

INEQUALITIES FOR GRADIENT EINSTEIN AND RICCI SOLITONS

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Abstract. This short note concerns with two inequalities in the geometry of gradient Einstein solitons (g, f, λ) on a smooth manifold M . These inequalities provide some relationships between the curvature of the Riemannian metric g and the behavior of the scalar field f through two quadratic equations satisfied by the scalar λ . The similarity with gradient Ricci solitons and a slight generalization involving a g -symmetric endomorphism A are provided.

Keywords: gradient Einstein solitons; smooth manifold; Riemannian metric; g -symmetric endomorphism.

1. Introduction

Let (M^n, g) be an n -dimensional Riemannian manifold endowed with a smooth function $f \in C^\infty(M)$; an excellent textbook in Riemannian geometry is [6]. The scalar field f yields the *Hessian endomorphism*: $h_f : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, $h_f(X) = \nabla_X \nabla f$, where ∇ is the Levi-Civita connection of g . Then we know the symmetry of the *Hessian tensor field* of f : $H_f(X, Y) := g(h_f(X), Y)$, namely $H_f(X, Y) = H_f(Y, X)$. What follows is the existence of a g -orthonormal frame field $E = \{E_i\}_{i=1, \dots, n} \subset \mathfrak{X}(M)$ and the existence of the eigenvalues $\lambda = \{\lambda_i\}_{i=1, \dots, n} \subset C^\infty(M)$:

$$(1.1) \quad h_f(E_i) = \lambda_i E_i.$$

Hence we express all the geometric objects related to f in terms of the pair (E, λ) which we call the *spectral data* of f :

$$(1.2) \quad \nabla f = \sum_{i=1}^n E_i(f) E_i, \quad \|\nabla f\|_g^2 = \sum_{i=1}^n [E_i(f)]^2, \quad h_f(X) = \sum_{i=1}^n (\lambda_i X^i) E_i,$$

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for $X = \sum_{i=1}^n X^i E_i$. Also the *Hessian* and the *Laplacian* of f are:

$$(1.3) \quad H_f(X, Y) = \sum_{i=1}^n \lambda_i (X^i Y^i), \quad \Delta f := \text{Tr}_g H_f = \sum_{i=1}^n \lambda_i.$$

Let us remark that if ∇f does not have zeros and E_1 is exactly its unit vector field i.e. $E_1 = \frac{\nabla f}{\|\nabla f\|_g}$, then ∇f is a *geodesic vector field*: $\nabla_{\nabla f} \nabla f = \lambda_1 \nabla f$ which means that the flow of ∇f consists in geodesics of g .

2. Results

Assume now that the triple $(g, f, \lambda \in \mathbb{R})$ is a *gradient Einstein soliton* on M , [2, p. 67]:

$$(2.1) \quad H_f + \text{Ric} + \left(\lambda - \frac{R}{2} \right) g = 0,$$

where Ric is the Ricci tensor field of g and R is the scalar curvature. Einstein solitons generate self-similar solutions of the Einstein flow (1.1) of [2] and are more rigid than the well-known Ricci solitons. By considering the Ricci endomorphism $Q \in \mathcal{T}_1^1(M)$ provided by:

$$(2.2) \quad \text{Ric}(X, Y) = g(QX, Y),$$

we can express (2.1) as:

$$(2.3) \quad h_f + Q + \left(\lambda - \frac{R}{2} \right) I = 0$$

with I the Kronecker endomorphism. From (2.3) we get that Q is also of diagonal form with respect to the frame E :

$$(2.4) \quad Q(X) = - \sum_{i=1}^n \left(\lambda_i + \lambda - \frac{R}{2} \right) X^i E_i, \quad \|Q\|_g^2 = \sum_{i=1}^n \left(\lambda_i + \lambda - \frac{R}{2} \right)^2.$$

By developing the second formula above we derive:

$$(2.5) \quad \begin{aligned} \| \text{Ric} \|_g^2 &= \sum_{i=1}^n \lambda_i^2 + (2\lambda - R) \sum_{i=1}^n \lambda_i + n \left(\lambda^2 - \lambda R + \frac{R^2}{4} \right) = \\ &= \| H_f \|_g^2 + (2\lambda - R) \Delta f + n \left(\lambda^2 - \lambda R + \frac{R^2}{4} \right). \end{aligned}$$

Hence the scalar λ is a solution of the quadratic equation:

$$(2.6) \quad n\lambda^2 + 2 \left(\Delta f - \frac{nR}{2} \right) \lambda + \left(\| H_f \|_g^2 - \| \text{Ric} \|_g^2 + \frac{nR^2}{4} - R\Delta f \right) = 0$$

which means the non-negativity:

$$(2.7) \quad 0 \leq \Delta' := \left(\Delta f - \frac{nR}{2} \right)^2 - n \left(\| H_f \|_g^2 - \| \text{Ric} \|_g^2 + \frac{nR^2}{4} - R\Delta f \right).$$

It follows a lower boundary of the geometry of g in terms of f :

$$(2.8) \quad \|Ric\|_g^2 \geq \|H_f\|_g^2 - \frac{1}{n}(\Delta f)^2.$$

An "exotic" consequence is provided by the case of strict inequality in (2.7), more precisely, it follows that the data (g, f, λ) is doubled by $(g, f, \frac{2\Delta f}{n} - R - \lambda = -\frac{2}{n}R - \lambda)$.

