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EXISTENCE AND BLOW UP FOR A NONLINEAR VISCOELASTIC HYPERBOLIC PROBLEM WITH VARIABLE EXPONENTS *

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Abstract. Our aim in this paper is to establish the weak existence theorem and find under suitable assumptions sufficient conditions on m, p and the initial data for which the blow up takes place for the following boundary value problem:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u.$$

This paper extends some of the results obtained by the authors and it is focused on new results which are consequence of the presence of variable exponents.
Keywords: Variable exponents; weak solutions; blow up.

1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded Lipschitz domain and $0 < T < \infty$. We consider the following initial boundary value problem:

$$(1.1) \qquad \begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ +|u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u, \quad (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in S_T, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases}$$

where $Q_T = \Omega \times (0,T]$ and S_T denote the lateral boundary of the cylinder Q_T. It is assumed throughout the paper that the exponents $m(x)$ and $p(x)$ are continuous in Ω with logarithmic module of continuity:

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$$(1.2)\qquad 1 < m^- = ess \inf_{x\in\Omega} m(x) \le m(x) \le m^+ = ess \sup_{x\in\Omega} m(x) < \infty,$$

$$(1.3)\qquad 1 < p^- = ess \inf_{x\in\Omega} p(x) \le p(x) \le p^+ = ess \sup_{x\in\Omega} p(x) < \infty,$$

$$(1.4)\qquad \forall z, \xi \in \Omega, |z-\xi| < 1, |m(x) - m(\xi)| + |p(z) - p(\xi)| \le \omega(|z-\xi|),$$

where

$$(1.5)\qquad \lim_{\tau\to 0^+} \sup \omega(\tau) ln\frac{1}{\tau} = C < +\infty.$$

Remark 1.1. We use the standard Lebesgue space $L^p(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar product and norms. We will use the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for $2 \le s \le 2n/(n-2)$ if $n \ge 3$ or $s \ge 2$ if $n = 1, 2$. The generic embedding constant, denoted by C_* is given by

$$(1.6)\qquad \|u\|_s \le C_* \|\nabla u\|_2.$$

And we also assume that
(H_1) : ρ is a constant that satisfies

$$0 < \rho \le \frac{2}{n-2} \text{ if } n \ge 3 \quad \text{and} \quad 0 < \rho \quad \text{if} \quad n = 1, 2.$$

(H_2) : $g : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded $\mathcal{C}^1$ function satisfying

$$g(0) > 0,\ 1 - \int_0^\infty g(s)ds = l > 0.$$

(H_3) : There exists $\xi > 0$ such that

$$g'(t) < -\xi(t)g(t), t \ge 0.$$

If m, p are constants, there have been many results about the existence and blow-up properties of the solutions, we refer the readers to the bibliography given in [5]-[25]. In recent years, a great attention has been focused on the study of mathematical models of electro-rheological fluids. These models include hyperbolic, parabolic or elliptic equations which are nonlinear with respect to gradient of the thought solution and with variable exponents of nonlinearity see ([3]-[12]-[15]-[23]-[24]) and the references therein. It should be mentioned that questions of existence, uniqueness and regularity of weak solutions for parabolic and elliptic equations have been studied by many authors under various conditions on the data and by different methods-(see [[1],[2]] and the further references therein).
To the best of our knowledge, there are only a few works about viscoelastic hyperbolic equations with variable exponents of nonlinearity. In [4] the authors investigated the finite time blow-up of solutions for viscoelastic hyperbolic equations, and in [5] the authors discussed only the viscoelastic hyperbolic problem with constant exponents. Motivated by the works of [[5],[4]], we shall study the existence and energy decay of the solutions to problem (1.1) and state some properties to the

solutions.
The present paper is organized as follows. In Section 2, we introduce the function spaces of Orlicz-Sobolev type and a brief description of their main properties, give the definition of the weak solution to the problem and prove the existence of weak solutions for problem (1.1) with Galerkin's method. In the last sections, we finally prove the desired results.

2. Existence of weak solutions

In this section, the existence of weak solutions is studied. Firstly, we introduce some Banach spaces
$L^{p(x)}(\Omega) = \left\{ u(x) : u \text{ is measurable in } \Omega, A_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}$,
with the following Luxembourg-type norm

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0, A_{p(.)}(u/\lambda) \le 1 \right\}.$$

We, next, define the variable-exponent Lebesgue Sobolev space $W^{1,p(.)}(\Omega)$ as follows:

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) \quad \text{such that} \quad \nabla u \quad \text{exists and} \quad |\nabla u| \in L^{p(.)}(\Omega) \right\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(.)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}$. Furthermore, we set $W_0^{1,p(.)}(\Omega)$ to be the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. Here we note that the space $W_0^{1,p(.)}(\Omega)$ is usually defined in a different way for the variable exponent case. However, both definitions are equivalent (see [10]). The dual of $W_0^{1,p(.)}(\Omega)$ is defined as $W^{-1,p'(.)}(\Omega)$; in the same way as the classical Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$.

Lemma 2.1. *([3]) For $u \in L^{p(x)}(\Omega)$, the following relations hold:*

1. $\|u\|_{p(.)} < 1 (= 1; > 1) \Leftrightarrow A_{p(.)}(u) < 1 (= 1; > 1)$;
2. $\|u\|_{p(.)} < 1 \Rightarrow \|u\|_{p(.)}^{p^+} \le A_{p(.)}(u) \le \|u\|_{p(.)}^{p^-}$;
   $\|u\|_{p(.)} > 1 \Rightarrow \|u\|_{p(.)}^{p^+} \ge A_{p(.)}(u) \ge \|u\|_{p(.)}^{p^-}$;
3. $\|u\|_{p(.)} \to 0 \Leftrightarrow A_{p(.)}(u) \to 0; \|u\|_{p(.)} \to \infty \Leftrightarrow A_{p(.)}(u) \to \infty$.

Lemma 2.2. *([26]) For $u \in W_0^{1,p(.)}(\Omega)$, if p satisfies condition* (1.2)*, the $p(.)$-Poincaré's inequality*

$$\|u\|_{p(x)} \le C \|\nabla u\|_{p(x)},$$

holds, where the positive constant C depends on p and Ω.

Remark 2.1. Note that the following inequality

$$\int_\Omega |u|^{p(x)}dx \leq C\int_\Omega |\nabla u|^{p(x)}dx,$$

does not in general hold.

Lemma 2.3. *([10]). Let Ω be an open domain (that may be unbounded) in $\mathbb{R}^n$ with cone property. If $p(x):\bar{\Omega}\to\mathbb{R}$ is Lipschitz continuous function satisfying $1<p^-\leq p^+<\frac{n}{k}$ and $r(x):\bar{\Omega}\to\mathbb{R}$ is measurable and satisfies*

$$p(x)\leq r(x)\leq p^*(x)=\frac{np(x)}{n-kp(x)}\ \ a.e\ x\in\bar{\Omega},$$

then there is a continuous embedding $W^{k,p(x)}(\Omega)\hookrightarrow L^{r(x)}(\Omega)$.

The main theorem in this section is the following:

Theorem 2.1. *Let $u_0,u_1\in H_0^1(\Omega)$ be given. Assume that the exponents $m(x)$ and $p(x)$ satisfy conditions (1.2)-(1.4). Then the problem (1.1) has at least one weak solution $u:\Omega\times(0,\infty)\to\mathbb{R}$ in the class*

$$u\in L^\infty(0,\infty;H_0^1(\Omega)),\ u'\in L^\infty(0,\infty;H_0^1(\Omega)),\ u''\in L^\infty(0,\infty;H_0^1(\Omega)).$$

And one of the following conditions holds:

$$(A_1)\ 2<p^-<p^+<\max\left\{n,\frac{np^-}{n-p^-}\right\},\quad 2<m^-<m^+<p^-;$$

$$(A_2)\ \max\left\{1,\frac{2n}{n+2}\right\}<p^-<p^+<2,\quad 1<m^-<m^+<\frac{3p^--2}{p^-}.$$

Proof. Let us take for $\{w_j\}_{j=1}^\infty$ the orthogonal basis of $H_0^1(\Omega)$ such that

$$-\Delta w_j=\lambda_j w_j, x\in\Omega,\ w_j=0,\ x\in\partial\Omega.$$

We denote by $V_k=span\ \{w_i,.....,w_k\}$ the subspace generated by the first k vectors of the basis $\{w\}_{j=1}^\infty$. By normalization, we have $\|w_j\|_2=1$. Let us define the operator:

$$\begin{aligned}<Lu,\phi> &=\int_\Omega\big[|u_t|^\rho u_{tt}\phi+\nabla u\nabla\phi+\nabla u_{tt}\nabla\phi-\int_0^t g(t-s)\nabla u\nabla\phi ds\\ &\quad+|u_t|^{m(x)-2}u_t\phi-\alpha|u|^{p(x)-2}u\phi\big]dx,\quad \phi\in V_k.\end{aligned}$$

For any given integer k, we consider the approximate solution $u_k = \sum_{i=1}^{k} c_i^k(t) w_i$, which satisfies

$$\left\{ \begin{array}{ll} < Lu_k, w_i >= 0 & i = 1, 2, ...k, \\ u_k(0) = u_{0k}, & u_{kt}(0) = u_{1k}, \end{array} \right. \tag{2.1}$$

where

$$u_{0k} = \sum_{i=1}^{k} (u_0, w_i) w_i, u_{1k} = \sum_{i=1}^{k} (u_1, w_i) w_i \text{ and } u_{0k} \to u_0, u_{1k} \to u_1 \; in \; H_0^1(\Omega).$$

Here we denote by (.,.) the inner product in $\mathbb{L}^2(\Omega)$.

