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ARENS REGULARITY AND STRONG IRREGULARITY OF CERTAIN BILINEAR MAPPINGS

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Abstract. In this paper, the relations between the topological centers of bounded bilinear mappings and some of their higher rank adjoints are investigated. Particularly, for a Banach algebra A, some results about the Banach $A-$modules and Arens regularity and strong Arens irregularity of module actions will be obtained.
Keywords: Normed algebra, Arens regular, strongly irregular, bounded bilinear map, Banach module.

1. Introduction

Suppose that $f : X \times Y \longrightarrow Z$ is a bounded bilinear mapping on normed spaces X, Y and Z and let X^* be the topological dual of X. The adjoint of f is the bounded bilinear map $f^* : Z^* \times X \longrightarrow Y^*$ defined by

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x, y)\rangle \quad (x \in X, y \in Y, z^* \in Z^*).$$

Using this method, the higher rank adjoints of f can be defined by setting $f^{**} = (f^*)^*$. This notion was first introduced by Arens [1]. The nth adjoint of f will be denoted by $f^{(n)}$ for $n > 3$.

The mapping f^t will be considered as the bounded bilinear map from $Y \times X$ into Z defined by $f^t(y, x) = f(x, y)$. The mapping f has two extensions f^{***} and f^{t***t} on $X^{**} \times Y^{**}$. The topological centers of the map f are defined as follows:

$$\begin{aligned} Z_\ell(f) : &= \{x^{**} \in X^{**}; f^{***}(x^{**}, y^{**}) = f^{t***t}(x^{**}, y^{**}) \text{ for every } y^{**} \in Y^{**}\} \text{ and} \\ Z_r(f) : &= \{y^{**} \in Y^{**}; f^{***}(x^{**}, y^{**}) = f^{t***t}(x^{**}, y^{**}) \text{ for every } x^{**} \in X^{**}\}. \end{aligned}$$

When f is the product π of a normed algebra A, π^{***} and π^{t***t} are really the first and second Arens products of A^{**} which will be denoted by $\Box$ and $\Diamond$, respectively. In

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this case the above topological centers are denoted by $Z_\ell(A^{**})$ and $Z_r(A^{**})$, respectively. The mapping f is called (Arens) regular when $f^{***} = f^{t***t}$ and the normed algebra A is said to be (Arens) regular if its product mapping is regular. The bilinear mapping f is said to be left (resp. right) strongly (Arens) irregular if $Z_\ell(f) = X$ (resp. $Z_r(f) = Y$). Let A be a normed algebra and let X be a normed $A-$module ,with module actions π_ℓ and π_r, denoted by (π_ℓ, X, π_r). Then $(\pi_r^{t*t}, X^*, \pi_\ell^*)$ is a normed $A-$module. Also, $(\pi_\ell^{***}, X^{**}, \pi_r^{***})$ and $(\pi_\ell^{t***t}, X^{**}, \pi_r^{t***t})$ are normed $A^{**}-$modules with first and second Arens products, respectively. For further details about these concepts one can refer to [2, 3, 8].

A bounded linear mapping $D : A \longrightarrow X^*$ is said to be a derivation if $D(ab) = \pi_\ell^*(D(a), b) + \pi_r^{t*t}(a, D(b))$ for all $a, b \in A$. The mapping D is said to be an inner derivation if there exists $x^* \in X^*$ such that $D(a) = \pi_\ell^*(x^*, a) - \pi_r^{t*t}(a, x^*)$ for each $a \in A$. Impose some conditions on the normed algebra A or the normed $A-$module (π_ℓ, X, π_r) implies that the bounded linear mapping $D^{**} : A^{**} \longrightarrow X^{***}$ is also a derivation. Some of these conditions have been demonstrated in [4] and [8].

In this paper, A will be considered as a normed algebra with product π and X, Y, Z are assumed to be the normed spaces. In the first section, some conditions will be discussed under which the range of the third adjoint of the bounded bilinear mapping $f : X \times Y \longrightarrow Z$ lies in the weak closure of Z; see Theorem 2.1. Many of the results of this paper are obtained by this theorem. In section 2, we discuss some conditions under which a Banach algebra A will be a (left or right) ideal in A^{**} and then we improve some results of [5]. In section 3, by using Theorem 2.1, we continue the studies of [8] about the second adjoint of derivations.

2. Topological centers and their relations

Suppose that $\bar{X}$ be the weak closure of normed space X which is equal to its original closure.

