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SOME RANDOM FIXED POINT RESULTS USING IMPLICIT RELATION IN HILBERT SPACES

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Abstract. In this paper, we obtain a common random fixed point theorem for six random operators satisfying an implicit relation and defined on a nonempty closed subset of a separable Hilbert space. Our results extend and generalize the corresponding results in the lectures.

1. Introduction

The theory of random operator is an important branch of probabilistic analysis which plays a key role in many applied areas. The study of random fixed points forms a central topic in this area. Random fixed point theorems for contraction mappings on separable complete metric spaces have been proved by several authors ([2]- [7], [9]-[11], [16], and [20]) and many others.

Beg et al. [2], [5], [6] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators and proved the random fixed points theorems for contraction random operators in Polish spaces. Some random fixed point theorems for weakly compatible random operators under generalized contractive conditions are proved by Beg [3], Beg and Abbas [4] and others.

In 1999, Popa [17] proved some fixed point theorems for compatible mappings satisfying an implicit relation. Many authors deal with establishment of some common fixed point results for compatible and weakly compatible mappings in different spaces (see for instance [1], [12], [13], [15], [18], [19] and many others. In [8], Dhagat et al. deal with some fixed point theorems for two random operators in Hilbert spaces by considering sequences of measurable functions satisfying implicit conditions.

In continuation of these results, we extend the contraction condition via implicit relation by Dhagat et al. [8] to six random mappings. We obtain a common random fixed point for six mappings by using implicit relation on a nonempty closed subset C of a separable Hilbert space H.

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2. preliminaries

Let (Ω, Σ) be a measurable space, H stands for a separable Hilbert space and C a nonempty closed subset of H.
A mapping $\xi : \Omega \to C$ is called measurable if $\xi^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of H.
A mapping $T : \Omega \times C \to C$ is said to be random mapping if for each fixed $x \in C$, the mapping $T(., x) : \Omega \to C$ is measurable.
A measurable mapping $\xi : \Omega \to C$ is called a random fixed point of the random mapping $T : \Omega \times C \to C$ if $T(w, \xi(w)) = \xi(w)$ for each $w \in \Omega$.

Definition 2.1. [12, 14] Let H be a separable Hilbert space. Random operators $S, T : \Omega \times H \to H$ are

1. Compatibles for each $w \in \Omega$ if,

$$\lim_{n\to\infty} \|S(w, T(w, \xi_n(w))) - T(w, S(w, \xi_n(w)))\| = 0$$

provided that $\lim_{n\to\infty} S(w, \xi_n(w))$ and $\lim_{n\to\infty} T(w, \xi_n(w))$ exist in H and

$$\lim_{n\to\infty} S(w, \xi_n(w)) = \lim_{n\to\infty} T(w, \xi_n(w)), w \in \Omega,$$

where ξ_n is a sequences of measurable mappings.

2. Weakly compatible if $T(w, \xi(w)) = S(w, \xi(w))$, for some measurable mappings ξ, then $T(w, S(w, \xi(w))) = S(w, T(w, \xi(w)))$ for every $w \in \Omega$.

Definition 2.2. Implicit Relation. Let Φ be the class of real valued continuous functions $\phi : (\mathbb{R}^+)^3 \to \mathbb{R}^+$ nondecreasing in the second argument and satisfying the following conditions:

$$x \le \phi(y, x + y, x) \tag{2.1}$$

or

$$x \le \phi(y, x + y, \frac{1}{2}(x + y)) \tag{2.2}$$

or

$$x \le \phi(y, x + y, x + y), \tag{2.3}$$

then there exists a real number $0 < k < 1$ such that $x \le ky$, for all $x, y \ge 0$.

Condition (A) Six random mappings E, F, S, T, A and $B : \Omega \times C \to C$, where C is a nonempty closed subset of a separable Hilbert space H are said to satisfy condition

A if
$\|E(w,x(w)) - F(w,y(w))\|^2 \leq$

$$\phi\left(\begin{array}{c} \|S(w,A(w,x(w))) - T(w,B(w,y(w)))\|^2, \\ \{\|S(w,A(w,x(w))) - E(w,x(w))\|^2 + \|T(w,B(w,y(w))) - F(w,y(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,x(w))) - F(w,y(w))\|^2 + \|T(w,B(w,y(w))) - E(w,x(w))\|^2\} \end{array}\right),$$

for $x, y \in H$ and $w \in \Omega$.

