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A STUDY ON SCREEN TRANSVERSAL LIGHTLIKE SUBMANİFOLDS OF GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. We have studied radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds. We have investigated several properties of such submanifolds and obtained necessary and sufficient conditions for the induced connection on these submanifolds to be metric connection. Moreover, we have studied totally umbilical radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds and given the examples.
Key words: Geometric structures on manifolds, semi-Riemannian maifolds, Global submanifolds

1. Introduction

It is well known that in case the induced metric on the submanifold of semi-Riemannian manifold is degenerate, the study becomes more different from the study of non-degenerate submanifolds. The primary difference between the lightlike submanifolds and non-degenerate submanifolds arises due to the fact that in the first case the normal vector bundle has non-trivial intersection with the tangent vector bundle and also in a lightlike hypersurface the normal vector bundle is contained in the tangent vector bundle. Lightlike submanifolds is developed by Duggal and Bejancu [5] and Duggal and Şahin [8]. The lightlike submanifolds have been studied by many authors in various spaces for example [1, 4, 13, 15, 17, 18, 19, 24, 27].

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Duggal and Bejancu [5] introduced CR-lightlike submanifolds of indefinite Kaehler manifolds. Similar to CR-lightlike submanifolds, Atçeken and Kılıç introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold [2]. Since CR-lightlike submanifolds exclude the complex and totally real submanifolds as subcases, Duggal and Şahin introduced Screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds [7]. As a generalization of real null curves of indefinite Kaehler manifolds, Şahin introduced the notion of screen transversal lightlike submanifolds and obtained many interesting results [22]. In [25], Yıldırım and Şahin introduced screen transversal lightlike submanifolds of indefinite almost contact manifolds and show that such submanifolds contain lightlike real curves. Yıldırım and Erdoğan studied screen transversal lightlike submanifolds of semi-Riemannian product manifolds [26]. Khursheed Haider, Advin and Thakur studied totally umbilical screen transversal lightlike submanifolds of semi-Riemannian product manifolds [16].

Manifolds which are considered as differential geometric structures (such as almost complex manifolds, almost contact manifolds and almost product manifolds) are convenient when it comes to studying submanifold theory. One of the most studied manifold types are Riemannian manifolds with golden structures. Golden structures on Riemannian manifolds allow many geometric results. Hretcanu introduced golden structure on manifolds [14]. Crasmareanu and Hretcanu investigated the geometry of the golden structure on a manifold by using the corresponding almost product structure [3]. The integrability of golden structures has been investigated in [11]. In [23], Şahin and Akyol introduced golden maps between golden Riemannian manifolds, give an example and show that such map is harmonic. Erdoğan and Yıldırım studied totally umbilical semi-invariant submanifolds of golden Riemannian manifolds [10]. Gök, Keleş and Kılıç studied Schouten and Vrănceanu connections on golden manifolds [12]. Poyraz and Yaşar introduced lightlike submanifolds of a golden semi-Riemannian manifold [21]. Erdoğan studied the geometry of screen transversal lightlike submanifolds and radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds [9].

In this paper, we study radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds. We investigate several properties of such submanifolds and obtain necessary and sufficient conditions for the induced connection on these submanifolds to be metric connection. Moreover, we study totally umbilical radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds and give examples. We also give different form of some theorems given in [9].

2. Preliminaries

Let $\tilde{M}$ be a $C^{\infty}-$differentiable manifold. If a tensor field $\tilde{P}$ of type $(1,1)$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I \tag{2.1}$$

then $\tilde{P}$ is named a golden structure on $\tilde{M}$, where I is the identity transformation [14].

Let $(\tilde{M},\tilde{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\tilde{M}$. If $\tilde{P}$ holds the following equation

$$\tilde{g}(\tilde{P}X,Y)=\tilde{g}(X,\tilde{P}Y) \tag{2.2}$$

then $(\tilde{M},\tilde{g},\tilde{P})$ is named a golden semi-Riemannian manifold [20].

If $\tilde{P}$ is a golden structure, then the equation (2.2) is equivalent with

$$\tilde{g}(\tilde{P}X,\tilde{P}Y)=\tilde{g}(\tilde{P}X,Y)+\tilde{g}(X,Y) \tag{2.3}$$

for any $X,Y\in\Gamma(T\tilde{M})$.

Let $(\tilde{M},\tilde{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold with index q, such that $m,n\geq 1$, $1\leq q\leq m+n-1$ and (M,g) be an $m-$dimensional submanifold of $\tilde{M}$, where g is the induced metric of $\tilde{g}$ on M. If $\tilde{g}$ is degenerate on the tangent bundle TM of M, then M is named a lightlike submanifold of $\tilde{M}$. For a degenerate metric g on M

$$TM^{\perp}=\cup\left\{u\in T_x\tilde{M}:\tilde{g}(u,v)=0,\forall v\in T_x\tilde{M},x\in M\right\} \tag{2.4}$$

is a degenerate $n-$dimensional subspace of $T_x\tilde{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xM)=T_xM\cap T_xM^{\perp}$ which is known as radical (null) space. If the mapping $Rad(TM):x\in M\longrightarrow Rad(T_xM)$, defines a smooth distribution, called radical distribution on M of rank $r>0$ then the submanifold M of $\tilde{M}$ is called an $r-$lightlike submanifold.