Problem (1.1) generates the system of k ordinary differential equations

$$\left\{ \begin{array}{l} \left| \sum_{i=1}^{k} (c_i^k(t))', w_i \right|^{\rho} (\sum_{i=1}^{k} c_i^k(t), w_i)'' = -\lambda_i c_i^k(t) + \lambda_i \int_0^t g(t-s) c_i^k(s) ds \\ + \left| (\sum_{i=1}^{k} (c_i^k(t))', w_i) \right|^{m(x)-2} (\sum_{i=1}^{k} (c_i^k(t))', w_i) \\ -\alpha \left| (\sum_{i=1}^{k} c_i^k(t), w_i) \right|^{p(x)-2} (\sum_{i=1}^{k} c_i^k(t), w_i), \\ c_i^k(0) = (u_0, w_i), \; (c_i^k(0))' = (u_1, w_i), \quad i = 1, 2, ...k. \end{array} \right. \tag{2.2}$$

By the standard theory of the ODE system, we infer that the problem (2.2) admits a unique solution $c_i^k(t)$ in $[0, t_k]$, where $t_k > 0$. Then we can obtain an approximate solution $u_k(t)$ for (1.1) in V_k, over $[0, t_k]$. This solution can be extended to $[0, T]$, for any given $T > 0$, by the estimate below. Multiplying (2.1) by $(c_i^k(t))'$ and summing with respect to i we arrive at the relation

$$\begin{aligned} 0 &= \tfrac{d}{dt} \left(\tfrac{1}{\rho+2} \|u_k'\|_{\rho+2}^{\rho+2} + \tfrac{1}{2} \|\nabla u_k\|_2^2 + \tfrac{1}{2} \|\nabla u_k'\|_2^2 \right) + \int_\Omega |u_k'|^{m(x)} dx \\ &- \tfrac{d}{dt} \left(\int_0^t g(t-s) \int_\Omega (\nabla u_k(s) \nabla u_k'(t) dx ds) \right) - \alpha \tfrac{d}{dt} \left(\int_\Omega \frac{1}{p(x)} |u_k|^{p(x)} dx \right). \end{aligned} \tag{2.3}$$

Multiplying (2.1) by $(c_i^k(t))'$, integrating over Q_T, using integration by part and Green formula, one obtains

$$\begin{aligned} &- \int_0^t g(t-s) \int_\Omega (\nabla u_k(s), \nabla u_k'(t)) dx ds = \frac{1}{2} \frac{d}{dt} (g \diamond \nabla u_k)(t) \\ &- \tfrac{1}{2} (g' \diamond \nabla u_k)(t) - \tfrac{1}{2} \tfrac{d}{dt} \int_0^t g(s) ds \|\nabla u_k\|_2^2 + \frac{1}{2} g(t) \|\nabla u_k\|_2^2, \end{aligned} \tag{2.4}$$

here

$$(\varphi \diamond \nabla \psi)(t) = \int_0^t \varphi(t-s)\|\nabla\psi(t) - \nabla\psi(s)\|_2^2 ds.$$

Combining (2.3)-(2.4) and $(H_2) - (H_3)$, we get

$$\begin{aligned} &\frac{d}{dt}\bigg(\frac{1}{\rho+2}\|u_k'\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_k'\|_2^2 + \frac{1}{2}(g\diamond\nabla u_k)(t) \\ &+\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u_k\|_2^2 - \alpha\int_\Omega \frac{1}{p(x)}|u_k|^{p(x)}dx\bigg) \\ &= \frac{1}{2}(g\diamond\nabla u_k)(t) - \frac{1}{2}g(t)\|\nabla u_k\|_2^2 - \int_\Omega |u_k'|^{m(x)}dx. \end{aligned} \tag{2.5}$$

Integrating (2.5) over $(0,t)$, and using the assumptions (1.2)-(1.4), it is easy to verify that

$$\begin{aligned} \tfrac{1}{\rho+2}\|u_k'\|_{\rho+2}^{\rho+2} \quad &+\frac{1}{2}\|\nabla u_k'\|_2^2 + \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u_k\|_2^2 + \frac{1}{2}(g\diamond\nabla u_k)(t) \\ &-\alpha\tfrac{1}{p(x)}|u_k|^{p(x)} \le C_1, \end{aligned}$$

where C_1 is a positive constant depending only on $\|u_0\|_{H_0^1}, \|u_1\|_{H_0^1}$.
According to the Lemma 2.1, we also have

$$\begin{aligned} \tfrac{1}{\rho+2}\|u_k'\|_{\rho+2}^{\rho+2} + \tfrac{1}{2}\|\nabla u_k'\|_2^2 \quad &+\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u_k\|_2^2 + \frac{1}{2}(g\diamond\nabla u_k)(t) \\ &-\max\left\{\alpha\tfrac{1}{p^-}\|u_k\|_{p(x)}^{p^-}, \alpha\tfrac{1}{p^-}\|u_k\|_{p(x)}^{p^+}\right\} \le C_1. \end{aligned} \tag{2.6}$$

In view of $(H_1) - (H_2) - (H_3)$ and $(A_1) - (A_2)$, we get

$$\|u_k'\|_{\rho+2}^{\rho+2} + \|\nabla u_k'\|_2^2 + (g\diamond\nabla u_k)(t) \le C_2, \tag{2.7}$$

where C_2 is positive constant depending only on $\|u_0\|_{H_0^1}, \|u_1\|_{H_0^1}, l, p^-, p^+$. It follows from (2.7) that

$$u_k \quad \text{is uniformly bounded in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)). \tag{2.8}$$

$$u_k' \quad \text{is uniformly bounded in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)). \tag{2.9}$$

Next, multiplying (1.1) by $(c_i^k(t))''$ and then summing with respect to i, we get the following

$$\begin{aligned} &\int_\Omega |u_k'|^\rho |u_k''|_2^2 dx + \|\nabla u_k''\|_2^2 + \frac{d}{dt}\left(\frac{1}{m(x)}|u_k'|^{m(x)}\right) = -\int_\Omega \nabla u_k \nabla u_k'' dx \\ &+\int_\Omega g(t-s)\int_\Omega \nabla u_k(s)\nabla u_k'' dxds + \alpha\int_\Omega |u_k|^{p(x)-2}u_k u_k'' dx. \end{aligned} \tag{2.10}$$

Note that we have the estimates for $\varepsilon > 0$

$$\int_\Omega |u_k'|^\rho |u_k''|^2 dx \le C_\varepsilon \||u_k'|^\rho\|_2^2 + \frac{1}{4\varepsilon}\|u_k''\|_2^2, \tag{2.11}$$

$$\left|-\int_\Omega \nabla u_k \nabla u_k'' dx\right| \le \varepsilon \|\nabla u_k''\|_2^2 + \frac{1}{4\epsilon}\|\nabla u_k\|_2^2, \tag{2.12}$$

$$\begin{aligned}
&\left|-\int_0^t g(t-s)\int_\Omega \nabla u_k(s)\nabla u_k''(t)dxds\right| \\
&\le \tfrac{1}{4\varepsilon}\int_\Omega \left(\int_0^t g(t-s)\nabla u_k(s)ds\right)^2 dx + \varepsilon\|\nabla u_k''\|_2^2 \\
&\le \varepsilon\|\nabla u_k''\|_2^2 + \tfrac{1}{4\varepsilon}\int_0^t g(s)ds\int_0^t g(t-s)\int_\Omega |\nabla u_k(s)|^2 dxds \\
&\le \varepsilon\|\nabla u_k''\|_2^2 + \tfrac{(1-l)g(0)}{4\varepsilon}\int_0^t \|\nabla u_k(s)\|_2^2 ds,
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
\alpha\||u_k|^{p(x)-2}u_k u_k''\| &\le \alpha\varepsilon\|u_k''\|_2^2 + \tfrac{\alpha}{4\varepsilon}\||u_k|^{p(x)-2}u_k\|_2^2 \\
&\le \alpha\varepsilon\|u_k''\|_2^2 + \tfrac{\alpha}{4\varepsilon}\int_\Omega (|u_k|^{p(x)-2}u_k)^2 dx.
\end{aligned} \tag{2.14}$$

