FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 34, No 4 (2019), 745–753
https://doi.org/10.22190/FUMI1904745E

DOUBLE EXPONENTIAL EULER–SINC COLLOCATION METHOD FOR A TIME–FRACTIONAL CONVECTION–DIFFUSION EQUATION

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Abstract. In this research, a new version of Sinc–collocation method incorporated with a Double Exponential (DE) transformation is implemented for a class of convectiondiffusion equations that involve time fractional derivative in the Caputo sense. Our approach uses the DE Sinc functions in space and the Euler polynomials in time, respectively. The problem is reduced to the solution of a system of linear algebraic equations. A comparison between the proposed approximated solution and numerical/exact/available solution reveals the reliability and significant advantages of our newly proposed method.
Keywords. Time-fractional convection–diffusion equation; Shifteted Legendre polynomials; Euler-Sinc collocation; Caputo fractional derivative; Double exponential.

1. Introduction

We focus on a time–fractional convection–diffusion equation with variable coefficients of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + a(x)\frac{\partial u(x,t)}{\partial x} + b(x)\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t),\ 0 < x < 1,\ 0 < t \leqslant 1, \tag{1.1}$$

with initial condition

$$u(x,0) = g(x),\ \ 0 < x < 1, \tag{1.2}$$

and boundary conditions

$$u(0,t) = h(t),\ \ u(1,t) = k(t),\ \ 0 < t \leqslant 1, \tag{1.3}$$

Received May 16, 2019; accepted August 05, 2019.
2010 *Mathematics Subject Classification.* Primary 65N35; Secondary 65M70, 26A33

where $a(x), b(x) \neq 0$ are continuous functions, $\alpha \in (0, 1]$ and the oprator $\frac{\partial^\alpha}{\partial t^\alpha}$ (or D_t^α) is defined in the Caputo sense as follows:

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}. \tag{1.4}$$

We recall that the Caputo fractional derivative of the power function satisfies

$$D_t^\alpha t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} t^{p-\alpha}, & p \in \mathbb{N}_0 \text{ and } p \geqslant \lceil \alpha \rceil \text{ or } p \notin \mathbb{N} \text{ and } p > \lfloor \alpha \rfloor, \\ 0, & p \in \mathbb{N}_0 \text{ and } p < \lceil \alpha \rceil. \end{cases} \tag{1.5}$$

where $\lceil \alpha \rceil$ and $\lfloor \alpha \rfloor$ denote the ceil and floor of α, respectively.

The problem (1.1)–(1.3) is a class of time–fractional diffusion/wave equations which appears frequently e.g., in earthquake modeling, non-Markov Processes and in the mathematical modelling of the earth surface transport [6, 4]. In recent years, due to its extensive engineering applications, much attention has been focused on providing an effective numerical method to solve it.

Izadkhah and Saberi-Najafi [2] expanded the required approximate solution as the elements of the Gegenbauer polynomials in time and Lagrange polynomials in space and by using a global collocation, reduced the problem to a system of linear algebraic equations. Other numerical/analytic methods, such as standard Sinc–Legendre collocation, finite difference, finite element, ADM, HAM and VIM have also been developed to solve time–fractional diffusion/wave equations and have been fully addressed in [6, 2].

The numerical methods based on Sinc approximations have been studied extensively during the last three decades. Recently, these approaches have been used for solution of fractional ordinary/partial differential equations and usually give a result with high accuracy even for problems with an algebraic singularity at the end point. Many of these methods have been found very effective and reliable and under some conditions, have convergency of order $\mathscr{O}\left(\exp\left(\frac{-cN}{\ln(N)}\right)\right)$, where $c > 0$ is a constant and $N \in \mathbb{N}$ depends on number of mesh points. For more historical remarks and technical details about Sinc numerical methods see [7, 8] and the references therein. In the present paper, we apply the Euler–Sinc collocation method coupled with Double Exponential (DE) transformation for solving equations (1.1)–(1.3). Our method consists of reducing the solution to a set of algebraic equations by expanding $u(x,t)$ as a combination of modified Sinc functions (in space) and Euler polynomials (in time) with a special boundary treatment. The numerical experiments are implemented in Maple 15 programming. The programs are executed on a Notebook System with 2.0 GHz Intel Core 2 Duo processor with 2 GB 533 MHz DDR2 SDRAM.