Let $S\left(TM\right)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM. This means that

$$TM=S\left(TM\right)\perp Rad\left(TM\right) \tag{2.5}$$

and $S\left(TM^{\perp}\right)$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr\left(TM\right))$ and $ltr\left(TM\right)$ be complementary (but not orthogonal) vector bundles to TM in $T\tilde{M}_{|M}$ and $Rad(TM)$ in $S\left(TM^{\perp}\right)^{\perp}$, respectively. Then we have

$$tr\left(TM\right) = ltr\left(TM\right)\perp S\left(TM^{\perp}\right), \tag{2.6}$$

$$\begin{aligned} T\tilde{M}\mid_M &= TM\oplus tr\left(TM\right) \\ &= \{Rad(TM)\oplus ltr\left(TM\right)\}\perp S\left(TM\right)\perp S(TM^{\perp}). \end{aligned} \tag{2.7}$$

Theorem 2.1. *Let $(M,g,S(TM),S\left(TM^{\perp}\right))$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$. Suppose U is a coordinate neighbourhood of M and ξ_i, $i\in\{1,..,r\}$ is a basis of $\Gamma(Rad\left(TM\right)_{|U})$. Then, there exist a complementary vector subbundle $ltr\left(TM\right)$ of $Rad(TM)$ in $S\left(TM^{\perp}\right)^{\perp}_{|U}$ and a basis $\{N_i\}$, $i\in\{1,..,r\}$ of $\Gamma(ltr\left(TM\right)_{|U})$ such that*

$$\tilde{g}\left(N_i,\xi_j\right)=\delta_{ij},\ \tilde{g}\left(N_i,N_j\right)=0, \tag{2.8}$$

for any $i, j \in \{1, .., r\}$ [5].

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\tilde{M}$ is
Case 1: $r-$lightlike if $r < min\{m, n\}$,
Case 2: Coisotropic if $r = n < m$, $S(TM^{\perp}) = \{0\}$,
Case 3: Isotropic if $r = m < n$, $S(TM) = \{0\}$,
Case 4: Totally lightlike if $r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$.

Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$. Then, using (2.7), the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.9}$$
$$\tilde{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{2.10}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. According to (2.7), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, (2.9) and (2.10) become

$$\tilde{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.11}$$
$$\tilde{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.12}$$
$$\tilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.13}$$

where $h^l(X, Y) = Lh(X, Y)$, $h^s(X, Y) = Sh(X, Y)$, $\{\nabla_X Y, A_N X, A_W X\} \in \Gamma(TM)$, $\{\nabla^l_X N, D^l(X, W)\} \in \Gamma(ltr(TM))$ and $\{\nabla^s_X W, D^s(X, N)\} \in \Gamma(S(TM^{\perp}))$. Thus taking account of (2.11)-(2.13) and the Levi-Civita connection $\tilde{\nabla}$ is a metric, we derive

$$g(h^s(X, Y), W) + g(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.14}$$
$$g(D^s(X, N), W) = g(A_W X, N). \tag{2.15}$$

Let J be a projection of TM on $S(TM)$. Thus using (2.5) we obtain

$$\nabla_X JY = \nabla^*_X JY + h^*(X, JY)\xi, \tag{2.16}$$
$$\nabla_X \xi = -A^*_\xi X - \nabla^{*t}_X \xi, \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\left\{\nabla^*_X JY, A^*_\xi X\right\}$ and $\{h^*(X, JY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively.

Using the equations given above, we derive

$$g(h^l(X, JY), \xi) = g(A^*_\xi X, JY), \tag{2.18}$$
$$g(h^*(X, JY), N) = g(A_N X, JY), \tag{2.19}$$
$$g(h^l(X, \xi), \xi) = 0, A^*_\xi \xi = 0. \tag{2.20}$$

Generally, the induced connection ∇ on M is not metric connection. Since $\tilde{\nabla}$ is a metric connection, from (2.11) we obtain

$$(\nabla_X g)(Y,Z) = \tilde{g}(h^l(X,Y),Z) + \tilde{g}(h^l(X,Z),Y). \tag{2.21}$$

But, ∇^* is a metric connection on $S(TM)$.

Theorem 2.2. [5] *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$. Then the induced connection ∇ is a metric connection iff $Rad(TM)$ is a parallel distribution with respect to ∇.*

A lightlike submanifold M of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$ is named totally umbilical in $\tilde{M}$, if there is a smooth transversal vector field $H \in \Gamma(ltr\,(TM))$ of M which is named the transversal curvature vector of M, such that

$$h(X,Y) = Hg(X,Y), \tag{2.22}$$

for any $X, Y \in \Gamma(TM)$.

It is known that M is totally umbilical if on each coordinate neighborhood U, there exists smooth vector fields $H^l \in \Gamma(ltr\,(TM))$ and $H^s \in \Gamma(S\left(TM^{\perp}\right))$ such that

$$h^l(X,Y) = g(X,Y)H^l, h^s(X,Y) = g(X,Y)H^s \text{ and } D^l(X,W) = 0, \tag{2.23}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S\left(TM^{\perp}\right))$ [6].