3. Main results

In this section, we prove a common random fixed point theorem for six random operators using implicit relation in separable Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed subset of a separable Hilbert space H. Let $E, F, S, , T, A$ and $B : \Omega \times C \to C$ be six random mappings defined on C such that for $w \in \Omega$, E, F, S, T, A and B satisfy condition (**A**) and the following conditions:*

$$E(w,H) \subset T(w,B(w,H)) \text{ and } F(w,H) \subset S(w,A(w,H)). \tag{3.1}$$

$$EA = AE, SA = AS, BF = FB \text{ and } TB = BT, \tag{3.2}$$

then E, F, S, T, A and B have a unique common random fixed point if one of the following conditions is satisfied:

1. *Either E or SA is continuous, the pair (E, SA) is compatible and the pair (F, TB) is weakly compatible.*
2. *Either F or TB is continuous, the pair (F, TB) is compatible and the pair (E, SA) is weakly compatible.*

Proof. Let the function $g_0 : \Omega \to C$ be an arbitrary measurable function on Ω. By (3.1) there exists a function $g_1 : \Omega \to C$ such that for $w \in \Omega$, $T(w,B(w,g_1(w))) = E(w,g_0(w))$ and for this function $g_1 : \Omega \to C$ we can choose another function $g_2 : \Omega \to C$ such that for $w \in \Omega$, $F(w,g_1(w)) = S(w,A(w,g_2(w)))$ and so on. By using the method of induction we can define a sequence of functions $\{y_n(w)\}$, $w \in \Omega$ as follows:

$$\begin{aligned} y_{2n+1}(w) &= T(w,B(w,g_{2n+1}(w))) = E(w,g_{2n}(w)), \\ y_{2n+2}(w) &= S(w,A(w,g_{2n+2}(w))) = F(w,g_{2n+1}(w)), w \in \Omega, n = 0,1,2,.... \end{aligned} \tag{3.3}$$

From condition (**A**) we have for $w \in \Omega$
$\|y_{2n+1}(w) - y_{2n}(w)\|^2 = \|E(w,g_{2n}(w)) - F(w,g_{2n-1}(w))\|^2$

$$\leq \phi\left(\begin{array}{c} \|S(w,A(w,g_{2n}(w)))-T(w,B(w,g_{2n-1}(w)))\|^2, \\ \{\|S(w,A(w,g_{2n}(w)))-E(w,g_{2n}(w))\|^2+\|T(w,B(w,g_{2n-1}(w)))-F(w,g_{2n-1}(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,g_{2n}(w)))-F(w,g_{2n-1}(w))\|^2+\|T(w,B(w,g_{2n-1}(w)))-E(w,g_{2n}(w))\|^2\} \end{array}\right).$$

It follows by (3.3) that
$\|y_{2n+1}(w) - y_{2n}(w)\|^2 \leq$

$$\phi\begin{pmatrix} \|y_{2n}(w) - y_{2n-1}(w)\|^2, \\ \{\|y_{2n}(w) - y_{2n+1}(w)\|^2 + \|y_{2n-1}(w) - y_{2n}(w\|^2\}, \\ \frac{1}{2}\{\|y_{2n}(w) - y_{2n}(w)\|^2 + \|y_{2n-1}(w) - y_{2n+1}(w\|^2\} \end{pmatrix}. \tag{3.4}$$

By a parallelogram law, $\|x + y\|^2 + \|x - y\|^2 \leq 2[\|x\|^2 + \|y\|^2]$, which implies that $\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - \|x - y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$, we can write

$$\begin{aligned} \|y_{2n-1}(w) - y_{2n+1}(w)\|^2 &= \|y_{2n-1}(w) - y_{2n}(w) + y_{2n}(w) - y_{2n+1}(w)\|^2 \\ &\leq 2\|y_{2n-1}(w) - y_{2n}(w)\|^2 + 2\|y_{2n}(w) - y_{2n+1}(w)\|^2. \end{aligned} \tag{3.5}$$

Applying (3.5) in (3.4) we get
$\|y_{2n+1}(w) - y_{2n}(w)\|^2 \leq$

$$\phi\begin{pmatrix} \|y_{2n}(w) - y_{2n-1}(w)\|^2, \\ \{\|y_{2n}(w) - y_{2n+1}(w)\|^2 + \|y_{2n-1}(w) - y_{2n}(w)\|^2\}, \\ \frac{1}{2}\{2\|y_{2n-1}(w) - y_{2n}(w)\|^2 + 2\|y_{2n}(w) - y_{2n+1}(w\|^2\} \end{pmatrix},$$

then by (2.3) we have

$$\|y_{2n+1}(w) - y_{2n}(w)\|^2 \leq k\|y_{2n}(w) - y_{2n-1}(w)\|^2.$$

Similarly, we can prove that

$$\|y_{2n}(w) - y_{2n-1}(w)\|^2 \leq k\|y_{2n-1}(w) - y_{2n-2}(w)\|^2.$$

Hence, in general,

$$\|y_n(w) - y_{n+1}(w)\| \leq k\|y_{n-1}(w) - y_n(w)\|.$$

Since $0 < k < 1$, then $\{y_n(w)\}$ is a Cauchy sequence and hence is convergent in the closed subset C of H. So that, $\{y_n(w)\} \to \{y(w)\}$ as $n \to \infty$ for $w \in \Omega$. Since C is closed, $\{y(w)\}$ is a function from C to C and consequently the subsequences of $\{y_n(w)\}$ also converge to $\{y(w)\}$ i. e.

$$\begin{aligned} y_{2n+1}(w) &= E(w, g_{2n}(w)) = T(w, B(w, g_{2n+1}(w))) \to y(w) \text{ as } (n \to \infty) \\ y_{2n}(w) &= F(w, g_{2n-1}(w)) = S(w, A(w, g_{2n}(w))) \to y(w) \text{ as } (n \to \infty), w \in \Omega. \end{aligned} \tag{3.6}$$