From Lemma 2.2, we have

$$\|u_k''\|_2^2 \le C^2\|\nabla u_k''\|_2^2, \tag{2.15}$$

and

$$\begin{aligned}
&\int_\Omega (|u_k'|^{p(x)-2}u_k)^2 dx = \int_\Omega |u_k|^{2(p(x)-1)}u_k dx \\
&\le \max\Big\{\int_\Omega |u_k|^{2(p^--1)}dx, \int_\Omega |u_k|^{2(p^+-1)}dx\Big\} \\
&\le \max\left\{C^{*\frac{1}{2(p^--1)}}\|\nabla u_k\|^{\frac{2}{2(p^--1)}}, C^{*\frac{1}{2p^+-1}}\|\nabla u_k'\|^{\frac{2}{2p^+-1}}\right\},
\end{aligned} \tag{2.16}$$

where C, C^* are embedding constants. Taking into account (2.10)-(2.16), we obtain

$$\begin{aligned}
&C_\varepsilon\int_\Omega |\nabla u_t|^{2\rho}dx + \frac{1}{4\varepsilon}\int_\Omega |u_k''|^2 dx + (1-2\varepsilon-\alpha\varepsilon C)\|\nabla u_k''\|_2^2 \\
&+\tfrac{d}{dt}(\tfrac{1}{m(x)}|u_k'|^{m(x)}) \le \tfrac{1}{4\varepsilon}\|\nabla u_k\|_2^2 + \tfrac{(1-l)g(0)}{4\varepsilon}\int_0^t \|\nabla u_k(s)\|_2^2 ds \\
&+\max\left\{C^{*\frac{1}{2(p^--1)}}\|\nabla u_k\|^{\frac{1}{p^--1}}, C^{*\frac{1}{2p^+-1}}\|\nabla u_k\|^{\frac{1}{p^+-1}}\right\}.
\end{aligned} \tag{2.17}$$

Integrating (2.17) over $(0,t)$ and using (2.7), Lemma 2.3 we get

$$\begin{aligned}
&C_\varepsilon T C_2^{2\rho} + \tfrac{1}{4\varepsilon}\int_0^t \|u_k''\|^2 dx + (1-2\varepsilon-\alpha\varepsilon C)\int_\Omega \|\nabla u_k''\|_2^2 ds \\
&+\int_\Omega \frac{1}{m(x)}|u_k'|^{m(x)}dx \le \frac{1}{4\varepsilon}(C_3 + (1-l)g(0)T) + C_4,
\end{aligned} \tag{2.18}$$

where C_4 is a positive constant depending only on $\|u_1\|_{H_0^1}$. Taking α,ε small enough in (2.18), we obtain the estimate

$$\frac{1}{4\varepsilon}\int_0^t \|u_k''\|^2 ds + \int_\Omega \frac{1}{m(x)}|u_k'|^{m(x)}dx \le C_5. \tag{2.19}$$

Hence according to the Lemma 2.1, we have that

$$\frac{1}{4\varepsilon}\int_0^t \|u_k''\|^2 ds + \min\left\{\frac{1}{m^+}\|u_k'\|_{m(x)}^{m^-}, \frac{1}{m^+}\|u_k'\|_{m(x)}^{m^+}\right\} \le C_5, \tag{2.20}$$

where C_5 is a positive constant depending only on $\|u_0\|_{H_0^1}$, $\|u_1\|_{H_0^1}$, l, $g(0)$, T. From estimate (2.20), we get

$$u_k'' \ \text{ is uniformly bounded in } \ \mathbb{L}^2(0,T;H_0^1(\Omega)). \tag{2.21}$$

By (2.7)-(2.9) and (2.25), we infer that there exists a subsequence u_i of u_k and function u such that

$$u_i \rightharpoonup u \quad \text{weakly star in} \quad L^\infty(0,T;H_0^1(\Omega)), \tag{2.22}$$

by (2.7)-(2.9) and (2.25), we infer that there exists a subsequence u_i of u_k and a function u such that

$$u_i \rightharpoonup u \quad \text{weakly star in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)), \tag{2.23}$$

$$u_i \rightharpoonup u \quad \text{weakly in} \quad \mathbb{L}^{p^-}(0,T;W^{1,p(x)}(\Omega)), \tag{2.24}$$

where C_5 is a positive constant depending only on $\|u_0\|_{H_0^1}$, $\|u_1\|_{H_0^1}$, l, $g(0)$, T. From estimate (2.20), we get

$$u_k'' \text{ is uniformly bounded in } \quad \mathbb{L}^2(0,T;H_0^1(\Omega)). \tag{2.25}$$

By (2.7)-(2.9) and (2.25), we infer that there exists a subsequence u_i of u_k and function u such that

$$u_i \rightharpoonup u \quad \text{weakly star in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)), \tag{2.26}$$

by (2.7)-(2.9) and (2.25), we infer that there exists a subsequence u_i of u_k and function u such that

$$u_i \rightharpoonup u \quad \text{weakly star in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)), \tag{2.27}$$

$$u_i \rightharpoonup u \quad \text{weakly in} \quad \mathbb{L}^{p^-}(0,T;W^{1,p(x)}(\Omega)), \tag{2.28}$$

$$u_i' \rightharpoonup u' \quad \text{weakly star in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)), \tag{2.29}$$

$$u_i'' \rightharpoonup u'' \quad \text{weakly in} \quad \mathbb{L}^2(0,T;H_0^1(\Omega)). \tag{2.30}$$

Next, we will deal with the nonlinear term. From the Aubin-Lions theorem, see ([20], pp.57-58], it follows from (2.29) and (2.30) that there exists a subsequence of u_i, still represented by the same notation, such that

$u_i' \to u'$ strongly in $\mathbb{L}^2(0,T;\mathbb{L}^2(\Omega))$, which implies that $u_i' \to u'$ almost everywhere in $\Omega \times (0,T)$. Hence, by (2.27) – (2.30), we have

$$|u_i'|^\rho u_i'' \to |u'|^\rho u'' \quad \text{weakly in} \quad \Omega \times (0,T), \tag{2.31}$$

$$|u_i|^{p(x)-2}u_i \rightharpoonup |u|^{p(x)-2}u \quad \text{weakly in} \quad \Omega \times (0,T), \tag{2.32}$$

$$|u_i'|^{m(x)-2}u_i' \to |u'|^{m(x)-2}u' \quad \text{almost everywhere in} \quad \Omega \times (0,T). \tag{2.33}$$

Multiplying (2.2) by $\phi(t) \in C(0,T)$(which $C(0,T)$ is space of C^∞ function with compact support in $(0,T)$) and integrating the obtained result over $(0,T)$, we obtain that

$$< Lu_k, w_i\phi(t) >= 0, \quad i = 1,2,........,k. \tag{2.34}$$

Note that $\{w_i\}_{i=1}^\infty$ is basis of $H_0^1(\Omega)$. Convergence (2.27)-(2.33) is sufficient to pass to the limit in (2.34) in order to get

$$|u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u \ \text{ in } \ \mathbb{L}^2(0,T;H_0^1(\Omega)),$$

for arbitrary $T > 0$. In view of (2.27) – (2.30) and Lemma 3.3.17 in [?], we derive that

$$u_k(0) \rightharpoonup u(0) \text{ weakly in } H_0^1(\Omega), \ u_k'(0) \rightharpoonup u'(0) \text{ weakly in } H_0^1(\Omega).$$

Hence, we get $u(0) = u_0$, $u_1(0) = u_1$. Then, we conclude the proof of the Theorem 2.1. □

3. Blow up

In this section, we shall prove our main result concerning the blow-up of solutions to Theorem 2.1. For this task, we define

$$\begin{aligned} E(t) = {} & \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\ & + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p, \end{aligned} \tag{3.1}$$

where

$$(g \circ v)(t) = \int_0^\infty g(s)ds < \frac{\frac{p}{2} - 1}{\frac{p}{2} - 1 + \frac{1}{2p}} \tag{3.2}$$

Lemma 3.1. *([22]) The modified energy functional satisfies the solution of (1.1)*

$$E'(t) \le \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - \|u_t\|_m^m \le \frac{1}{2}(g' \circ \nabla u)(t). \tag{3.3}$$

Theorem 3.1. *Suppose that*

$$\max\{m,p\} \le \frac{2(n-1)}{n-2}, \quad n \ge 3, \tag{3.4}$$

holds. Assume further that $u_0, u_1 \in H_0^1(\Omega)$ *and* $E(0) < 0$. *Then the solution of theorem 2.1 blows up in finite time*

$$T^* \leq \frac{C(1-\alpha)}{\epsilon\gamma\alpha L^{\frac{\alpha}{1-\alpha}}(0)}$$

Lemma 3.2. *Suppose that (3.4) holds. Then there exists a positive constant* $C > 1$ *depending on* Ω *only such that for any* $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p$, *we have*

$$\|u\|_p^s \leq C\left(\|\nabla u\|_2^2 + \|u\|_p^p\right). \tag{3.5}$$

Proof. 1. If $\|u\|_p \leq 1$ then $\|u\|_p^s \leq \|u\|_p^2 \leq C\|\nabla u\|_2^2$ by Sobolev embedding theorems.