3. Radical Screen Transversal Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

Definition 3.1. Let M be a lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then we say that M is a screen transversal lightlike submanifold of $\tilde{M}$ if there exists a screen transversal bundle $S(TM^{\perp})$ such that

$$\tilde{P}(Rad(TM)) \subset S(TM^{\perp}). \tag{3.1}$$

Definition 3.2. Let M be a screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then M is said to be a radical screen transversal lightlike submanifold if $S(TM)$ is invariant with respect to $\tilde{P}$.

Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Thus, for any $X \in \Gamma(TM)$ we derive

$$\tilde{P}X = PX + wX, \tag{3.2}$$

where PX and wX are tangential and transversal parts of $\tilde{P}X$.

For any $V \in \Gamma(tr(TM))$ we write

$$\tilde{P}V = BV + CV, \tag{3.3}$$

where BV and CV are tangential and transversal parts of $\tilde{P}V$.

Throughout this paper, we assume that $\tilde{\nabla}\tilde{P} = 0$.

Lemma 3.1. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then we have*

$$P^2X = PX + X - BwX, \tag{3.4}$$

$$wPX = wX - CwX, \tag{3.5}$$

$$PBV = BV - BCV, \tag{3.6}$$

$$C^2 = CV + V - wBV, \tag{3.7}$$

$$g(PX, Y) - g(X, PY) = g(X, wY) - g(wX, Y), \tag{3.8}$$

$$\begin{aligned} g(PX, PY) &= g(PX, Y) + g(X, Y) + g(wX, Y) - g(PX, wY) \\ &\quad - g(wX, PY) - g(wX, wY). \end{aligned} \tag{3.9}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Applying $\tilde{P}$ in (3.2) and using (2.1), we obtain

$$P^2X + wPX + BwX + CwX = PX + wX + X, \tag{3.10}$$

for any $X \in \Gamma(TM)$. From (3.10) we obtain (3.4) and (3.5). Using (2.1) and (3.3) we get

$$PBV + wBV + BCV + C^2V = BV + CV + V. \tag{3.11}$$

From (3.11) we get (3.6) and (3.7). From (2.2) and (3.2) we obtain

$$g(PX + wX, Y) = g(X, PY + wY), \tag{3.12}$$

for any $X, Y \in \Gamma(TM)$ and from this we obtain (3.8). Also, from (2.3) and (3.2) we derive

$$g(PX + wX, PY + wY) = g(PX + wX, Y) + g(X, Y), \tag{3.13}$$

for any $X, Y \in \Gamma(TM)$ and we get (3.9). □

Proposition 3.1. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then P is golden structure on $S(TM)$.*

Proof. By the definition of radical screen transversal lightlike submanifold we have $wX = 0$, for any $X \in \Gamma(S(TM))$. Then from (3.4) we have $P^2X = PX + X$. Thus P is golden structure on $S(TM)$. □

Proposition 3.2. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then C is golden structure on $ltr(TM)$.*

Proof. By the definition of radical screen transversal lightlike submanifold we have $BN = 0$, for any $N \in \Gamma(ltr(TM))$ From (3.7) we have $C^2N = CN + N$. Thus C is golden structure on $ltr(TM)$. □

Example 3.1. Let $(\tilde{M} = \mathbb{R}^7_3, \tilde{g})$ be a 7−dimensional semi-Euclidean space with signature $(-,+,-,+,-,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ be the standard coordinate system of $\mathbb{R}^7_3$. If we set $\tilde{P}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = ((1-\phi)x_1, (1-\phi)x_2, \phi x_3, \phi x_4, (1-\phi)x_5, (1-\phi)x_6, (1-\phi)x_7)$, then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\tilde{M}$. Suppose M is a submanifold of $\tilde{M}$ defined by

$$\begin{aligned} x_1 &= \phi u_1 + \cos u_2, x_2 = \phi u_1 - u_3, x_3 = \sqrt{2}u_1, x_4 = \sqrt{2}u_1, \\ x_5 &= \phi u_1 - \cos u_2, x_6 = \phi u_1 + \sqrt{2}u_3, x_7 = \cos u_2 - u_3, \end{aligned}$$

where u_i, $1 \le i \le 3$, are real parameters. Thus $TM = Span\{U_1, U_2, U_3\}$, where

$$\begin{aligned} U_1 &= \phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} + \sqrt{2}\frac{\partial}{\partial x_4} + \phi\frac{\partial}{\partial x_5} + \phi\frac{\partial}{\partial x_6}, \\ U_2 &= -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7}, \\ U_3 &= -\frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7}. \end{aligned}$$