Example 1 i) (*Gaussian soliton*) We have $(M = \mathbb{R}^n, g_{can})$ and $f(x) = -\frac{\lambda}{2}\|x\|^2$. It results $h_f = -\lambda I_n$ and $\Delta f = -n\lambda$. Since $\|H_f\|^2 = n\lambda^2$, the left hand side of (2.6) is:

$$n\lambda^2 + 2 \left(\Delta f - \frac{nR}{2} \right) \lambda + \left(\|H_f\|_g^2 - \|Ric\|_g^2 + \frac{nR^2}{4} - R\Delta f \right) = n\lambda^2 + 2(-n\lambda)\lambda + n\lambda^2$$

which is exactly zero. Also: $\Delta' = (n\lambda)^2 - n(n\lambda^2 - 0) = 0$ which means the uniqueness of λ and the equality case in (2.8): $0 = n\lambda^2 - \frac{(n\lambda)^2}{n}$.

ii) A generalization of the previous example is provided on a Ricci-flat manifold by a smooth function f satisfying a generalization of Hessian structures:

$$(2.9) \quad H_f = -\lambda g.$$

Then $\Delta f = -n\lambda$ and $\|H_f\|^2 = n\lambda^2$ exactly as for the Gaussian soliton. Using Lemma 4.1. of [3, p. 1540] it results from (2.9) that ∇f is a particular *concircular vector field*: $h_f = -\lambda I$; hence $\lambda_1 = \dots = \lambda_n = -\lambda$ is the spectral part of the spectral data of f . If ∇f is without zeros it follows from Theorem 3.1. of [3, p. 1539] that (M, g) is locally a warped product manifold with a 1-dimensional basis: $(M, g) = (I \subseteq \mathbb{R}, g_{can}) \times_{\varphi} (F^{n-1}, g_F)$. In fact, $\nabla f = \varphi(s) \frac{\partial}{\partial s}$ with $\varphi'(s) = -\lambda$ which means an affine warping function, $\varphi(s) = -\lambda s + C$. \square

A new quadratic equation, similar to (2.6), follows from:

$$(2.10) \quad \Delta f + \left(1 - \frac{n}{2}\right) R + n\lambda = 0$$

obtained by tracing (2.1). Hence the companion equation of (2.6) is:

$$(2.11) \quad n\lambda^2 + 2 \left(1 - \frac{n}{2}\right) R\lambda + \left(\|Ric\|_g^2 - \|H_f\|_g^2 + \frac{n-4}{4} R^2 \right) = 0.$$

The new inequality is then:

$$(2.12) \quad 0 \leq \Delta' := \left(1 - \frac{n}{2}\right)^2 R^2 - n \left(\|Ric\|_g^2 - \|H_f\|_g^2 + \frac{n-4}{4} R^2 \right)$$

and it results a lower boundary of the behavior of f in terms of the geometry of g :

$$(2.13) \quad \|H_f\|_g^2 \geq \|Ric\|_g^2 - \frac{R^2}{n} = \frac{1}{n} \sum_{i \neq j} (\lambda_i - \lambda_j)^2.$$

We remark that (2.8) and (2.13) can be unified in the double inequality:

$$(2.14) \quad \|H_f\|_g^2 - \frac{1}{n}(\Delta f)^2 \leq \|Ric\|_g^2 \leq \|H_f\|_g^2 + \frac{R^2}{n}$$

and the simultaneous equalities for $n \geq 3$ hold if and only if $R = \Delta f = 0 = \lambda$ and $H_f = -Ric$; hence f is a harmonic map on a steady gradient Einstein soliton. The vanishing of the right-hand side of (2.13) means that g is an Einstein metric; other interesting aspects concerning the functional $F_g := \frac{R^2}{\|Ric\|_g^2}$ on the space of non-flat metrics appear in [5]. This raises the first future problem to study the similar functional $F_f^g := \frac{(\Delta f)^2}{\|H_f\|_g^2}$ on the space of smooth functions which are not *linear on* M after the name from [6, p. 283]. Remark that for the Hessian structures (2.9) we have a constant and maximal $F_f^g = n$.