2. If $\|u\|_p > 1$ then $\|u\|_p^s \leq \|u\|_p^p$. Therefore (3.5) follows.

□

We set

$$H(t) = -E(t).$$

We use, throughout this paper, C to denote a generic positive constant depending on Ω only. As a result of (3.1) and (3.5) we have

Corollary 3.1. *Let the assumptions of the lemma 3.2 hold. Then we have the following for all* $t \in [0, T)$,

$$\|u\|_p^s \leq C\left(-H(t) - \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 - \|\nabla u\|_2^2 - (g \circ \nabla u)(t) + \|u\|_p^p\right). \tag{3.6}$$

Proof. (Theorem 3.1) By multiplying equation (1.1) by $-u_t$ and integrating over Ω we obtain

$$\begin{aligned} &\frac{d}{dt}\left\{-\frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx - \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + \frac{1}{p}\int_\Omega |u|^p dx\right\} \\ &+ \int_0^t g(t-s)\int_\Omega \nabla u_t(t).\nabla u(s)dxds = \int_\Omega |u_t|^m dx, \end{aligned} \tag{3.7}$$

for any regular solution. This result can be extended to weak solutions by density argument. But

$$(3.8)\quad \begin{aligned}&\int_0^t g(t-s)\int_\Omega \nabla u_t(t).\nabla u(\tau)dxds\\&=\int_0^t\int_\Omega \nabla u_t(t).|\nabla u(s)-\nabla u(t)|dxds+\int_0^t g(t-s)\int_\Omega \nabla u_t(t).\nabla u(t)dxds\\&=-\frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\int_\Omega |\nabla u(s)-\nabla u(t)|^2dxds\\&+\int_0^t g(s)\left(\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla u(t)|^2dx\right)ds\\&=-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2dxds\right]\\&+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)\int_\Omega |\nabla u(t)|^2dxds\right]\\&+\frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2dxds-\frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2dxds.\end{aligned}$$

We then insert (3.8) in (3.7) to get

$$(3.9)\quad \begin{aligned}&\frac{d}{dt}\left\{-\frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx-\frac{1}{2}\int_\Omega |\nabla u|^2dx-\frac{1}{2}\int_\Omega |\nabla u_t|^2dx+\frac{1}{p}\int_\Omega |u|^pdx\right\}\\&-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2dxd\tau\right]\\&+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)\int_\Omega |\nabla u(t)|^2dxds\right]\\&=\int_\Omega |u_t|^mdx-\frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2dxds+\frac{1}{2}g(t)\|\nabla u(t)\|^2.\end{aligned}$$

By using the definition of $H(t)$ the estimate (3.9) becomes

$$(3.10)\quad \begin{aligned}H'(t)=&\int_\Omega |u_t|^mdx-\frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2dxds\\&+\frac{1}{2}g(t)\|\nabla u(t)\|^2\geq 0.\end{aligned}$$

Consequently, we have

$$(3.11)\quad 0<H(0)\leq H(t)\leq \frac{1}{p}\|u\|_p^p.$$

We define

$$(3.12)\quad L(t)=H^{1-\alpha}(t)+\frac{\epsilon}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx+\epsilon\int_\Omega \nabla u_t.\nabla u dx,$$

where ϵ small to be chosen later and

$$0<\alpha\leq \frac{p-m}{m-1}$$

By taking derivative of (3.12) and using (1.1) we obtain

$$
\begin{aligned}
&L'(t) = -\frac{1}{2}(1-\alpha)H^{-\alpha}(t)\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds \\
&+(1-\alpha)H^{-\alpha}(t)\left\{\int_\Omega |u_t|^m dx + \frac{1}{2}g(t)\|\nabla u(t)\|^2\right\} \\
&+\frac{\epsilon}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - \epsilon\|\nabla u\|_2^2 + \epsilon\|\nabla u_t\|_2^2 + \epsilon\int_0^t g(t-s)\int_\Omega \nabla u(s).\nabla u(t)ds \\
&-\epsilon\int_\Omega |u_t|^{m-2}u_t u dx + \epsilon\|u\|_p^p.
\end{aligned}
\tag{3.13}
$$

We then exploit Young's inequality, and use (3.1) to substitute for $\int_\Omega |u(x,t)|^p dx$ hence (3.13) becomes

$$
\begin{aligned}
&L'(t) \geq (1-\alpha)H^{1-\alpha}(t)\|u_t\|_m^m + \frac{\epsilon}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} \\
&-\epsilon\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \epsilon\|\nabla u_t\|_2^2 - \epsilon\eta(g\circ\nabla u) \\
&-\frac{\epsilon}{4\eta}\int_0^t g(s)ds\|\nabla u\|_2^2 - \epsilon\int_\Omega |u_t|^{m-2}u_t u dx + \epsilon\frac{p}{2}(g\circ\nabla u) \\
&+\epsilon\left(pH(t) + \frac{p}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{p}{2}\|\nabla u_t\|_2^2 + \frac{p}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right) \\
&\geq (1-\alpha)H^{1-\alpha}(t)\|u_t\|_m^m + \epsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon pH(t) \\
&+\epsilon\left(\left(\frac{p}{2}-1\right) - \left(\frac{p}{2}-1+\frac{1}{4\eta}\right)\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\
&+\epsilon\left(\frac{p}{2}-\eta\right)(g\circ\nabla u) + \epsilon\left(\frac{p}{2}+1\right)\|\nabla u_t\|_2^2,
\end{aligned}
\tag{3.14}
$$

for some number η with $0 < \eta < \frac{p}{2}$. By recalling (3.2), the estimate (3.14) is reduced to

$$
\begin{aligned}
&L'(t) \geq (1-\alpha)H^{-\alpha}(t)\|u_t\|_m^m + \epsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} \\
&+\epsilon pH(t) + \epsilon a_1(g\circ\nabla u) + \epsilon a_2\|\nabla u\|_2^2 + \epsilon a_3\|\nabla u_t\|_2^2 - \epsilon\int_\Omega |u_t|^{m-2}u_t u dx,
\end{aligned}
\tag{3.15}
$$

where

$$
a_1 = \frac{p}{2}-\eta > 0, \quad a_2 = \left(\frac{p}{2}-1\right) - \left(\frac{p}{2}-1+\frac{1}{4\eta}\right)\int_0^t g(s)ds > 0, \quad a_3 = \frac{p}{2}+1 > 0.
$$

To estimate the last term of (3.15), we use again Young's inequality

$$
XY \leq \frac{\delta^r}{r}X^r + \frac{\delta^{-q}}{q}Y^q, \quad X,Y \geq 0, \quad for \quad all \quad \delta > 0, \quad \frac{1}{r}+\frac{1}{q} = 1,
$$

with $r = m$ and $q = \frac{m}{(m-1)}$. So we have

$$\int_\Omega |u_t|^{m-1}|u|dx \geq \frac{\delta^m}{m}\|u\|_m^m + \frac{m-1}{m}\delta^{\frac{-m}{(m-1)}}\|u_t\|_m^m,$$

which yields, by substitution in (3.15), for all $\delta > 0$

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha)H^{-\alpha}(t) - \frac{m-1}{m}\delta^{\frac{-m}{m-1}}\right]\|u_t\|_m^m + \epsilon a_3\|\nabla u_t\|_2^2 - \epsilon\frac{\delta^m}{m}\|u\|_m^m \\ & +\epsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon pH(t) + \epsilon a_1(g \circ \nabla u) + \epsilon a_2\|\nabla u\|_2^2. \end{aligned} \tag{3.16}$$