Then M is a 1−lightlike submanifold. We have $Rad(TM) = Span\{U_1\}$ and $S(TM) = Span\{U_2, U_3\}$. Moreover, $\tilde{P}U_2 = \phi U_2$, $\tilde{P}U_3 = \phi U_3$ implies that $\tilde{P}(S(TM)) = S(TM)$. Lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = -\frac{1}{4(2+\phi)}(\phi\frac{\partial}{\partial x_1} - \phi\frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} - \sqrt{2}\frac{\partial}{\partial x_4} + \phi\frac{\partial}{\partial x_5} - \phi\frac{\partial}{\partial x_6}).$$

Also, screen transversal bundle $S(TM^\perp)$ is spanned by

$$\begin{aligned} W_1 &= \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7}, \\ W_2 &= -\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \sqrt{2}\phi\frac{\partial}{\partial x_3} + \sqrt{2}\phi\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6}, \\ W_3 &= -\frac{1}{4(2+\phi)}(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{2}\phi\frac{\partial}{\partial x_3} - \sqrt{2}\phi\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}). \end{aligned}$$

Then it is easy to see that $\tilde{P}\xi = W_2$, $\tilde{P}N = W_3$ and $\tilde{P}W_1 = \phi W_1$. Thus M is a radical screen transversal lightlike submanifold of $\tilde{M}$.

Theorem 3.1. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the screen distribution is integrable iff*

$$h^s(X, \tilde{P}Y) = h^s(Y, \tilde{P}X), \tag{3.14}$$

for any $X, Y \in \Gamma(S(TM))$ [9].

Theorem 3.2. *Let M be a totally umbilicial radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the screen distribution is always integrable.*

Proof. Using the definition of a radical screen transversal lightlike submanifold, $S(TM)$ is integrable iff $\tilde{g}([X,Y],N)=0$, for any $X,Y\in\Gamma(S(TM))$ and $N\in\Gamma(ltr(TM))$. Using (2.3) and (2.11) and taking into account that M is a totally umbilicial, we obtain

$$\begin{aligned}\tilde{g}([X,Y],N) &= \tilde{g}(\tilde{\nabla}_X Y-\tilde{\nabla}_Y X,N)\\ &= \tilde{g}(\tilde{\nabla}_X\tilde{P}Y-\tilde{\nabla}_Y\tilde{P}X,\tilde{P}N)-\tilde{g}(\tilde{\nabla}_X Y-\tilde{\nabla}_Y X,\tilde{P}N)\\ &= \tilde{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),\tilde{P}N)-\tilde{g}(h^s(X,Y)-h^s(Y,X),\tilde{P}N)\\ &= \tilde{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),\tilde{P}N)\\ &= (\tilde{g}(X,\tilde{P}Y)-\tilde{g}(Y,\tilde{P}X))\tilde{g}(H^s,\tilde{P}N)\end{aligned}$$

which completes the proof. □

Theorem 3.3. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then the radical distribution is integrable iff*

$$A_{\tilde{P}\xi_1}\xi_2-A_{\tilde{P}\xi_2}\xi_1=A^*_{\xi_1}\xi_2-A^*_{\xi_2}\xi_1, \tag{3.15}$$

$\xi_1,\xi_2\in\Gamma(Rad(TM))$ [9].

Theorem 3.4. *Let M be a totally umbilical radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then the radical distribution is always integrable.*

Proof. Using the definition of a radical screen transversal lightlike submanifold, $Rad(TM)$ is integrable iff $\tilde{g}([\xi_1,\xi_2],X)=0$, for any $X\in\Gamma(S(TM))$ and $\xi_1,\xi_2\in\Gamma(Rad(TM))$. Using (2.3), (2.11), and (2.23) and taking into account that $\tilde{\nabla}$ is a metric connection, we get

$$\begin{aligned}\tilde{g}([\xi_1,\xi_2],X) &= \tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2-\tilde{\nabla}_{\xi_2}\xi_1,X)=\tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2,X)-\tilde{g}(\tilde{\nabla}_{\xi_2}\xi_1,X)\\ &= -\tilde{g}(\xi_2,\tilde{\nabla}_{\xi_1}X)+\tilde{g}(\xi_1,\tilde{\nabla}_{\xi_2}X)\\ &= -\tilde{g}(\tilde{P}\xi_2,\tilde{\nabla}_{\xi_1}\tilde{P}X)+\tilde{g}(\tilde{P}\xi_2,\tilde{\nabla}_{\xi_1}X)\\ &\quad +\tilde{g}(\tilde{P}\xi_1,\tilde{\nabla}_{\xi_2}\tilde{P}X)-\tilde{g}(\tilde{P}\xi_1,\tilde{\nabla}_{\xi_2}X)\\ &= -\tilde{g}(\tilde{P}\xi_2,h^s(\xi_1,\tilde{P}X))+\tilde{g}(\tilde{P}\xi_2,h^s(\xi_1,X))\\ &\quad +\tilde{g}(\tilde{P}\xi_1,h^s(\xi_2,\tilde{P}X))-\tilde{g}(\tilde{P}\xi_1,h^s(\xi_2,X))\\ &= -g(\xi_1,\tilde{P}X)\tilde{g}(\tilde{P}\xi_2,H^s)+g(\xi_1,X)\tilde{g}(\tilde{P}\xi_2,H^s)\\ &\quad +g(\xi_2,\tilde{P}X)\tilde{g}(\tilde{P}\xi_1,H^s)-g(\xi_2,X)\tilde{g}(\tilde{P}\xi_1,H^s).\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then the screen distribution defines a totally geodesic foliation iff $h^s(X,\tilde{P}Y)-h^s(X,Y)$ has no compenents in $\tilde{P}(Rad(TM))$, for any $X,Y\in\Gamma(S(TM))$* [9].