2. Basic Definitions and Theorems of the Method

In this section, we introduce some basic definitions and derive preliminary results for developing our method.

2.1. Sinc Functions

The sine cardinal or Sinc function on $\mathbb{C}$ defined by

$$\mathrm{Sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

For step size $h > 0$, and any integer k, the translated Sinc function on uniform meshes is denoted $S(k,h)(z)$ and defined by

$$S(k,h)(z) = \mathrm{Sinc}\left(\frac{z - kh}{h}\right).$$

For target equation (1.1) on spatial interval $0 < x < 1$, we employ a conformal map

$$w = \phi(x) = \ln\left(\frac{1}{\pi}\ln\left(\frac{x}{1-x}\right) + \sqrt{1 + \left(\frac{1}{\pi}\ln\left(\frac{x}{1-x}\right)\right)^2}\right),$$

with inverse

$$x = \psi(w) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\pi}{2}\sinh w\right), \tag{2.1}$$

as the DE transformation, on a subinterval $\Gamma = (0,1) = \psi(\mathbb{R})$ with $\phi(0) = -\infty$ and $\phi(1) = \infty$. Now we have

$$f(x) \simeq \sum_{k=-N}^{N} f(x_k) S_k(x),$$

as a method of interpolation where, $x_k = \psi(kh) \in (0,1)$ and $S_k(x) = S(k,h) \circ \phi(x)$ are defined as Sinc grid points and the translated Sinc basic functions, respectively. Also, the pth order derivative of $S_j(x)$ with respect ot ϕ at the node x_k is denoted by $\delta_{j,k}^{(p)}$ and computed by the following relation

$$\delta_{j,k}^{(p)} = \frac{d^p}{d\phi^p}\left[S(j,h) \circ \phi(x)\right]\Big|_{x=x_k} = \frac{\pi^p}{h^p}\begin{cases} \lim\limits_{t\to 0} \dfrac{d^p}{dt^p}\left(\dfrac{\sin t}{t}\right), & j = k \\ \dfrac{d^p}{dt^p}\left(\dfrac{\sin t}{t}\right)\Big|_{t=\pi(k-j)}, & j \neq k \end{cases}, \quad p = 0, 1, \ldots \tag{2.2}$$

Leibniz's rule for higher-order derivatives of products provides the following recurrence formulas of (2.2) [5]:

$$\delta_{j,k}^{(2r)} = \frac{1}{h^{2r}}\begin{cases} \frac{(-1)^r \pi^{2r}}{2r+1}, & j = k, \\ \frac{(-1)^{k-j}(2r)!}{(k-j)^{2r}} \sum\limits_{s=0}^{r-1} \frac{(-1)^{s+1}\pi^{2s}}{(2s+1)!}(k-j)^{2s}, & j \neq k, \end{cases}$$

$$\delta_{j,k}^{(2r+1)} = \frac{1}{h^{2r+1}} \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}(2r+1)!}{(k-j)^{2r+1}} \sum_{s=0}^{r} \frac{(-1)^s \pi^{2s}}{(2s+1)!} (k-j)^{2s}, & j \neq k, \end{cases}$$

with $r = 0, 1, 2, \ldots$.
Hence for $p = 0, 1, 2$ these quantities are as following

$$\delta_{j,k}^{(0)} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{2.3}$$

$$\delta_{j,k}^{(1)} = \frac{1}{h} \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases} \tag{2.4}$$

$$\delta_{j,k}^{(2)} = \frac{1}{h^2} \begin{cases} -\frac{\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k, \end{cases} \tag{2.5}$$

Definition 2.1. [8] A function f is said to decay double exponentially with respect to ψ, if there exist positive constants α and β such that

$$|f(\psi(\xi))| \leq \alpha \exp\left(-\beta \exp(|\xi|)\right), \text{ for all } \xi \in \mathbb{R}.$$

Moreover, under some conditions on f, $f(\psi(\xi))$ decays double exponentially with respect to ψ.

