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k-TYPE SLANT HELICES FOR SYMPLECTIC CURVE IN 4-DIMENSIONAL SYMPLECTIC SPACE

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Abstract. In this study, we have expressed the notion of k-type slant helix in 4-symplectic space. Also, we have generated some differential equations for k-type slant helix of symplectic regular curves.

Keywords: Symplectic curve; k-Type slant helix.

1. Introduction

The helix concept is an important area for differential geometers due to its numerous applications in many areas from physics to engineering. So, many authors are interested in helices to study in Euclidean 3-space and Euclidean 4-space. In [7, 6, 9], the authors gave new characterizations for an helix. The notion of a slant helix belongs to Izumiya and Takeuchi [4]. They consider the principle normal vector field of the curve instead of tangent vector field and they defined a new kind of helix which is called slant helix. Recently, some studies have been done to extend the definitions of helix and slant helix to Minkowski space (see [1, 2, 3]) and other frames [8].

2. Preliminaries

Let us give a brief related to symplectic space. One can found a brief account of the symplectic space in [10, 5]. The symplectic space $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ is the vector space $\mathbb{R}^4$ endowed with the standard symplectic form Ω, given in global Darboux coordinates by $\Omega = \sum\limits_{i=1}^{2} dx_i \wedge dy_i$. Each tangent space is endowed with symplectic inner product defined in canonical basis by

$$\begin{aligned} \langle u, v \rangle &= \Omega(u, v) \\ &= x_1\eta_1 + x_2\eta_2 - y_1\xi_1 - y_2\xi_2 \end{aligned}$$

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where $u = (x_1, x_2, y_1, y_2)$ and $v = (\xi_1, \xi_2, \eta_1, \eta_2)$.

A symplectic frame is a smooth section of the bundle of linear frames over $\mathbb{R}^4$ which assigns to every point $z \in \mathbb{R}^4$ an ordered basis of tangent vectors $a_1, a_2, a_3, a_4$with the property that

$$\langle a_i, a_j \rangle = \langle a_{2+i}, a_{2+j} \rangle = 0, \quad 1 \leq i, j \leq 2, \tag{2.1}$$

$$\langle a_i, a_{2+j} \rangle = 0, \quad 1 \leq i \neq j \leq 2, \tag{2.2}$$

$$\langle a_i, a_{2+i} \rangle = 1, \quad 1 \leq i \leq 2.$$

Let $z(t) : \mathbb{R} \to \mathbb{R}^4$ denotes a local parametrized curve. In our notation, we allow z to be defined on an open interval of $\mathbb{R}$. As it is customary in classical mechanics, we use the notation $\dot{z}$ to denote differentiation with respect to the parameter t

$$\dot{z} = \frac{dz}{dt}.$$

Definition 2.1. A curve $z(t)$ is said to be symplectic regular if it satisfies the following non-degeneracy condition

$$\langle \dot{z} \ , \ \ddot{z} \rangle \neq 0 \ , \quad for \ all \ t \in \mathbb{R}. \tag{2.3}$$

Definition 2.2. Let $t_0 \in \mathbb{R}$, then the symplectic arc length s of a symplectic regular curve starting at t_0 is defined by

$$s(t) = \int_{t_0}^{t} \langle \dot{z} \ , \ \ddot{z} \rangle^{1/3} dt, \quad for \ t \geqslant t_0. \tag{2.4}$$

Taking the extrerior differential of (2.4) we obtain the symplectic arc length element as

$$ds = \langle \dot{z} \ , \ \ddot{z} \rangle^{1/3} \, dt.$$

Dually, the arc length derivative operator is

$$D = \frac{d}{ds} = \langle \dot{z} \ , \ \ddot{z} \rangle^{-1/3} \, \frac{d}{dt}. \tag{2.5}$$

In the following, primes are used to denote differentiation with respect to the symplectic arc length derivative operator (2.5)

$$z' = \frac{dz}{ds}.$$

Definition 2.3. A symplectic regular curve is parametrized by symplectic arc length if

$$\langle \dot{z}, \ddot{z} \rangle = 1, \quad for \ all \ t \in \mathbb{R}.$$

Let $z(s)$ be a symplectic regular curve in $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$. In this case there exist only one Frenet frame $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ for which $z(s)$ is a symplectic regular curve with Frenet equations

$$
\begin{aligned}
&a_1^{'}(s) = a_3(s), \quad a_2^{'}(s) = H_2(s)a_4(s), \\
&a_3^{'}(s) = k_1(s)a_1(s) + a_2(s), \quad a_3^{'}(s) = a_1(s) + k_2(s)a_2(s),
\end{aligned} \tag{2.6}
$$

where $H_2(s) = constant(\neq 0)$ [10].