Now, we give different form of theorem given in [9].

Theorem 3.6. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the radical distribution defines a totally geodesic foliation iff $h^s(\xi_1, X)$ has no compenents in $\tilde{P}(ltr(TM))$, for any $X \in \Gamma(S(TM))$ and $\xi_1 \in \Gamma(Rad(TM))$.*

Proof. Since $S(TM)$ is invariant, if $X \in \Gamma(S(TM))$ then $\tilde{P}X \in \Gamma(S(TM))$. Using the definition of radical screen transversal lightlike submanifold, $Rad(TM)$ defines a totally geodesic foliation iff $g(\nabla_{\xi_1}\xi_2, \tilde{P}X) = 0$, for any $X \in \Gamma(S(TM))$ and $\xi_1, \xi_2 \in \Gamma(Rad(TM))$. Since $\tilde{\nabla}$ is a metric connection, from (2.2) and (2.11), we derive

$$\begin{aligned} g(\nabla_{\xi_1}\xi_2, \tilde{P}X) &= \tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, \tilde{P}X) = \tilde{g}(\tilde{\nabla}_{\xi_1}\tilde{P}\xi_2, X) \\ &= -\tilde{g}(\tilde{P}\xi_2, \tilde{\nabla}_{\xi_1}X) = -\tilde{g}(\tilde{P}\xi_2, h^s(\xi_1, X)). \end{aligned}$$

Therefore we derive our theorem. □

Taking into account that M is a totally umbilicial in Theorem 3.6 we get following theorem.

Theorem 3.7. *Let M be a totally umbilicial radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the radical distribution always defines a totally geodesic foliation.*

Theorem 3.8. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the induced connection ∇ on M is a metric connection iff there is no component of $h^s(X, Y)$ in $\tilde{P}(ltr(TM))$ or $A_{\tilde{P}\xi}X$ in $S(TM)$ for any $X, Y \in \Gamma(S(TM))$ and $\xi \in \Gamma(Rad(TM))$* [9].

Theorem 3.9. *Let M be a totally umbilicial radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the induced connection ∇ on M is a metric connection iff H^s has no compenents in $\tilde{P}(ltr(TM))$.*

Proof. Considering Theorem 2.2, using (2.2), (2.11), (2.23) and taking into account that $\tilde{\nabla}$ is a metric connection, we obtain

$$\begin{aligned} g(\nabla_X\xi, \tilde{P}Y) &= \tilde{g}(\tilde{\nabla}_X\xi, \tilde{P}Y) = \tilde{g}(\tilde{\nabla}_X\tilde{P}\xi, Y) = -\tilde{g}(\tilde{P}\xi, \tilde{\nabla}_XY) \\ &= -\tilde{g}(\tilde{P}\xi, h^s(X, Y)) = -g(X, Y)\tilde{g}(H^s, \tilde{P}\xi), \end{aligned} \tag{3.16}$$

for any $X, Y \in \Gamma(S(TM))$ and $\xi \in \Gamma(Rad(TM))$, which completes the proof. □

4. Screen Transversal Anti-invariant Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

Definition 4.1. Let M be a screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then M is said to be a screen transversal anti-invariant lightlike submanifold if $S(TM)$ is screen transversal with respect to $\tilde{P}$, i.e.

$$\tilde{P}(S(TM)) \subset S(TM^{\perp}).$$

Let M be a screen transversal anti-invariant lightlike submanifold. Thus we have

$$S(TM^{\perp}) = \tilde{P}(Rad(TM)) \oplus \tilde{P}(ltr(TM)) \perp \tilde{P}(S(TM)) \perp D_0,$$

where D_0 is a non-degenerate orthogonal complementary distribution to

$$\tilde{P}(Rad(TM)) \oplus \tilde{P}(ltr(TM)) \perp \tilde{P}(S(TM)).$$

Proposition 4.1. *The distribution D_0 is an invariant distribution with respect to $\tilde{P}$* [9].

Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then we have

$$\tilde{P}X = wX. \tag{4.1}$$

Let T_1, T_2, T_3 and T_4 be the projection morphisms on $\tilde{P}(Rad(TM))$, $\tilde{P}(S(TM))$, $\tilde{P}(ltr(TM))$ and D_0, respectively. Thus, for any $V \in \Gamma(S(TM^{\perp}))$ we obtain

$$V = T_1V + T_2V + T_3V + T_4V. \tag{4.2}$$

On the other hand, for any $V \in \Gamma(S(TM^{\perp}))$ we write

$$\tilde{P}V = BV + CV, \tag{4.3}$$

where BV and CV are tangential and transversal parts of $\tilde{P}V$. Then applying $\tilde{P}$ to (4.2), we derive

$$\tilde{P}V = \tilde{P}T_1V + \tilde{P}T_2V + \tilde{P}T_3V + \tilde{P}T_4V. \tag{4.4}$$