If the function f is double exponentially decreasing, then the interpolation formula of it over [0, 1] takes the form

$$\sum_{j=-N}^{N} f(x_j) S_j(x)$$

and under some restrictions on f, it is shown by both theoretical analysis and numerical experiments that the approximation error on $x \in [0, 1]$ can be estimated by

$$\left\| f(x) - \sum_{j=-N}^{N} f(x_j) S_j(x) \right\|_{\infty} \leqslant C \exp\left[\frac{-\pi d N}{\ln(\pi d N/\beta)}\right] \tag{2.6}$$

where h is taken as

$$h = \frac{\ln(\pi d N/\beta)}{N}, \tag{2.7}$$

and C is a constant independent of f and N [8].

2.2. Euler Functions

We end this section by introducing the classical Euler polynomials $E_n(t)$ and deriving some of their features. The classical Euler polynomials denoted by $E_n(t)$, are usually defined by means of the following generating function:

$$\frac{2e^{tx}}{e^x+1} = \sum_{n=0}^{\infty} E_n(t)\frac{x^n}{n!}, \qquad |x| < \pi. \tag{2.8}$$

It is possible to get explicit expressions for the Euler polynomials as following [3]:

$$E_n(t) = \frac{1}{n+1}\sum_{k=0}^{n}\left(2-2^{n+2-k}\right)\binom{n+1}{k}B_{n+1-k}\, t^k, \quad n = 0, 1, 2, 3, \ldots \tag{2.9}$$

where B_k's are the Bernoulli numbers.

Lemma 2.1. *Let $\alpha > 0$. Then the fractional derivative of $E_j(t)$ of order α is*

$$D_t^{\alpha}E_j(t) = \sum_{k=\lceil\alpha\rceil}^{j} e_{j,k}t^{k-\alpha}, \quad j = 0, 1, 2, 3, \ldots \tag{2.10}$$

where $e_{j,k} = \frac{1}{j+1}\left(2-2^{j+2-k}\right)\binom{j+1}{k}B_{j+1-k}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}$.

Proof. The proof is straightforward and deduces from (1.5) and (2.9). □

3. Euler–Sinc Collocation Method

In order to discretize equations (1.1)–(1.3) by using Euler–Sinc collocation approach, we approximate $u(x,t)$ as

$$u_{m,n}(x,t) = \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{i,j}S_i(x)E_j(t) + B(x,t). \tag{3.1}$$

Here, $S_k(x)$'s convege to zero as x tends to 0 or 1. Hence we can pick a nice function satisfying the boundary conditions (1.3), say $B(x,t) = (1-x)h(t) + xk(t)$. The $(2m+1)(n+1)$ unknown expansion coefficients $\{c_{i,j}\}$ in (3.1) are determined by substituting $u_{m,n}(x,t)$ into equations (1.1)–(1.2) and evaluating the results at the collocation points $x_k = \psi(kh)$, $k = -m, \ldots, m$, and t_l, $l = 1, \ldots, n$ (as the n first roots of the shifteted Legendre polynomial $P_{n+1}(t)$ in $[0, 1]$).

Lemma 3.1. *If the assumed approximate solution of the initial–boundary value problem (1.1)–(1.3) is (3.1), then the discrete Euler–Sinc collocation system for the* *determination of the unknown coefficients* $\{c_{i,j}\}$ *is given by the following* $(2m+1)\times(n+1)$ *equations*

$$\sum_{j=0}^{n}\sum_{r=\lceil\alpha\rceil}^{j} c_{k,j}e_{j,r}t_l^{r-\alpha} + a\left(x_k\right)\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{i,j}\lambda_{i,k}E_j\left(t_l\right)$$