Proposition 2.1. *Let $f : X \times Y \longrightarrow Z$ be a bounded bilinear map. Then $\bar{X} \subseteq Z_\ell(f)$ and $\bar{Y} \subseteq Z_r(f)$.*

Proof. Let $x^{**} \in \bar{X}$, then there is a net $\{x_\alpha\} \subseteq X$ which converges to x^{**} in weak topology of X^{**}. Now for every $y^{**} \in Y^{**}$ and $z^* \in Z^*$ and bounded net $\{y_\beta\} \subseteq Y$, which is w^*-convergent to y^{**}, we have:

$$\begin{aligned}
\langle f^{t***t}(x^{**}, y^{**}), z^* \rangle &= \langle f^{t****}(z^*, y^{**}), x^{**} \rangle \\
&= \lim_\alpha \langle f^{t****}(z^*, y^{**}), x_\alpha \rangle \\
&= \lim_\alpha \langle f^{t***}(y^{**}, x_\alpha), z^* \rangle \\
&= \lim_\alpha \langle y^{**}, f^{t**}(x_\alpha, z^*) \rangle \\
&= \lim_\alpha \lim_\beta \langle f^{t**}(x_\alpha, z^*), y_\beta \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \lim_\alpha \lim_\beta \langle f^{t*}(z^*, y_\beta), x_\alpha \rangle \\
&= \lim_\alpha \lim_\beta \langle z^*, f(x_\alpha, y_\beta) \rangle \\
&= \langle f^{***}(x^{**}, y^{**}), z^* \rangle.
\end{aligned}
$$

We thus have $x^{**} \in Z_\ell(f)$; that is $\bar{X} \subseteq Z_\ell(f)$, as required. A similar proof may apply for the inclusion $\bar{Y} \subseteq Z_r(f)$. □

Applying the above proposition for the multiplication of a normed algebra A we have:

Corollary 2.1. *For every normed algebra A, $\bar{A} \subseteq Z_\ell(A^{**}) \cap Z_r(A^{**})$.*

[2, Theorem 3.4] indicates some cases for which the inclusion relations in Proposition 2.1 are converted to the equality. In fact, for every approximately unital left and right normed $A-$modules (π_ℓ, X) and (X, π_r),

$$Z_r(\pi_\ell^{t*}) = \bar{X} = Z_r(\pi_r^*),$$

whereas

$$Z_\ell({\pi_\ell}^{t*}) = X^* = Z_\ell({\pi_r}^*).$$

obviously, if X is a Banach space, then π_ℓ^{t*} and π_r^* are both strongly irregular. In this case the above conclusions will be trivial.

Theorem 2.1. *Let X, Y and Z be normed spaces and let $f : X \times Y \longrightarrow Z$ be a bounded bilinear mapping. Then*

(i) $f^{***}(x^{**}, y^{**}) \in \bar{Z}$, for every $x^{**} \in Z_r(f^*)$ and $y^{**} \in Z_r(f^{t*})$;

(ii) $f^{***}(x, y^{**}) \in \bar{Z}$, for every $x \in X$ and $y^{**} \in Z_r(f^{t*})$;

(iii) $f^{***}(x^{**}, y) \in \bar{Z}$, for every $x^{**} \in Z_r(f^*)$ and $y \in Y$.

Proof. Let $x^{**} \in Z_r(f^*)$, $y^{**} \in Z_r(f^{t*})$ and $z^{***} \in Z^{***}$. If $\{x_\alpha\}$, $\{y_\beta\}$ and $\{z^*_\gamma\}$ be bounded nets which converge to these elements in w^*-topology, respectively, then

$$
\begin{aligned}
\langle z^{***}, f^{***}(x^{**}, y^{**}) \rangle &= \langle f^{****}(z^{***}, x^{**}), y^{**} \rangle \\
&= \langle f^{*t****t}(z^{***}, x^{**}), y^{**} \rangle \\
&= \lim_\alpha \lim_\gamma \lim_\beta \langle f^*(z^*_\gamma, x_\alpha), y_\beta \rangle \\
&= \lim_\alpha \lim_\gamma \lim_\beta \langle f^{t*}(z^*_\gamma, y_\beta), x_\alpha \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle f^{t*}(z^*_\gamma, y_\beta), x_\alpha \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle f(x_\alpha, y_\beta), z^*_\gamma \rangle \\
&= \lim_\alpha \lim_\beta \langle z^{***}, f(x_\alpha, y_\beta) \rangle.
\end{aligned}
$$

Therefore $f^{***}(x^{**}, y^{**})$ belongs to the weak closure of Z which equals to its norm closure. Parts (ii) and (iii) are direct results of part (i) □

It is easy to show that $f^{***}\mid_{X\times Y^{**}}= f^{*t*t}$ and $f^{***}\mid_{X^{**}\times Y}= f^{t*t*}$. Consider the normed algebra A and the normed $A-$module (π_ℓ, X, π_r). The following theorem reveals a relation between the topological center of such mappings and that of f.