If we put $\tilde{P}T_1V = B_1V + C_1V$, $\tilde{P}T_2V = B_2V + C_2V$, $\tilde{P}T_3V = C_3^lV + C_3^sV$ and $\tilde{P}T_4V = C_4V$, we can rewrite (4.4) as follows:

$$\tilde{P}V = B_1V + B_2V + C_1V + C_2V + C_3^lV + C_3^sV + C_4V. \tag{4.5}$$

$B_1V \in \Gamma(S(TM))$, $B_2V \in \Gamma(Rad(TM))$, $C_1V \in \Gamma(\tilde{P}S(TM))$, $C_2V \in \Gamma(\tilde{P}Rad(TM))$, $C_3^lV \in \Gamma(ltr(TM))$, $C_3^sV \in \Gamma(\tilde{P}ltr(TM))$ and $C_4V \in \Gamma(D_0)$. From (4.3) and (4.5), we can write

$$BV = B_1V + B_2V, CV = C_1V + C_2V + C_3^lV + C_3^sV + C_4V. \tag{4.6}$$

Similar to the proof of Lemma 3.1, we have the following lemma.

Lemma 4.1. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then we have*

$$BwX = X, \tag{4.7}$$

$$CwX = wX, \tag{4.8}$$

$$BCV = BV - PBV, \tag{4.9}$$

$$C^2V = CV + V - wBV, \tag{4.10}$$

$$g(X, wY) = g(wX, Y), \tag{4.11}$$

$$g(wX, wY) = g(wX, Y) + g(X, Y), \tag{4.12}$$

for any $X, Y \in \Gamma(TM)$.

Example 4.1. Let $(\tilde{M} = \mathbb{R}^9_3, \tilde{g})$ be a 7−dimensional semi-Euclidean space with signature $(-,-,+,+,-,+,+,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ be the standard coordinate system of $\mathbb{R}^9_3$. If we set $\tilde{P}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = (\phi x_1, \phi x_2, \phi x_3, (1-\phi)x_4, (1-\phi)x_5, (1-\phi)x_6, (1-\phi)x_7, (1-\phi)x_8, \phi x_9)$, then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\tilde{M}$. Suppose M is a submanifold of $\tilde{M}$ defined by

$$\begin{aligned} x_1 &= u_1 + u_2, x_2 = u_1 - u_2, x_3 = u_1, x_4 = \phi u_1, \\ x_5 &= \sqrt{2}\phi u_1, x_6 = -\phi u_2, x_7 = \phi u_2, x_8 = \phi u_3, x_9 = u_3 \end{aligned}$$

where u_i, $1 \le i \le 3$, are real parameters. Thus $TM = Span\{U_1, U_2, U_3\}$, where

$$\begin{aligned} U_1 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \phi\frac{\partial}{\partial x_4} + \sqrt{2}\phi\frac{\partial}{\partial x_5}, \\ U_2 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \phi\frac{\partial}{\partial x_6} + \phi\frac{\partial}{\partial x_7}, \\ U_3 &= \phi\frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_9}. \end{aligned}$$

Then M is a 1−lightlike submanifold. We have $Rad(TM) = Span\{U_1\}$ and $S(TM) = Span\{U_2, U_3\}$. Lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = -\frac{1}{3(\phi+2)}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \phi\frac{\partial}{\partial x_4} + \sqrt{2}\phi\frac{\partial}{\partial x_5}\right).$$

Also, screen transversal bundle $S(TM^\perp)$ is spanned by

$$\begin{aligned} W_1 &= \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7}, W_2 = \phi\frac{\partial}{\partial x_1} - \phi\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7}, \\ W_3 &= -\frac{\partial}{\partial x_8} + \phi\frac{\partial}{\partial x_9}, W_4 = \phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} + \phi\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} - \sqrt{2}\frac{\partial}{\partial x_5}, \\ W_5 &= -\frac{1}{3(2+\phi)}\left(\phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} - \phi\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} - \sqrt{2}\frac{\partial}{\partial x_5}\right). \end{aligned}$$

It is easy to see that $\tilde{P}U_1 = W_4$, $\tilde{P}U_2 = W_2$, $\tilde{P}U_3 = W_3$, $\tilde{P}N = W_5$ and $\tilde{P}W_1 = \phi W_1$. Thus we have $\tilde{P}(S(TM)) \subset S(TM^\perp)$, $\tilde{P}(Rad(TM)) \subset S(TM^\perp)$ and $\tilde{P}(ltr(TM)) \subset S(TM^\perp)$. Then M is a screen transversal anti-invariant lightlike submanifold of $\tilde{M}$.

Proposition 4.2. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then w is golden structure on TM.*

Proof. From (4.12), we have

$$g(wX, wY) = g(wX, Y) + g(X, Y),$$

for any $X, Y \in \Gamma(TM)$, which completes the proof. □

Proposition 4.3. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then C is golden structure on $ltr(TM)$.*

Proof. By the definition of screen transversal anti-invariant lightlike submanifold we have $BN = 0$, for any $N \in \Gamma(ltr(TM))$. From (4.10) we have $C^2N = CN + N$. Thus C is golden structure on $ltr(TM)$. □

In the similar way, we have the following.