The inequality (3.16) remains valid even if δ is time dependant since the integral is taken over the x variable. Therefore by taking δ so that $\delta^{\frac{-m}{m-1}} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in (3.16) we arrive at

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right]H^{-\alpha}(t)\|u_t\|_m^m + \epsilon\left(\frac{1}{\rho+1}\right)\|u_t\|_{\rho+2}^{\rho+2} \\ & +\epsilon a_1(g \circ \nabla u) + \epsilon a_2\|\nabla u\|_2^2 + \epsilon a_3\|\nabla u_t\|_2^2 + \epsilon\left(\frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} \\ & +\epsilon\left[pH(t) - \frac{k^{1-m}}{m}H^{\alpha(m-1)}(t)\|u\|_m^m\right]. \end{aligned} \tag{3.17}$$

By exploiting (3.11) and inequality $\|u\|_m^m \leq C\|u\|_p^m$, we obtain

$$H^{\alpha(m-1)}(t)\|u\|_m^m \leq \left(\frac{1}{p}\right)^{\alpha(m-1)} C\|u\|_p^{m+\alpha p(m-1)},$$

therefore, from (3.17), one obtains

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right]H^{-\alpha}(t)\|u_t\|_m^m \\ & +\epsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon a_1(g \circ \nabla u) + \epsilon a_2\|\nabla u\|_2^2 + \epsilon a_3\|\nabla u_t\|_2^2 \\ & +\epsilon\left[pH(t) - \frac{k^{1-m}}{m}\left(\frac{1}{p}\right)^{\alpha(m-1)} C\|u\|_p^{m+\alpha p(m-1)}\right]. \end{aligned} \tag{3.18}$$

At this stage, we use Corollary 3.1 for $s = m + \alpha(m-1) \le p$, to deduce from (3.18)

$$\begin{aligned}
L'(t) \ge & \left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right] H^{-\alpha}(t)\|u_t\|_m^m \\
& +\epsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon a_1(g\circ\nabla u) + \epsilon a_2\|\nabla u\|_2^2 + \epsilon a_3\|\nabla u_t\|_2^2 \\
& +\epsilon\Bigg[pH(t) - C_1 k^{1-m}\Big\{-H(t) - \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 - \|\nabla u\|_2^2 \\
& -(g\circ\nabla u)(t) + \|u\|_p^p\Big\}\Bigg] \qquad (3.19) \\
\ge & \left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right] H^{-\alpha}(t)\|u_t\|_m^m \\
& +\epsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2} + C_1 k^{1-m}\right)\|u_t\|_{\rho+2}^{\rho+2} \\
& +\epsilon\left(a_1 + C_1 k^{1-m}\right)(g\circ\nabla u) + \epsilon\left(a_2 + C_1k^{1-m}\right)\|\nabla u\|_2^2 \\
& +\epsilon\left(a_3 - C_1k^{1-m}\right)\|\nabla u_t\|_2^2 + \epsilon\left(p + C_1k^{1-m}\right)H(t) - \epsilon C_1 k^{1-m}\|u\|_p^p,
\end{aligned}$$

where $C_1 = \left(\frac{1}{p}\right)^{\alpha(m-1)} C/m$. By noting that

$$H(t) \ge -\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} - \frac{1}{2}\|\nabla u_t\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u) + \frac{1}{p}\|u\|_p^p,$$

and writing $p = 2a_4 + (p - 2a_4)$, where $a_4 = \min\{a_1, a_2, a_3\}$, the estimate (3.19) yields

$$\begin{aligned}
L'(t) \ge & \left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right] H^{-\alpha}(t)\|u_t\|_m^m \\
& +\epsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2} + C_1k^{1-m} - a_4\right)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon\left(\frac{2a_4}{p} - C_1k^{1-m}\right)\|u\|_p^p \qquad (3.20) \\
& +\epsilon\left(a_1 + C_1k^{1-m} - a_4\right)(g\circ\nabla u) + \epsilon\left(a_2 + C_1k^{1-m} - a_4\right)\|\nabla u\|_2^2 \\
& +\epsilon\left(a_3 - C_1k^{1-m} - a_4\right)\|\nabla u_t\|_2^2 + \epsilon\left(p + C_1k^{1-m} - 2a_4\right)H(t).
\end{aligned}$$

We choose k large enough so that (3.20) becomes

$$\begin{aligned}
L'(t) \ge & \left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right] H^{-\alpha}(t)\|u_t\|_m^m \\
& +\epsilon\gamma\left[H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u)(t) + \|u\|_p^p\right], \qquad (3.21)
\end{aligned}$$

where $\gamma > 0$ is the minimum of the coefficients of $H(t)$, $\|u_t\|_2^2$, $\|u\|_p^p$, and $(g\circ\nabla u)(t)$ in (3.21). Once k is fixed (hence γ), we pick ϵ small enough so that

$$(1-\alpha) - \frac{\epsilon k(m-1)}{m} \ge 0,$$

and

$$L(0) = H^{1-\alpha}(0) + \frac{\epsilon}{\rho+1}\int_\Omega |u_t|^\rho u_1 u_0 dx + \epsilon\int_\Omega \nabla u_1.\nabla u_0 dx > 0.$$

Therefore (3.21) takes the form

$$L'(t) \geq \epsilon\gamma \left[H(t) + \|u_t\|_{\rho 2}^{\rho+2} + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 + (g \circ \nabla u)(t) + \|u\|_p^p \right]. \tag{3.22}$$

Consequently, we have

$$L(t) \geq L(0) > 0, \quad for \quad all \quad t \geq 0.$$

We now estimate

$$\left| \int_\Omega |u_t|^\rho u_t u dx \right| \leq \|u_t\|_{\rho+2}^{\rho+1} \|u\|_{\rho+2} \leq C \|u_t\|_{\rho+2}^{\rho+2} \|u\|_p,$$

we have

$$\left| \int_\Omega |u_t|^\rho u_t u dx \right|^{\frac{1}{1-\alpha}} \leq C \|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\alpha}} \|u\|_p^{\frac{1}{1-\alpha}} \leq C \left(\|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\alpha}\mu} + \|u\|_p^{\frac{\theta}{1-\alpha}} \right).$$

Where $\frac{1}{\mu} + \frac{1}{\theta} = 1$. Choose $\mu = \frac{(1-\alpha)(\rho+2)}{\rho+1} (> 1)$, then

$$\frac{\theta}{1-\alpha} = \frac{\rho+2}{(1-\alpha)(\rho+2) - (\rho+1)} < p.$$

Using Corollary 3.1, we obtain for all $t \geq 0$

$$\left| \int_\Omega |u_t|^\rho u_t u dx \right|^{\frac{1}{1-\alpha}} \leq C \left[-H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 - \|\nabla u\|_2^2 - (g \circ \nabla u)(t) + \|u\|_p^p \right].$$

Therefore,

$$\begin{aligned} L^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha}(t) + \frac{\epsilon}{\rho+1} \int_\Omega |u_t|^\rho u_t u dx + \epsilon \int_\Omega \nabla u_t . \nabla u dx \right)^{\frac{1}{1-\alpha}} \\ &\leq C \left[\|u_t\|_{\rho+2}^{\rho+2} + H(t) + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^{\frac{2}{1-2\alpha}} + \|u\|_p^p \right], \quad \forall t \geq 0. \end{aligned} \tag{3.23}$$

Noting that

$$\|\nabla u\|_2^{\frac{2}{1-2\alpha}} \leq C^{\frac{1}{1-2\alpha}} \leq \frac{C^{\frac{1}{1-2\alpha}}}{H(0)} H(t), \tag{3.24}$$

it follows from (3.23) and (3.24) that

$$L^{\frac{1}{1-\alpha}}(t) \leq C \left[\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 + \|u\|_p^p \right], \quad \forall t \geq 0. \tag{3.25}$$

Combining (3.22) and (3.25), we arrive at

$$L'(t) \geq \frac{\epsilon\gamma}{C} L^{\frac{1}{1-\alpha}}(t), \quad \forall t \geq 0. \tag{3.26}$$

A simple integration of (3.26) over $(0,t)$ yields

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{\frac{-\alpha}{1-\alpha}}(0) - \epsilon\gamma t\alpha/|C(1-\alpha)|} \tag{3.27}$$

This shows that $L(t)$ blows up in finite time.

$$T^* \leq \frac{C(1-\alpha)}{\epsilon\gamma\alpha L^{\frac{\alpha}{1-\alpha}}(0)} \tag{3.28}$$