$$+\, b\left(x_k\right)\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{i,j}\mu_{i,k}E_j\left(t_l\right) = F\left(x_k, t_l\right), \tag{3.2}$$

$$\sum_{j=0}^{n} c_{k,j}E_j(0) = g\left(x_k\right), \quad k=-m,\,-m+1,\,\dots,m, \quad l=0,\,1,\,\dots,n,$$

where,

$$\lambda_{i,k} = \delta_{i,k}^{(1)}\phi'\left(x_k\right), \qquad \mu_{i,k} = \delta_{i,k}^{(2)}\left(\phi'\left(x_k\right)\right)^2 + \delta_{i,k}^{(1)}\phi''\left(x_k\right), \tag{3.3}$$

and

$$F\left(x_k, t_l\right) = f\left(x_k, t_l\right) - D_t^{\alpha}B\left(x_k, t_l\right) - \frac{\partial B\left(x_k, t_l\right)}{\partial x} - \frac{\partial^2 B\left(x_k, t_l\right)}{\partial x^2}. \tag{3.4}$$

Proof. The process of proof is similar to the proof of lemma 1 in [6] and left to the reader. □

Finally, the linear system (3.2) for the unknown coefficients $\{c_{i,j}\,:\, i=-m,\,-m+1,\,\dots,m,\; j=0,\,1,\,\dots,n\}$ can be solved by using `fsolve` command in MAPLE.

4. Illustrative Examples

In this section we show numerical results of the Euler–Sinc collocation method. In order to verify the performance and reliability of the proposed method, three examples are examined in this section. In all examples, we heuristically choose $d=\frac{\pi}{6}$ and $\beta=\frac{\pi}{2}$ which leads to $h=\frac{1}{m}\ln\left(\frac{m\pi}{3}\right)$. In the presence of exact solutions, we also define maximum absolute error $e_{m,n}$ as

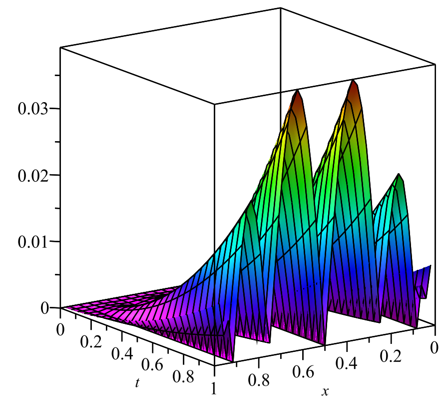
$$e_{m,n} := \max\left\{\left|u(x,t)-u_{m,n}(x,t)\right| :\; 0\leqslant x\leqslant 1,\; 0<t\leqslant 1\right\}$$

.

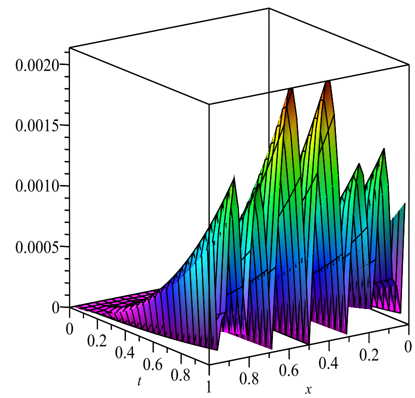
Example 4.1. [6] Consider the following time–fractional diffusion equation

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad 0<x<1,\; 0<t\leqslant 1,\; 0<\alpha\leqslant 1 \tag{4.1}$$

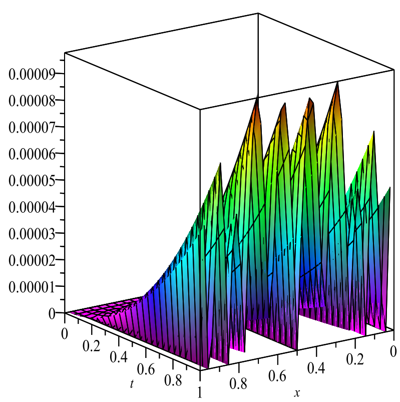
with $h(t)=0$, $k(t)=0$, $f(x,t)=\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}\sin(2\pi x)+4\pi^2t^2\sin(2\pi x)$ and $u(x,0)=0$. This problem has the exact solution $u(x,t)=t^2\sin(2\pi x)$.