Proposition 4.4. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then C is golden structure on D_0.*

Theorem 4.1. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the screen distribution is integrable iff*

$$\nabla^s_X \tilde{P}Y = \nabla^s_Y \tilde{P}X, \tag{4.13}$$

for any $X, Y \in \Gamma(S(TM))$ [9].

Theorem 4.2. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the radical distribution is integrable iff*

$$\nabla^s_{\xi_1} \tilde{P}\xi_2 = \nabla^s_{\xi_2} \tilde{P}\xi_1, \tag{4.14}$$

for any $\xi_1, \xi_2 \in \Gamma(Rad(TM))$ [9].

Theorem 4.3. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the screen distribution is parallel iff*

$$\tilde{g}(\nabla^s_X \tilde{P}Y, \tilde{P}N) = \tilde{g}(h^s(X, Y), \tilde{P}N), \tag{4.15}$$

for any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. Using the definition of screen transversal anti-invariant lightlike submanifold, $S(TM)$ is parallel iff $g(\nabla_X Y, N) = 0$, for any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. From (2.3), (2.11) and (2.13), we obtain

$$\begin{aligned} g(\nabla_X Y, N) &= \tilde{g}(\tilde{\nabla}_X Y, N) = \tilde{g}(\tilde{\nabla}_X \tilde{P}Y, \tilde{P}N) - \tilde{g}(\tilde{\nabla}_X Y, \tilde{P}N) \\ &= \tilde{g}(\nabla^s_X \tilde{P}Y, \tilde{P}N) - \tilde{g}(h^s(X, Y), \tilde{P}N), \end{aligned} \tag{4.16}$$

which completes the proof. □

Theorem 4.4. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the radical distribution is parallel iff*

$$\tilde{g}(\nabla^s_{\xi_1} \tilde{P}\xi_2, \tilde{P}X) = \tilde{g}(h^s(\xi_1, \xi_2), \tilde{P}X), \tag{4.17}$$

for any $X \in \Gamma(S(TM))$ and $\xi_1, \xi_2 \in \Gamma(Rad(TM))$.

Proof. Using the definition of screen transversal anti-invariant lightlike submanifold $Rad(TM)$ is parallel iff $g(\nabla_{\xi_1} \xi_2, X) = 0$ for any $X \in \Gamma(S(TM))$ and $\xi_1, \xi_2 \in \Gamma(Rad(TM))$. From (2.3), (2.11) and (2.13), we get

$$\begin{aligned} g(\nabla_{\xi_1} \xi_2, X) &= \tilde{g}(\tilde{\nabla}_{\xi_1} \xi_2, X) = \tilde{g}(\tilde{\nabla}_{\xi_1} \tilde{P}\xi_2, \tilde{P}X) - \tilde{g}(\tilde{\nabla}_{\xi_1} \xi_2, \tilde{P}X) \\ &= \tilde{g}(\nabla^s_{\xi_1} \tilde{P}\xi_2, \tilde{P}X) - \tilde{g}(h^s(\xi_1, \xi_2), \tilde{P}X), \end{aligned} \tag{4.18}$$

which completes the proof. □

Taking into account that M is a totally umbilicial in Theorem 4.4 we get following theorem.

Theorem 4.5. *Let M be a totally umbilical screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the radical distribution is parallel iff $\nabla^s_{\xi_1} \tilde{P}\xi_2$ has no compenents in $\tilde{P}(S(TM))$, for any $\xi_1, \xi_2 \in \Gamma(Rad(TM))$.*

Now, we give different form of theorem given in [9].

Theorem 4.6. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the induced connection ∇ on M is a metric connection iff $B_1 \nabla^s_X \tilde{P}\xi = B_1 h^s(X, \tilde{P}\xi)$, for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.*

Proof. Since $\tilde{\nabla}\tilde{P} = 0$, we have

$$\tilde{\nabla}_X \tilde{P}\xi = \tilde{P}\tilde{\nabla}_X \xi, \tag{4.19}$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Applying $\tilde{P}$ in this equation and using (2.1), we get

$$\tilde{P}\tilde{\nabla}_X \tilde{P}\xi = \tilde{P}\tilde{\nabla}_X \xi + \tilde{\nabla}_X \xi. \tag{4.20}$$

From (2.11), (2.13), (4.5) and (4.20), we have

$$
\begin{aligned}
& -\tilde{P}A_{\tilde{P}\xi}X + B_1\nabla^s_X\tilde{P}\xi + B_2\nabla^s_X\tilde{P}\xi + C_1\tilde{\nabla}^s_X\tilde{P}\xi + C_2\nabla^s_X\tilde{P}\xi \\
& +C^l_3\nabla^s_X\tilde{P}\xi + C^s_3\nabla^s_X\tilde{P}\xi + C_4\nabla^s_X\tilde{P}\xi + \tilde{P}h^l(X,\tilde{P}\xi) \\
= \; & \tilde{P}\nabla_X\xi + \tilde{P}h^l(X,\xi) + B_1h^s(X,\xi) + B_2h^s(X,\xi) + C_1h^s(X,\xi) + C_2h^s(X,\xi) \\
& +C^l_3h^s(X,\xi) + C^s_3h^s(X,\xi) + C_4h^s(X,\xi) + \nabla_X\xi + h^l(X,\xi) + h^s(X,\xi).
\end{aligned} \tag{4.21}
$$