Summarizing, the proof is completed. □

4. Asymptotic Behavior

In this section, we investigate the asymptotic behavior of the problem (1.1). We define

$$G(t) = ME(t) + \epsilon\psi(t) + \chi(t), \tag{4.1}$$

where ϵ and M are positive constants which shall be determined later, and

$$\psi(t) = \frac{1}{\rho+1}\xi(t)\int_\Omega |u_t|^\rho u_t u dx + \xi(t)\int_\Omega \nabla u_t.\nabla u dx, \tag{4.2}$$

$$\chi(t) = \xi(t)\int_\Omega \left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^t g(t-s)\left[u(t)-u(s)\right] dsdx. \tag{4.3}$$

Theorem 4.1. *Let $(u_0,u_1) \in H_0^1(\Omega)\times H_0^1(\Omega)$ be given. Assume that $(H_1)-(H_3)$ and (3.4) hold. Then for each $t_0>0$, there exists two positive constants K and κ such that the solution of (1.1) satisfies*

$$E(t) \leq Ke^{-\kappa\int_0^t \xi(s)ds}, \quad t \geq t_0. \tag{4.4}$$

For our purposes, we need:

Theorem 4.2. *([22]) Suppose that $(H_1)-(H_3)$ and (3.4) hold. If $u_0,u_1 \in H_0^1(\Omega)$ and*

$$\frac{C_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{4.5}$$

where C_* is the best Poincare's constant. Then the solution of the problem (1.1) is global in time and satisfies

$$l\|\nabla u(t)\| + \|\nabla u_t(t)\| \leq \frac{2p}{p-2}E(0). \tag{4.6}$$

The proof of the theorem 4.2 is detailed in [22].

Lemma 4.1. *Let $u \in L^\infty(0,T;H_0^1(\Omega))$ be the solution of (1.1), then we have*

$$\int_\Omega \left(\int_0^t g(t-s)\left[u(t)-u(s)\right]ds\right)^{\rho+2} dx \le C_*^{\rho+2}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}} \times (g\circ\nabla u)(t). \tag{4.7}$$

Proof. Here, we point out that

$$\int_0^t g(t-s)\left[u(t)-u(s)\right]ds = \int_0^t \left[g(t-s)\right]^{\frac{\rho+1}{\rho+2}}\left[g(t-s)\right]^{\frac{1}{\rho+2}}\left[u(t)-u(s)\right]ds,$$

then by using Hölder's inequality, we get

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^{\rho+2} dx \le \\
&\le \left(\int_0^\infty g(s)ds\right)^{\rho+1}\int_0^t g(t-s)\int_\Omega |u(t)-u(s)|^{\rho+2}dxds \\
&\le C_*^{\rho+2}(1-l)^{\rho+1}\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_2^{\rho+2}ds \\
&\le C_*^{\rho+2}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}(g\circ\nabla u)(t).
\end{aligned}$$

This ends the proof. □

Lemma 4.2. *For $\epsilon > 0$ small enough while $M > 0$ is large enough, the relation*

$$\alpha_1 G(t) \le E(t) \le \alpha_2 G(t),$$

holds for two positive constants α_1 and α_2.

Proof. By using Young's inequality, the Sobolev embedding theorem, (1.6),(4.6) and Lemma 4.1, we can derive that

$$\begin{aligned}
\left|\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx\right| &\le \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\|u\|_{\rho+2}^{\rho+2} \\
&\le \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{2pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}\|\nabla u\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g(t-s)[u(t)-u(s)]dsdx\right| \\
&\le \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\int_\Omega\left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)^{\rho+2}dx \\
&\le \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}(g\circ\nabla u)(t).
\end{aligned}$$

It follows that

$$\begin{aligned}
G(t) &\le ME(t) + \left(\frac{1}{\rho+1} + \frac{\epsilon}{\rho+2}\right)\xi(t)\|u_t\|_{\rho+2}^{\rho+2} \\
&+\epsilon\left[\frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{2pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}} + \frac{1}{2}\right]\xi(t)\|\nabla u\|_2^2 + \frac{\epsilon+1}{2}\xi(t)\|\nabla u_t\|_2^2 \\
&+\left[\frac{1-l}{2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}\right]\xi(t)(g\circ\nabla u)(t) \\
&\le ME(t) + \left(\frac{1}{\rho+1} + \frac{\epsilon}{\rho+2}\right)N\|u_t\|_{\rho+2}^{\rho+2} \\
&+\epsilon\left[\frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{2pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}} + \frac{1}{2}\right]N\|\nabla u\|_2^2 + \frac{\epsilon+1}{2}N\|\nabla u_t\|_2^2 \\
&+\left[\frac{1-l}{2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}\right]N(g\circ\nabla u)(t) \le \frac{1}{\alpha_1}E(t),
\end{aligned}$$

and

$$\begin{aligned}
G(t) &\ge ME(t) - \left(\frac{1}{\rho+1} + \frac{\epsilon}{\rho+2}\right)\xi(t)\|u_t\|_{\rho+2}^{\rho+2} \\
&-\epsilon\left[\frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{2pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}} + \frac{1}{2}\right]\xi(t)\|\nabla u\|_2^2 - \frac{\epsilon+1}{2}\xi(t)\|\nabla u_t\|_2^2 \\
&-\left[\frac{1-l}{2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}\right]\xi(t)(g\circ\nabla u)(t) \\
&\ge \left[\frac{M}{\rho+2} - \left(\frac{1}{\rho+1} + \frac{\epsilon}{\rho+2}\right)N\right]\|u_t\|_{\rho+2}^{\rho+2} + \left(\frac{M}{2} - \frac{\epsilon+1}{2}N\right)\|\nabla u_t\|_2^2 \\
&+\left\{\frac{M}{2}l - \epsilon\left[\frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{2pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}} + \frac{1}{2}\right]N\right\} - \frac{M}{p}\|u\|_p^p \\
&+\left\{\frac{M}{2} - \left[\frac{1-l}{2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}\right]N\right\}(g\circ\nabla u)(t) \ge \frac{1}{\alpha_2}E(t).
\end{aligned}$$

For $\epsilon > 0$ small enough while $M > 0$ is large enough. This completes the proof. $\square$

Lemma 4.3. *Under the assumptions $(H_1) - (H_3)$ and (4.6), the functional*

$$\psi(t) = \frac{1}{\rho+1}\xi(t)\int_\Omega |u_t|^\rho u_t u dx + \xi(t)\int_\Omega \nabla u_t.\nabla u dx,$$

satisfies the solutions of (1.1),

$$\begin{aligned}
\psi'(t) &\le -\left[\frac{l}{2} - \delta C_*^2 - k\delta\left(1 + \frac{C_*^2}{\rho+1}\right)\right]\xi(t)\|\nabla u\|_2^2 \\
&+\left\{1 + \frac{A}{4\delta} + \frac{k}{4\delta}\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right]\right\}\xi(t)\|\nabla u_t\|_2^2 \\
&+\frac{1-l}{2l}\xi(t)(g\circ\nabla u)(t) + \frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2} + \xi(t)\|u\|_p^p.
\end{aligned} \tag{4.8}$$

Proof. By using the equation of (1.1), we easily see that

$$
\begin{aligned}
\psi'(t) &= \frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2} + \xi(t)\int_\Omega |u_t|^\rho u_{tt} u dx + \xi(t)\|\nabla u_t\|_2^2 \\
&+\xi(t)\int_\Omega \nabla u.\nabla u_{tt} dx + \frac{1}{\rho+1}\xi'(t)\int_\Omega |u_t|^\rho u_t u dx + \xi'(t)\int_\Omega \nabla u_t.\nabla u dx \\
&= \frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2} + \xi(t)\|\nabla u_t\|_2^2 - \xi(t)\|\nabla u\|_2^2 \\
&+\xi(t)\int_\Omega \nabla u.\int_0^t g(t-s)\nabla u(s) ds dx - \xi(t)\int_\Omega |u_t|^{m-2} u_t u dx + \xi(t)\|u\|_p^p \\
&+\frac{1}{\rho+1}\xi'(t)\int_\Omega |u_t|^\rho u_t u dx + \xi'(t)\int_\Omega \nabla u_t.\nabla u dx.
\end{aligned} \tag{4.9}
$$

Now we estimate

$$
\begin{aligned}
&\int_\Omega \nabla u(t).\int_0^t g(t-s)\nabla u(s) ds dx \le \frac{1}{2}\|\nabla u\|_2^2 \\
&+\frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|) ds\right)^2 dx.
\end{aligned} \tag{4.10}
$$