Proposition 2.2. *For every bounded bilinear mapping $f : X \times Y \longrightarrow Z$,*

$Z_\ell(f^{***}\mid_{X\times Y^{**}}) \subseteq Z_\ell(f)$ *and* $Z_r(f^{***}\mid_{X^{**}\times Y}) \subseteq Z_r(f)$.

Proof. We only prove the first inclusion. Let $x^{**} \in Z_\ell(f^{*t*t})$; then by Theorem 2.1

$$f^{(4)}(z^*, x^{**}) \in Y^* \qquad (\forall z^* \in Z^*)$$

and applying the proof of [8, Theorem 2.1] implies that $x^{**} \in Z_\ell(f)$. □

As an straightforward consequence of the above theorem, the following corollary is obtained; although the part (iii) is a known result .

Corollary 2.2. *Let $f : X \times Y \longrightarrow Z$ be a bounded bilinear mapping.*

(i) *If f is left strongly irregular, then so does $f^{***}\mid_{X\times Y^{**}}$.*

(ii) *If f is right strongly irregular, then so does $f^{***}\mid_{X^{**}\times Y}$.*

(iii) *If one of $f^{***}\mid_{X\times Y^{**}}$ or $f^{***}\mid_{X^{**}\times Y}$ are Arens regular, then f is Arens regular.*

Corollary 2.3. *If π is the product of a strongly irregular Banach algebra, then $\pi^{***}\mid_{A\times A^{**}}$ (resp. $\pi^{***}\mid_{A^{**}\times A}$) is left (resp. right) strongly irregular.*

For example, the above corollary holds for the group algebras $L^1(G)$ and $M(G)$, as the known strongly irregular Banach algebras; see [6] and [7].

3. X and X^* as $A^{**}-$modules

Suppose that A is an Arens regular Banach algebra and let (π_ℓ, X, π_r) be a Banach $A-$module. It has been shown in [8] that $(\pi_r^{**}, X^*, \pi_\ell^{t**t})$ is a Banach $A^{**}-$module if and only if for every $x \in X$, the bilinear mapping $\theta_x : A \times A \longrightarrow X$ defined by

$$\theta_x(a, b) = \pi_\ell(a, \pi_r(x, b)) = \pi_r(\pi_\ell(a, x), b) \quad (a, b \in A)$$

is regular. Now the following proposition as another application of Theorem 2.1, improves this result.

Proposition 3.1. *Let A be an Arens regular Banach algebra and let (π_ℓ, X, π_r) be a Banach $A-$module. If either π_r^{t*t} or π_ℓ^* is regular, then $(\pi_r^{**}, X^*, \pi_\ell^{t**t})$ is a Banach $A^{**}-$module.*

Proof. Suppose that $x \in X$. The natural extensions of θ_x on $A^{**} \times A^{**}$ satisfies

$$\begin{aligned}\theta_x^{***}(a^{**}, b^{**}) &= \pi_\ell^{***}(a^{**}, \pi_r^{***}(x, b^{**})) = \pi_r^{***}(\pi_\ell^{***}(a^{**}, x), b^{**}) \text{ and} \\ \theta_x^{t****t}(a^{**}, b^{**}) &= \pi_\ell^{t****t}(a^{**}, \pi_r^{t****t}(x, b^{**})) = \pi_r^{t****t}(\pi_\ell^{t****t}(a^{**}, x), b^{**}),\end{aligned}$$

for all $a^{**}, b^{**} \in A^{**}$.
In the case that π_r^{t*t} is regular, Theorem 2.1 implies that $\pi_r^{***}(x, b^{**}) \in \bar{X} = X$, for every $b^{**} \in A^{**}$ and so the above equalities demonstrate the regularity of θ_x. A similar argument may applies for the case that π_ℓ^* is regular. □

Let A be a Banach algebra and let (π_ℓ, X, π_r) be a Banach $A-$module. Theorem 2.1 implies that

$$\pi_\ell^{***}(Z_r(\pi_\ell^*), X) \subseteq X \ , \ \pi_r^{***}(X, Z_\ell(\pi_r^{t*t})) \subseteq X \text{ and}$$

$$\pi_\ell^{t****t}(Z_r(\pi_\ell^*), X) \subseteq X \ , \ \pi_r^{t****t}(X, Z_\ell(\pi_r^{t*t})) \subseteq X.$$

Therefore, we obtain the following proposition.

Proposition 3.2. *let X be a Banach $A-$module as above.*

(i) If π_r^{t*t} is regular, then X is a right Banach $A^{**}-$module with either Arens products of A^{**}.

(ii) If π_ℓ^* is regular, then X is a left Banach $A^{**}-$module with either Arens products.

(iii) If both module actions π_r^{t*t} and π_ℓ^* are regular, then X is a Banach $A^{**}-$module (with either Arens products).

For the special case $X = A$, this implies the next result.

Corollary 3.1. *Suppose that A is a Banach algebra.*

(i) If π^{t*t} is regular, then A is a right ideal of A^{**}.