Then, taking the tangential parts of (4.21), we derive

$$
\nabla_X\xi = B_1\nabla^s_X\tilde{P}\xi + B_2\nabla^s_X\tilde{P}\xi - B_1h^s(X,\xi) - B_2h^s(X,\xi). \tag{4.22}
$$

Considering Theorem 2.2, the equation (4.22) completes the proof. □

Taking into account that M is a totally umbilicial in Theorem 4.6 we get following theorem.

Theorem 4.7. *Let M be a totally umbilicial screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then the induced connection ∇ on M is a metric connection iff $\nabla^s_X\tilde{P}\xi$ has no component in $\tilde{P}(S(TM))$, for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.*

Theorem 4.8. *Let M be a totally umbilicial screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then $H^l = 0$ iff $\nabla^s_X\tilde{P}X$ has no component in $\tilde{P}(ltr(TM))$, for any $X \in \Gamma(S(TM))$.*

Proof. Using (2.3) and (2.11) and taking into account that M is a totally umbilicial screen transversal anti-invariant lightlike submanifold of $\tilde{M}$, we get

$$
\begin{aligned}
g(\nabla^s_X\tilde{P}X,\tilde{P}\xi) &= \tilde{g}(\tilde{\nabla}_X\tilde{P}X,\tilde{P}\xi) = \tilde{g}(\tilde{\nabla}_X\tilde{P}X,\xi) + \tilde{g}(\tilde{\nabla}_XX,\xi) \\
&= \tilde{g}(h^l(X,\tilde{P}X),\xi) + \tilde{g}(h^l(X,X),\xi) \\
&= \tilde{g}(X,\tilde{P}X)\tilde{g}(H^l,\xi) + \tilde{g}(X,X)\tilde{g}(H^l,\xi) \\
&= \tilde{g}(X,X)\tilde{g}(H^l,\xi),
\end{aligned}
$$

for any $X \in \Gamma(S(TM))$ and $\xi \in \Gamma(Rad(TM))$, which completes the proof. □

Theorem 4.9. *Let M be a totally umbilicial screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then H^s has no component in $\tilde{P}(S(TM))$ or $dim(S(TM)) = 1$.*

Proof. Using (2.2) and (2.11) and taking into account that $\tilde{\nabla}$ is a metric connection, we derive

$$
\tilde{g}(\tilde{\nabla}_X\tilde{P}X,Y) = \tilde{g}(\tilde{\nabla}_XX,\tilde{P}Y) = \tilde{g}(h^s(X,X),\tilde{P}Y), \tag{4.23}
$$

$$
\tilde{g}(\tilde{\nabla}_X\tilde{P}X,Y) = -\tilde{g}(\tilde{P}X,\tilde{\nabla}_XY) = -\tilde{g}(\tilde{P}X,h^s(X,Y)), \tag{4.24}
$$

for any $X, Y \in \Gamma(S(TM))$. Combining (4.23) and (4.24), we obtain

$$\tilde{g}(h^s(X,X), \tilde{P}Y) = -\ \tilde{g}(\tilde{P}X, h^s(X,Y)). \tag{4.25}$$

Using (2.23) in equation (4.25), we get

$$g(X,X)\tilde{g}(H^s, \tilde{P}Y) = -g(X,Y)\ \tilde{g}(H^s, \tilde{P}X). \tag{4.26}$$

Interchanging X and Y in (4.26) and rearranging the terms, we derive

$$\tilde{g}(H^s, \tilde{P}X) = -\frac{g(X,Y)}{g(Y,Y)}\ \tilde{g}(H^s, \tilde{P}Y). \tag{4.27}$$

From (4.26) and (4.27), we conclude that

$$\tilde{g}(H^s, \tilde{P}X) = \frac{g(X,Y)^2}{g(X,X)g(Y,Y)}\ \tilde{g}(H^s, \tilde{P}X). \tag{4.28}$$

This completes the proof. □

Theorem 4.10. *Let M be a totally umbilicial screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then H^s has no component in $\tilde{P}(ltr(TM))$.*

Proof. From (2.2), (2.11) and (2.23), we get

$$\begin{aligned}\tilde{g}(D^l(X, \tilde{P}Y), \xi) &= \tilde{g}(\tilde{\nabla}_X \tilde{P}Y, \xi) = \tilde{g}(\tilde{\nabla}_X Y, \tilde{P}\xi)\\ &= \tilde{g}(h^s(X,Y), \tilde{P}\xi) = g(X,Y)\ \tilde{g}(H^s, \tilde{P}\xi),\end{aligned}$$

for any $X, Y \in \Gamma(S(TM))$, which completes the proof. □

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