We use Young's inequality and the fact that

$$\int_0^t g(s) ds \le \int_0^\infty g(s) ds = 1-l,$$

it follows from (4.10) for $\eta = \frac{l}{1-l} > 0$ that

$$
\begin{aligned}
&\int_\Omega \nabla u(t).\int_0^t g(t-s)\nabla u(s) ds dx \le \frac{1}{2}(1+\eta)\int_\Omega \left(\int_0^t g(t-s)|\nabla u(t)| ds\right)^2 dx \\
&+\frac{1}{2}\left(1+\frac{1}{\eta}\right)\int_\Omega \left(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)| ds\right)^2 dx + \frac{1}{2}\|\nabla u\|_2^2 \\
&\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(1+\eta)(1-l)^2\|\nabla u(t)\|_2^2 + \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t) \\
&\le \frac{2-l}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2l}(1-l)(g\circ\nabla u)(t),
\end{aligned}
$$

and

$$\int_\Omega |u_t|^\rho u_t u dx \le \frac{1}{4\delta}\|u_t\|_{2(\rho+1)}^{2(\rho+1)} + \delta C_*^2\|\nabla u\|_2^2, \tag{4.11}$$

for any $\delta > 0$. In view of (4.6) and the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega), \quad for \quad 0 < \rho \le \frac{2}{n-2} \quad if \quad n \ge 3 \quad and \quad \rho > 0 \quad if \quad n = 1, 2,$$

we get

$$\|u_t\|_{2(\rho+1)}^{2(\rho+1)} \le C_*^{2(\rho+1)}\left(\frac{2pE(0)}{p-2}\right)^\rho \|\nabla u_t\|_2^2. \tag{4.12}$$

It follows from (4.11) and (4.12) that

$$\begin{aligned}\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx &\le \frac{1}{4\delta}\frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho \|\nabla u_t\|_2^2 \\ &+\frac{1}{\rho+1}\delta C_*^2\|\nabla u\|_2^2,\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}&\left|\int_\Omega |u_t|^{m-2}u_t u dx\right| \le \delta\int_\Omega |u|^2dx + \frac{1}{4\delta}\int_\Omega |u_t|^{2m-2}dx \\ &\le \delta C_*^2\|\nabla u\|_2^2 + \tfrac{1}{4\delta}C_*^{2m-2}\|\nabla u_t\|_{2m-2}^{2m-2} \\ &\le \delta C_*^2\|\nabla u\|_2^2 + \tfrac{1}{4\delta}C_*^{2m-2}\left((\tfrac{2pE(0)}{p-2}\right)^{\frac{m-2}{2}}\|\nabla u_t\|_2^2 \\ &\le \delta C_*^2\|\nabla u\|_2^2 + \tfrac{A}{4\delta}\|\nabla u_t\|_2^2.\end{aligned} \tag{4.14}$$

Also

$$\int_\Omega \nabla u_t.\nabla u dx \le \frac{1}{4\delta}\|\nabla u_t\|_2^2 + \delta\|\nabla u\|_2^2. \tag{4.15}$$

By combining (4.9),(4.10),(4.13),(4.14) and (4.15), we deduce easily the estimate (4.8). This completes our proof. □

Lemma 4.4. *Under the assumptions $(H_1)-(H_3)$, the functional*

$$\chi(t) = \xi(t)\int_\Omega\left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^t g(t-s)[u(t)-u(s)]dsdx,$$

satisfies, along solutions of (1.1) and for $\delta > 0$

$$\begin{aligned}&\chi'(t) \le \delta_1\left[1+2(1-l)^2+C_*^{2p-2}\left(\tfrac{2pE(0)}{(p-2)l}\right)^{p-2}\right]\xi(t)\|\nabla u\|_2^2 \\ &\left[\frac{(\rho+2)k}{4\delta_1(\rho+1)}+2\delta_1+\frac{1}{2\delta_1}+\frac{C_*^2}{2\delta_1}\right](1-l)\xi(t)(g\circ\nabla u)(t) \\ &-\frac{g(0)}{4\delta_1}\left(1+\frac{C_*^2}{\rho+1}\right)\xi(t)(g'\circ\nabla u)(t) - \frac{1}{\rho+1}\xi(t)\left(\int_0^t g(s)ds\right)\|u_t\|_{\rho+2}^{\rho+2} \\ &+\left\{(k+1)\delta_1\left[1+\frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right]+\delta_1\tilde{A}-\int_0^t g(s)ds\right\}\xi(t)\|\nabla u_t\|_2^2.\end{aligned} \tag{4.16}$$

Proof. Applying (1.1), the computation yields

$$\begin{aligned}\chi'(t) &= \xi(t)\int_\Omega(\Delta u_{tt} - |u_t|^\rho u_{tt})\int_0^t g(t-s)[u(t)-u(s)]dsdx \\ &+\xi(t)\int_\Omega\left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^t g'(t-s)[u(t)-u(s)]dsdx \\ &+\xi(t)\int_0^t g(s)ds\int_\Omega\left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)u_t dx \\ &+\xi'(t)\int_\Omega \Delta u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx \\ &-\frac{1}{\rho+1}\xi'(t)\int_\Omega |u_t|^\rho u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx.\end{aligned}$$

By integrating the parts, it follows that

$$
\begin{aligned}
\chi'(t) &= \xi(t)\int_\Omega \nabla u(t).\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx\\
&-\xi(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)ds.\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx\\
&-\xi(t)\int_0^t g(s)ds\|\nabla u_t\|_2^2-\xi(t)\int_\Omega \nabla u_t.\int_0^t g'(t-s)[\nabla u(t)-\nabla u(s)]dsdx\\
&-\frac{1}{\rho+1}\xi(t)\int_\Omega |u_t|^\rho u_t\int_0^t g'(t-s)[u(t)-u(s)]dsdx\\
&-\frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2}\int_0^t g(s)ds\\
&+\xi(t)\int_\Omega |u_t|^{m-2}u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx\\
&-\xi(t)\int_\Omega |u|^{p-2}u\int_0^t g(t-s)[u(t)-u(s)]dsdx\\
&-\xi'(t)\int_\Omega \nabla u_t.\int_0^t g(t-s)[\nabla u(t)-u(s)]dsdx\\
&-\frac{1}{\rho+1}\xi'(t)\int_\Omega |u_t|^\rho u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx.
\end{aligned}
\tag{4.17}
$$

In fact, by exploiting Young's inequality, we get that for any $\delta_1>0$

$$
\begin{aligned}
\int_\Omega \nabla u(t).\left(\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]ds\right)dx &\le \delta_1\|\nabla u\|_2^2\\
&+\frac{1}{4\delta_1}(1-l)(g\circ\nabla u)(t),
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&\int_\Omega\int_0^t g(t-s)\nabla u(s)ds.\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx\\
&\le \delta_1\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|ds\right)^2dx\\
&+\frac{1}{4\delta_1}\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2dx\\
&\le\left(2\delta_1+\frac{1}{4\delta_1}\right)\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2dx\\
&\le\left(2\delta_1+\frac{1}{4\delta_1}\right)(1-l)(g\circ\nabla u)(t)+2\delta_1(1-l)^2\|\nabla u\|_2^2+2\delta_1(1-l)^2\|\nabla u\|_2^2,
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
\int_\Omega \nabla u_t.\int_0^t g'(t-s)[\nabla u(t)-\nabla u(s)]dsdx &\le \delta_1\|\nabla u_t\|_2^2\\
&+\frac{g(0)}{4\delta_1}(-g'\circ\nabla u)(t).
\end{aligned}
\tag{4.20}
$$

Also

$$\begin{aligned}(4.21)\qquad &\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g'(t-s)[u(t)-u(s)]dsdx \\ &\le \delta_1 \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho \|\nabla u_t\|_2^2 + \frac{g(0)}{4\delta_1(\rho+1)} C_*^2(-g'\circ\nabla u)(t).\end{aligned}$$