In [2], the authors introduced the k-type slant helix in Minkowski 4-dimensional space E_1^4. Now, we extend the concept of slant helix for symplectic regular curve as follows:

Definition 2.4. Let z be a symplectic regular curve with the Frenet frame $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$. We say that z is a k-type slant helix if there exists a (non-zero) constant vector field $U \in R^4$ such that

$$\langle a_{k+1}(s), U\rangle = const.$$

for $0 \leq k \leq 3$ where U is an axis of the curve.

In particular, 0-type slant helices are general helices and 1-type slant helices are slant helices.

3. k-Type Slant Helices

Theorem 3.1. *Let z be a symplectic regular curve in $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$. Then z is 0-type slant helix(or general helix) if and only if*

$$\frac{k_1(s)}{k_2(s)} = const. \tag{3.1}$$

Morever, z is also a k-type slant helix, for $k \in \{1, 2, 3\}$.

Proof. Assume that z is a 0-type slant helix. Then for a constant vector field U, we have $\langle a_1(s), U\rangle = c$ is constant. Differentiating this equation and using Frenet equations, we obtain $\langle a_3(s), U\rangle = 0$. So U is orthogonal to $a_3(s)$ and we can decompose U as differentiating (3.1) and using Frenet equations, one arrives to

$$
\begin{aligned}
ck_1(s) + U_4(s) &= 0, \\
U_4'(s) &= 0, \\
c + U_4 k_2(s) &= 0.
\end{aligned} \tag{3.2}
$$

Thus U_4 is constant. By (3.1) and (3.2) can easily obtained. Converse of proof is obvious. □

Theorem 3.2. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$, *where* $k_1(s) \neq const(\neq 0)$, $k_2(s) \neq const(\neq 0)$ *and* $H_2(s) = const = c_0$. *If z is a* 0*-slant helix, then*

$$k_1^{''}(s) - c_0 k_1(s) k_2(s) + c_0 = 0. \tag{3.3}$$

Proof. Assume that z is a 0-type slant helix. Then for a constant vector field U, we have

$$\langle a_1(s), U\rangle = c. \tag{3.4}$$

Differentiating this equation and using Frenet equations, we obtain

$$\langle a_3(s), U\rangle = 0. \tag{3.5}$$

Taking the derivative of equation (3.5) with respect to s, we have

$$\langle a_2(s), U\rangle = -ck_1(s). \tag{3.6}$$

Now, if we differentiate (3.6) and use the Frenet frame, we get

$$\langle a_4(s), U\rangle = \frac{-c}{H_2(s)} k_1'(s). \tag{3.7}$$

Hence, we differentiate (3.7) for the last time. Taking into account of hypotesis of the Theorem and the Frenet frame, we obtain (3.3). □

Corollary 3.1. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$. *If z is a* 0*-type slant helix, then we have following diffrential equation*

$$k_2^{''}(s) - c_0 k_2^2(s) - 1 = 0, \tag{3.8}$$

where $c_0 = H_2(s) = const(\neq 0)$ *and* $k_1(s) \neq const(\neq 0)$ *and* $k_2(s) \neq const(\neq 0)$.

Proof. From (3.3) and (3.1) we obtain (3.8). □

Corollary 3.2. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$. *a) If z is a 0-type slant helix with* $k_1(s) = const(\neq 0)$, *then we have*

$$k_2(s) = \frac{1}{k_1(s)}, \tag{3.9}$$

b) If z is a 0-type slant helix with $k_2(s) = const(\neq 0)$, *then we have*

$$H_2(s) = -\frac{1}{k_2^2(s)}. \tag{3.10}$$

Similarly, we can give the following conclusions:

Theorem 3.3. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ *with* $k_1(s) \neq const(\neq 0), k_2(s) \neq const(\neq 0)$ *and* $H_2(s) = const(\neq 0)$. *If z is a* 1*-type slant helix, then*

$$k_2''(s) - k_1(s)k_2(s) + 1 = 0. \tag{3.11}$$

Corollary 3.3. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ *with non-zero constant* $k_1(s), k_2(s), H_2(s)$. *If z is a* 1*-type slant helix, then*

$$k_1(s) = \frac{1}{k_2(s)}. \tag{3.12}$$

Morever z is also a k-type slant helix, for $k \in \{2, 3\}$. In this case, we have the following:

Theorem 3.4. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ *with* $k_1(s) \neq const(\neq 0), k_2(s) \neq const(\neq 0)$ *and* $H_2(s) = const(\neq 0)$. *If z is a* 2*-type slant helix, then we get*

$$(k_1''(s) + H_2(s) - H_2(s)k_1(s)k_2(s)\,\langle a_1(s), U\rangle = -2k_1'(s)c, \tag{3.13}$$

where c is a constant.

Theorem 3.5. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ *with* $k_1(s) \neq const(\neq 0), k_2(s) \neq const(\neq 0)$ *and* $H_2(s) = const(\neq 0)$. *If z is a* 3*-type slant helix, then we have*

$$(k_2''(s) - k_1(s)k_2(s) + 1)\,\langle a_2(s), U\rangle = -2ck_2'(s)H_2(s), \tag{3.14}$$

where c is a constant.

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