Example 1 revisited i) (*Gaussian soliton*) The inequality (2.13) becomes $n\lambda^2 \geq 0$.

ii) Again, (2.13) means $n\lambda^2 \geq 0$.

iii) (*relationship with gradient Ricci solitons*) If $R = 0$, then the gradient Einstein soliton becomes a gradient Ricci soliton and we remark that (2.14) is exactly the double inequality (20) of [4, p. 3339]. The explication of this fact is provided by the following remark. \square

Remark An unified proof of the double inequality (2.14) is provided by the following relation satisfied by an Einstein soliton, which is a direct consequence of the equations (2.5) and (2.10):

$$(2.15) \quad n(\|H_f\|_g^2 - \|Ric\|_g^2) = (\Delta f)^2 - R^2$$

and it is important to point out that this equation does not involves the scalar λ . In other words, (2.15) is a universal formula of the gradient Einstein solitons. With $\lambda \rightarrow \lambda + \frac{R}{2}$ we get that (2.15) holds also for gradient Ricci solitons and hence we obtain the similarity between gradient Ricci and Einstein solitons with respect to (2.14). \square

Returning to (2.3) we remark that the Ricci endomorphism Q commutes with h_f for an Einstein or Ricci gradient soliton. It results the commuting property also for the Einstein endomorphism:

$$(2.16) \quad Einst_g := Q - \frac{R}{n}I$$

which is the trace-free part of Q . We will assume now that the data $(g, f, \lambda, \mu \in \mathbb{R})$ satisfies:

$$(2.17) \quad h_f + Q + \lambda I + \mu Einst_g = 0.$$

The corresponding relation in terms of Ricci endomorphism is:

$$(2.18) \quad h_f + (1 + \mu)Q + \left(\lambda - \frac{\mu R}{n}\right)I = 0$$

or, for $\mu \neq -1$:

$$(2.19) \quad h_{\frac{f}{1+\mu}} + Q + \left(\frac{\lambda}{1+\mu} - \frac{\mu R}{n(1+\mu)} \right) I = 0.$$

This last equation is an example of ρ -Einstein soliton as is introduced in Definition 1.1 of [2, p. 67] with $\rho = \frac{\mu}{n(1+\mu)}$ and (f, λ) of [2] replaced by $\frac{1}{1+\mu}(f, \lambda)$.

Hence we naturally arrive to the following slight generalization of all the above considerations. Fix a g -symmetric endomorphism $A \in \mathcal{T}_1^1(M)$ which is also diagonal with respect to the frame E :

$$(2.20) \quad A(E_i) = \rho_i E_i, \quad \rho_i \in C^\infty(M).$$

Hence A and h_f commutes: $A \circ h_f = h_f \circ A$. We introduce:

Definition The data $(g, f, \lambda, \mu \in \mathbb{R})$ is an *A-Ricci gradient soliton* if:

$$(2.21) \quad h_f + Q + \lambda I + \mu A = 0.$$

We get that A commutes also with Q and the corresponding generalization of (2.15) is:

$$(2.22) \quad n \left[\|H_f\|_g^2 - \|Ric\|_g^2 + \mu^2 \|A\|_g^2 + 2\mu Tr_g(h_f \circ A) \right] = (\Delta f + \mu Tr_g A)^2 - R^2$$

yielding the double inequality:

$$(2.23) \quad \begin{aligned} & \|H_f\|_g^2 - \frac{1}{n} (\Delta f + \mu Tr_g A)^2 + \mu^2 \|A\|_g^2 + 2\mu Tr_g(h_f \circ A) \leq \|Ric\|_g^2 \leq \\ & \leq \|H_f\|_g^2 + \frac{R^2}{n} + \mu^2 \|A\|_g^2 + 2\mu Tr_g(h_f \circ A). \end{aligned}$$

There is another problem: to find remarkable endomorphisms commuting with a given h_f . We will finish this note with an example.

Example 2 Suppose that (M, g) is a hypersurface in (N^{n+1}, g) and let $A = S$ be the shape endomorphism of M commuting with h_f for the fixed scalar field $f \in C^\infty(M)$. If $(g, f, \lambda, \mu \in \mathbb{R})$ is a *shape-Ricci gradient soliton* on M i.e. (2.21) holds for S , then denoting by H the mean curvature of M , we get:

$$(2.24) \quad \begin{aligned} & \|H_f\|_g^2 - \frac{1}{n} (\Delta f + \mu H)^2 + \mu^2 \|S\|_g^2 + 2\mu Tr_g(h_f \circ S) \leq \|Ric\|_g^2 \leq \\ & \leq \|H_f\|_g^2 + \frac{R^2}{n} + \mu^2 \|S\|_g^2 + 2\mu Tr_g(h_f \circ S). \end{aligned}$$

We point out that immersions of (almost) Ricci solitons into another Riemannian manifold are studied in [1]. \square

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