Similarly, we get

$$\begin{aligned}(4.22)\qquad &\left|\int_\Omega |u_t|^{m-2}u_t \int_0^t g(t-s)(u(t)-u(s))dsdx\right| \\ &\le \delta_1\|u_t\|_{2m-2}^{2m-2} + \frac{1}{4\delta_1}\int_\Omega \left(\int_0^t g(t-s)|u(t)-u(s)|ds\right)^2 dx \\ &\le \delta_1 C_*^{2m-2}\|\nabla u_t\|_{2m-2}^{2m-2} + \frac{1}{4\delta_1}\int_0^t g(s)ds \int_0^t g(t-s)\int_\Omega |u(t)-u(s)|^2 dsdx \\ &\le \delta_1 \tilde{A}\|\nabla u_t\|_2^2 + \frac{1}{4\delta_1}(1-l)C_*^2(g\circ\nabla u)(t),\end{aligned}$$

$$\begin{aligned}(4.23)\qquad &-\int_\Omega |u|^{p-2}u\int_0^t g(t-s)[u(t)-u(s)]dsdx \\ &\le \frac{1}{4\delta_1}\int_\Omega\left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)^2 dx \\ &+\delta_1\|u\|_{2p-2}^{2p-2} \le \delta_1\|u\|_{2p-2}^{2p-2} + \frac{C_*^2(1-l)}{4\delta_1}(g\circ\nabla u)(t) \\ &\le \delta_1 C_*^{2p-2}\left(\frac{2pE(0)}{(p-2)l}\right)^{p-2}\|\nabla u\|_2^2 + \frac{C_*^2(1-l)}{4\delta_1}(g\circ\nabla u)(t),\end{aligned}$$

and

$$\begin{aligned}(4.24)\qquad \int_\Omega \nabla u_t.\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx \quad &\le \delta_1\|\nabla u_t\|_2^2 \\ &+\frac{1}{4\delta_1}(1-l)(g\circ\nabla u)(t).\end{aligned}$$

We estimate

$$\begin{aligned}(4.25)\qquad &\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx \\ &\le \frac{1}{\rho+1}\delta_1\|u_t\|_{2(\rho+1)}^{2(\rho+1)} + \frac{1}{4\delta_1(\rho+1)}\int_\Omega\left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)^2 dx \\ &\le \delta_1\frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho \|\nabla u_t\|_2^2 + \frac{1-l}{4\delta_1(\rho+1)}(g\circ\nabla u)(t).\end{aligned}$$

Combining the estimates (4.18)-(4.25) and (4.17), the assertion of the lemma 4.4 is established. □

Proof. (Theorem 4.1). Since g is positive, we have that, for any $t_0 > 0$,

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0, \quad t \geq t_0.$$

By using (4.1),(4.8),(4.16) and Lemma 4.1, a series of computations yields, for $t \geq t_0$,

$$\begin{aligned}
G'(t) &\leq \frac{M}{2}(g' \circ \nabla u)(t) - \epsilon \left[\frac{l}{2} - \delta C_*^2 - k\delta \left(1 + \frac{C_*^2}{\rho+1}\right)\right] \xi(t)\|\nabla u\|_2^2 \\
&+\epsilon \frac{1-l}{2l}\xi(t)(g \circ \nabla u)(t) + \epsilon \frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon\xi(t)\|u\|_p^p \\
&+\epsilon \left[1 + \frac{A}{4\delta} + \frac{k}{4\delta}\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right]\right] \xi(t)\|\nabla u_t\|_2^2 \\
&+\delta_1 \left[1 + 2(1-l)^2 + C_*^{2p-2}\left(\tfrac{2pE(0)}{(p-2)l}\right)^{p-2}\right] \xi(t)\|\nabla u\|_2^2 \\
&+ \left[\frac{(\rho+2)k}{4\delta_1(\rho+1)} + 2\delta_1 + \frac{1}{2\delta_1} + \frac{C_*^2}{2\delta_1}\right](1-l)\xi(t)(g \circ \nabla u)(t) \\
&- \frac{g(0)}{4\delta_1}\left(1 + \frac{C_*^2}{\rho+1}\right)\xi(t)(g' \circ \nabla u)(t) - \frac{1}{\rho+1}\xi(t)\left(\int_0^t g(s)ds\right)\|u_t\|_{\rho+2}^{\rho+2} \\
&+\left\{(k+1)\delta_1\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right] + \delta_1\tilde{A} - \int_0^t g(s)ds\right\}\xi(t)\|\nabla u_t\|_2^2 \\
&\leq -\left\{\left[g_0 - \epsilon\left(1 + \frac{A}{4\delta} + \frac{k}{4\delta}\left(1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right)\right)\right]\right. \\
&\left.-(k+1)\delta_1\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho + \delta_1\tilde{A}\right]\right\}\xi(t)\|u_t\|_2^2 \\
&-\left\{\epsilon\left[\frac{l}{2} - \delta C_*^2 - k\delta\left(1 + \frac{C_*^2}{\rho+1}\right)\right]\right. \\
&\left.-\delta_1\left[1 + 2(1-l)^2 + C_*^{2p-2}\left(\tfrac{2pE(0)}{(p-2)l}\right)^{p-2}\right]\right\}\xi(t)\|\nabla u\|_2^2 + \epsilon\xi(t)\|u\|_p^p \\
&- \left[\frac{M}{2} - \frac{g(0)}{4\delta_1}\left(1 + \frac{C_*^2}{\rho+1}\right)N\right](-g' \circ \nabla u)(t) - (g_0 - \epsilon)\frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2} \\
&+ \left[\frac{\epsilon}{2l} + \frac{(\rho+2)k}{4\delta_1(\rho+1)} + 2\delta_1 + \frac{1}{2\delta_1} + \frac{C_*^2}{2\delta_1}\right](1-l)\xi(t)(g \circ \nabla u)(t).
\end{aligned} \tag{4.26}$$

At this point, we choose $\delta > 0$ so small that

$$\frac{l}{2} - \delta C_*^2 - k\delta\left(1 + \frac{C_*^2}{\rho+1}\right) > \frac{l}{4}.$$

Hence δ is fixed, we choose $\epsilon > 0$ small enough so that Lemma 4.2 holds and that

$$\epsilon < \frac{g_0}{2\left[1 + \frac{A}{4\delta} + \frac{k}{4\delta}\left(1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right)\right]}$$

Once δ and ϵ are fixed, we choose a positive constant δ_1 satisfying

$$\delta_1 < \min\{\xi_1, \xi_2\},$$

where

$$\xi_1 = \frac{g_0}{2(k+1)\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho + \delta_1 \tilde{A}\right]},$$

and

$$\xi_2 = \frac{g_0}{2(k+1)\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho + \delta_1 \tilde{A}\right]}.$$

and be such that

$$\begin{aligned} g_0 - \epsilon &\left[1 + \frac{A}{4\delta} + \frac{k}{4\delta}\left(1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right)\right] \\ &-(k+1)\delta_1\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho + \delta_1\tilde{A}\right] > 0, \end{aligned}$$

also

$$\epsilon\left[\frac{l}{2} - \delta C_*^2 - k\delta\left(1 + \frac{C_*^2}{\rho+1}\right)\right] - \delta_1\left[1 + 2(1-l)^2 + C_*^{2p-2}\left(\frac{2pE(0)}{(p-2)l}\right)^{p-2}\right] > 0.$$

We then pick M sufficiently large so that Lemma 4.2 holds and that

$$\left[\frac{M}{2} - \frac{g(0)}{4\delta_1}\left(1 + \frac{C_*^2}{\rho+1}\right)N\right] - \left[\frac{\epsilon}{2l} + \frac{(\rho+2)k}{4\delta_1(\rho+1)} + 2\delta_1 + \frac{1}{2\delta_1} + \frac{C_*^2}{2\delta_1}\right](1-l) > 0.$$

Hence, using (H_3), we get

$$\begin{aligned} &\left[\frac{M}{2} - \frac{g(0)}{4\delta_1}\left(1 + \frac{C_*^2}{\rho+1}\right)N\right](-g' \circ \nabla u)(t) \\ &-\left[\frac{\epsilon}{2l} + \frac{(\rho+2)k}{4\delta_1(\rho+1)} + 2\delta_1 + \frac{1}{2\delta_1} + \frac{C_*^2}{2\delta_1}\right](1-l)\xi(t)(g \circ \nabla u)(t) \geq k_3\xi(t)(g \circ \nabla u)(t). \end{aligned}$$

By using Lemma 4.2 and (4.26), we arrive $\forall t \geq t_0$ at

$$G'(t) \leq -\beta_1\xi(t)E(t) \leq \alpha_1\beta_1\xi(t)G(t), \tag{4.27}$$

for some positive constant β_1. A simple integration of (4.27) leads to

$$G(t) \leq G(t_0)e^{-\alpha_1\beta_1\int_{t_0}^t \xi(s)ds}, \quad \forall t \geq t_0. \tag{4.28}$$

Thus, from Lemma 4.2 and (4.28), we get

$$E(t) \leq \alpha_2 G(t_0)e^{-\alpha_1\beta_1\int_{t_0}^t \xi(s)ds} = Ke^{-\kappa\int_{t_0}^t \xi(s)ds}, \quad \forall t \geq t_0. \tag{4.29}$$

This completes our proof. □

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