(t, 1),$$

and

$$\begin{aligned} \underline{y}(t, 1) - \underline{y}(t, \alpha) &= (1 - \alpha) \left(\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda t}) \left(\frac{\bar{b} - \underline{b}}{2} \right) + \cos(\sqrt{\lambda t}) \left(\frac{\bar{a} - \underline{a}}{2} \right) \right) \\ &= \bar{y}(t, \alpha) - \bar{y}(t, 1), \end{aligned}$$

(1,2) solution of the initial value problem (3.1)-(3.2) is a symmetric triangular fuzzy number for any $t > 0$. For (1,2) and (2,2) solutions, the proof is similar. \square

Example 3.1. Consider the solutions of the fuzzy initial value problem

$$(3.3) \quad y''(t) = -y(t), \quad y(0) = [1]^\alpha, \quad y'(0) = [2]^\alpha$$

by fuzzy Laplace transform, where $[1]^\alpha = [\alpha, 2 - \alpha]$, $[2]^\alpha = [1 + \alpha, 3 - \alpha]$ with supports $[0, 2]$ and $[1, 3]$, respectively.

For (1,1) solution, the lower and the upper solutions are

$$\underline{y}(t, \alpha) = e^t (\alpha - 1) + 2 \sin t + \cos t,$$

$$\bar{y}(t, \alpha) = e^t (1 - \alpha) + 2 \sin t + \cos t.$$

For (1,2) solution, the lower and the upper solutions are

$$\underline{y}(t, \alpha) = (1 + \alpha) \sin(t) + \alpha \cos(t),$$

$$\bar{y}(t, \alpha) = (3 - \alpha) \sin(t) + (2 - \alpha) \cos(t).$$

For (2,1) solution, the lower and the upper solutions are

$$\underline{y}(t, \alpha) = (3 - \alpha) \sin(t) + \alpha \cos(t),$$

$$\bar{y}(t, \alpha) = (1 + \alpha) \sin(t) + (2 - \alpha) \cos(t).$$

For (2,2) solution, the lower and the upper solutions are

$$\underline{y}(t, \alpha) = e^{-t}(\alpha - 1) + 2 \sin t + \cos t,$$

$$\bar{y}(t, \alpha) = e^{-t}(1 - \alpha) + 2 \sin t + \cos t.$$

(1,1) solution is a valid α -level set since $e^t > 0$. (1,2) solution is a valid α -level set since $t - \tan^{-1}(-1) > 0$ for $t \in (0, \frac{\pi}{2})$ according to figure 3.1. (2,1) solution is a valid α -level set since $t - \tan^{-1}(1) \leq 0$ for $t \in (0, 0.785398]$ on $(0, \frac{\pi}{2})$ according to figure 3.2. (2,2) solution is a valid α -level set since $e^{-t} > 0$. All of the solutions are symmetric triangular fuzzy numbers. Also, we can see graphics of solutions for $\alpha = 0.2$ in figure 3.3-figure 3.6.