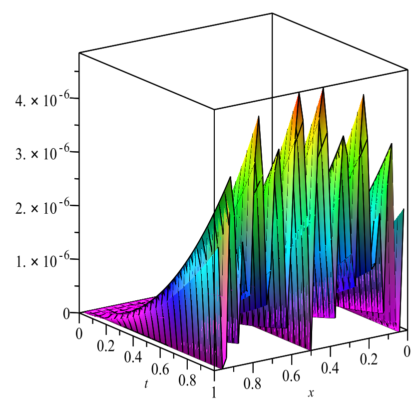
$m = 5$



$m = 10$



$m = 15$



$m = 20$

FIG. 4.1: The convergence of the sequence $e_{m,3}$ for different values of m.

In Figure 4.1, as a convergence criterion of $u_{m,n}(x,t)$ to $u(x,t)$, we illustrate the error function $e_{m,n}$ for $n=3$ and $m=5$, 10, 15, 20. This figure illustrates that the error function $e_{m,3}$ gets smaller and smaller values by increasing spatial resolution m.

Example 4.2. [6, 2] Consider the problem (1.1)–(1.3) of order $0<\alpha<1$ with $a(x)=x$, $b(x)=1$, $f(x,t)=2t^{\alpha}+2x^2+2$, $g(x)=x^2$, $h(t)=\frac{2\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}$, $k(t)=1+\frac{2\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}$ and the exact solution $u(x,t)=x^2+\frac{2\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}$.

Table 4.1: Comparison of absolute error $e_{m,n}$ for $\alpha=0.5$, $n=3$ and $t=\frac{1}{2}$, for Example 4.2.

x	Method of [1] for $m=64$	Method of [6] for $m=25$ and $n=7$	Present method $m=5$	$m=10$
0.1	1.210×10^{-03}	6.462×10^{-06}	7.816×10^{-05}	5.648×10^{-07}
0.2	1.259×10^{-03}	1.578×10^{-05}	1.100×10^{-04}	5.198×10^{-07}
0.3	1.865×10^{-03}	2.272×10^{-05}	9.353×10^{-05}	4.891×10^{-07}
0.4	7.412×10^{-03}	2.674×10^{-05}	8.441×10^{-05}	5.835×10^{-07}
0.5	1.000×10^{-06}	2.759×10^{-05}	8.440×10^{-05}	5.951×10^{-07}
0.6	7.460×10^{-03}	2.534×10^{-05}	8.604×10^{-05}	5.736×10^{-07}
0.7	1.724×10^{-03}	2.035×10^{-05}	9.540×10^{-05}	4.833×10^{-07}
0.8	4.990×10^{-03}	1.320×10^{-05}	1.101×10^{-04}	5.227×10^{-07}
0.9	1.678×10^{-02}	4.653×10^{-06}	7.636×10^{-05}	5.591×10^{-07}

Taking $\alpha=0.5$ and temporal resolution $n=3$, in Table 4.1, we compare our method for moderate values of spatial resolution, say $m=5$ and $m=10$, together with the results obtained by using the wavelet method [1] for $m=64$ and Sinc-Legendre collocation method [6] with $n=7$ and $m=25$. The results of this Table show that our computations are in good agreement with those obtained by the existing methods and a slight increase in m significantly improves our numerical results.

5. Conclusion

The present work exhibits the reliability of the Euler–Sinc method to solve a Caputo time fractional convection-diffusion equation that arises frequently in the mathematical modeling of real-world physical problems such as earthquake modeling, traffic flow model and financial option pricing problems. Approximated results are in close agreement with numerical/exact solutions and the results of the previous section reveal that our scheme can be used to obtain accurate numerical solutions of problem (1.1)-(1.3) with very little computational effort.

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