(ii) If π^* is regular, then A is a left ideal of A^{**}.

(iii) If both π^{t*t} and π^* are regular, then A is an ideal of A^{**}.

Let A be a Banach algebra and let X be a Banach $A-$module. We say that X factors A on the left (resp. right) if $\pi_r(X, A) = X$ (resp. $\pi_\ell(A, X) = X$). Some relationships between the factorization property and Arens regularity are stated in [2] and [5]. Proposition 3.3 and Theorem 3.1 from [5] are of these cases which together with Corollary 3.1 provide conditions for the Arens regularity of A.

Proposition 3.3. *[5, Corollary 4.1] Let A be a left ideal in A^{**}.*

(1) *If A^* factors A on the right, then A is Arens regular.*

(2) *If A^{**} factors A on the right, then $Z_A(A^{***}) = A^{***}$.*

As an immediate consequence of Corollary 3.1 and the part (1) of the above proposition we have the following result.

Corollary 3.2. *If π^* is regular and A^* factors A on the right, then A is Arens regular.*

Theorem 3.1. *[5, Theorem 4.2] Let A be a right ideal in A^{**}. In each of the following situations, A is Arens regular.*

(1) *A^* factors A on the left.*

(2) *A^{**} factors A on the left.*

As another application of Corollary 3.1, the next result is obtained by applying the above theorem.

Corollary 3.3. *If π^{t*t} is regular and A^* or A^{**} factors A on the left, then A is Arens regular.*

4. The second adjoint of a derivation

The following result as a consequence of [4, Proposition 6.2] and Theorem 2.1 indicates some conditions under which the second adjoint of a derivation is a derivation too.

Proposition 4.1. *Suppose that A is a Banach algebra. If the mappings π^* and π^{t*} are both regular, then the second adjoint of every inner derivation $D : A \longrightarrow A^*$ is also a derivation.*

Proof. As it was shown in [4, Proposition 6.2], the second adjoint of every inner derivation $D : A \longrightarrow A^*$ is a derivation if and only if

$$(b^{**}\Box c^{**})\Diamond a^{**} + (c^{**}\Box a^{**})\Diamond b^{**} - c^{**}\Diamond(a^{**}\Box b^{**}) - b^{**}\Box(c^{**}\Box a^{**}) = 0$$

for every $a^{**}, b^{**}, c^{**} \in A^{**}$. Now if π^* and π^{t*} are regular, then Theorem 2.1 implies that

$$(b^{**}\Box c^{**})\Diamond a^{**} = (b^{**}\Box c^{**})\Box a^{**} = b^{**}\Box(c^{**}\Box a^{**});$$

and also

$$(c^{**}\Box a^{**})\Diamond b^{**} = (c^{**}\Box a^{**})\Box b^{**} = c^{**}\Box(a^{**}\Box b^{**}) = c^{**}\Diamond(a^{**}\Box b^{**}).$$

These equalities complete the proof. $\Box$

Let (π_ℓ, X, π_r) be a Banach $A-$module and let $D : A \longrightarrow X^*$ be a derivation. [8, Theorem 4.2] indicates that the second adjoint of D is a derivation on $(A^{**}, \Box)$ (resp. $(A^{**}, \Diamond)$) iff $\pi_r^{****}(D^{**}(A^{**}), X^{**}) \subseteq A^*$ (resp. $\pi_\ell^{t****}(D^{**}(A^{**}), X^{**}) \subseteq A^*$). This theorem together with Theorem 2.1, implies the following corollary.

Corollary 4.1. *By the above hypothesis,*

(i) If π_r^{*t*} is regular and $D^{**}(A^{**}) \subseteq Z_r(\pi_r^{**})$, then D^{**} is a derivation on $(A^{**}, \Box)$.

(ii) If π_ℓ^{t*t*} is regular and $D^{**}(A^{**}) \subseteq Z_\ell(\pi_\ell^{t**t})$, then D^{**} is a derivation on $(A^{**}, \Diamond)$.

Applying [8, Corollary 4.4] for the case $X = A$ reveals that D^{**} is a derivation on $(A^{**}, \Box)$ (resp. $(A^{**}, \Diamond)$) if both π and π^{t*} (resp. π^*) are Arens regular. Now by using corollaries 3.2 and 3.3 we obtain the next result.

Corollary 4.2. *Suppose that A is a Banach algebra and $D : A \longrightarrow A^*$ is a derivation.*

(i) If π^* is regular and A^* factors A on the right, then $D^{**} : (A^{**}, \Diamond) \longrightarrow A^{***}$ is a derivation.

(ii) If π^{t*t} is regular and A^* factors A or A^{**} on the left, then $D^{**} : (A^{**}, \Box) \longrightarrow A^{***}$ is a derivation.

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