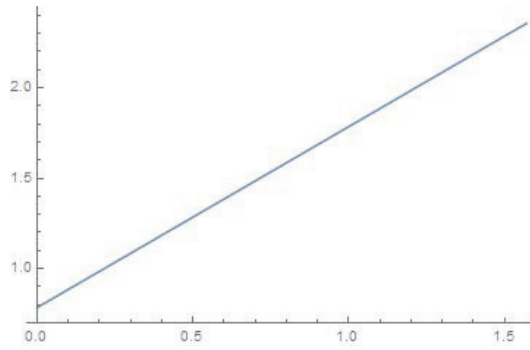


Figure 3.1. Graphic of the function $t - \tan^{-1}(-1)$

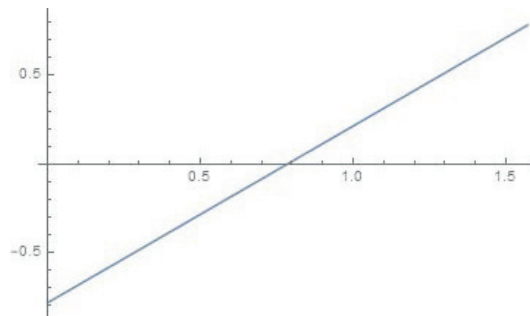
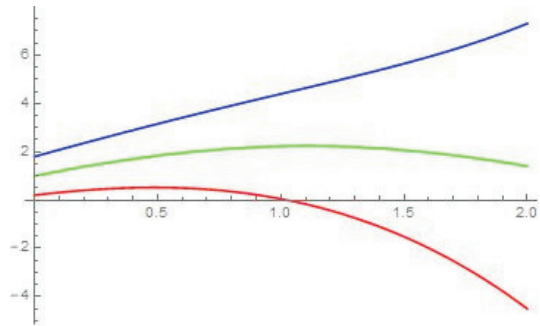
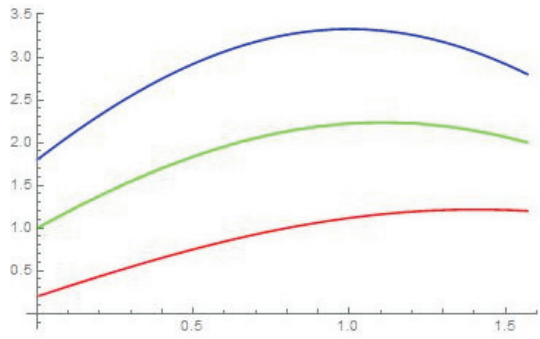
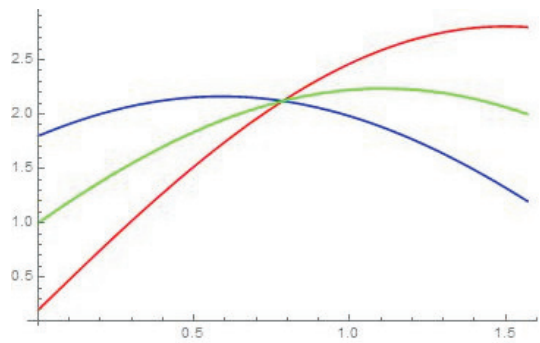


Figure 3.2. Graphic of the function $t - \tan^{-1}(1)$

Figure 3.3. Graphic of (1,1) solution for $\alpha = 0.2$ Figure 3.4. Graphic of (1,2) solution for $\alpha = 0.2$ Figure 3.5. Graphic of (2,1) solution for $\alpha = 0.2$

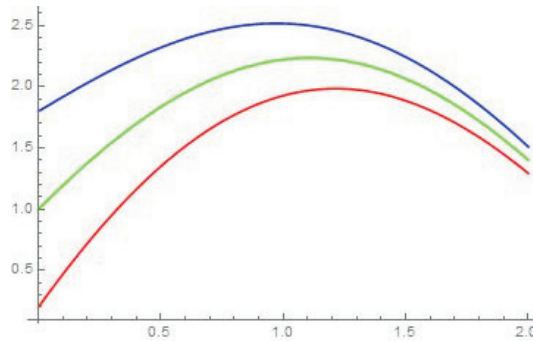


Figure 3.6. Graphic of (2,2) solution for $\alpha = 0.2$

Red $\rightarrow y_{\alpha}(t)$, Blue $\rightarrow \bar{y}_{\alpha}(t)$, Green $\rightarrow \underline{y}_1(t) = \bar{y}_1(t)$

4. Conclusions

In this paper, fuzzy initial value problem with negative coefficient is studied by fuzzy Laplace transform and generalized differentiability. Solutions are found and comparison results are given. It has been proved that the solutions are valid fuzzy functions, which has been shown on an example. It has also been found that when (1,1), (2,1) and (2,2) solutions are valid α -level sets, (1,2) solution is a valid α -level set for $t \in (0, 0.785398]$. However, we can see that (1,1) solution behaves differently from the crisp solution in figure 3.3. It means that (1,1) solution becomes fuzzier as time goes by.

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