Now, since (E, SA) is compatible, then by (3.6) we obtain

$$\|E(w, S(w, A(w, g_{2n}(w)))) - S(w, A(w, E(w, g_{2n}(w))))\| \to 0 \text{ as } (n \to \infty), w \in \Omega. \tag{3.7}$$

Suppose SA is continuous, it follows that

$$S(w, A(w, E(w, g_{2n}(w)))) \to S(w, A(w, y(w))).$$

Then by (3.7), we have

$$E(w,S(w,A(w,g_{2n}(w)))) \to S(w,A(w,y(w))), w \in \Omega. \tag{3.8}$$

Using condition (**A**) we obtain
$\|E(w,S(w,A(w,g_{2n}(w)))) - F(w,g_{2n-1}(w))\|^2 \le$

$$\phi\left(\begin{array}{c} \|S(w,A(w,S(w,A(w,g_{2n}(w)))))-T(w,B(w,g_{2n-1}(w)))\|^2, \\ \{\|S(w,A(w,S(w,A(w,g_{2n}(w)))))-E(w,S(w,A(w,g_{2n}(w))))\|^2+\|T(w,B(w,g_{2n-1}(w)))-F(w,g_{2n-1}(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,S(w,A(w,g_{2n}(w)))))-F(w,g_{2n-1}(w))\|^2+\|T(w,B(w,g_{2n-1}(w)))-E(w,S(w,A(w,g_{2n}(w))))\|^2\} \end{array}\right).$$

Letting $(n \to \infty)$ and using (3.8), we obtain

$$\|S(w,A(w,y(w)))-y(w)\|^2 \le \phi\left(\begin{array}{c} \|S(w,A(w,y(w)))-y(w)\|^2, \\ \{\|S(w,A(w,y(w)))-S(w,A(w,y(w)))\|^2+\|y(w)-y(w)\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,y(w)))-y(w)\|^2+\|y(w)-S(w,A(w,y(w)))\|^2\} \end{array}\right),$$

using (2.1) (taking into a count the property of ϕ that is ϕ is nondecreasing in the second argument), we get

$$\|S(w,A(w,y(w)))-y(w)\|^2 \le k\|S(w,A(w,y(w)))-y(w)\|^2.$$

Hence,

$$S(w,A(w,y(w))) = y(w),\ \ w \in \Omega. \tag{3.9}$$

Again using condition (**A**),
$\|E(w,y(w)) - F(w,g_{2n-1}(w))\|^2 \le$

$$\phi\left(\begin{array}{c} \|S(w,A(w,y(w)))-T(w,B(w,g_{2n-1}(w)))\|^2, \\ \{\|S(w,A(w,y(w)))-E(w,y(w))\|^2+\|T(w,B(w,g_{2n-1}(w)))-F(w,g_{2n-1}(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,y(w)))-F(w,g_{2n-1}(w))\|^2+\|T(w,B(w,g_{2n-1}(w)))-E(w,y(w))\|^2\} \end{array}\right).$$

Letting $(n \to \infty)$ and using (3.9), we obtain

$$\|E(w,y(w))-y(w)\|^2 \quad \le \quad \phi\left(\begin{array}{c} \|y(w)-y(w)\|^2, \\ \{\|y(w)-E(w,y(w))\|^2+\|y(w)-y(w)\|^2\}, \\ \frac{1}{2}\{\|y(w)-y(w)\|^2+\|y(w)-E(w,y(w))\|^2\} \end{array}\right),$$

using (2.2), we get

$$E(w,y(w)) = y(w), w \in \Omega. \tag{3.10}$$

Both (3.9) and (3.10) give

$$E(w,y(w)) = S(w,A(w,y(w))) = y(w), w \in \Omega. \tag{3.11}$$

Now, since $y(w) = E(w,y(w)) \in E(w,H) \subset T(w,B(w,H))$, there exists $h(w) \in C$ such that

$$y(w) = T(w,B(w,h(w))) \quad for \quad w \in \Omega. \tag{3.12}$$

Using condition (**A**), we obtain

$$\|y(w)-F(w,h(w))\|^2 = \|E(w,y(w))-F(w,h(w))\|^2$$

$$\leq \phi\begin{pmatrix} \|S(w,A(w,y(w)))-T(w,B(w,h(w)))\|^2, \\ \{\|S(w,A(w,y(w)))-E(w,y(w))\|^2+\|T(w,B(w,h(w)))-F(w,h(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,y(w)))-F(w,h(w))\|^2+\|T(w,B(w,h(w)))-E(w,y(w))\|^2\} \end{pmatrix}.$$

From (3.11) and (3.12), we obtain

$$\|y(w)-F(w,h(w))\|^2 \leq \phi\begin{pmatrix} \|y(w)-y(w)\|^2, \\ \{\|y(w)-y(w)\|^2+\|y(w)-F(w,h(w))\|\}, \\ \frac{1}{2}\{\|y(w)-F(w,h(w))\|^2+\|y(w)-y(w)\|^2\} \end{pmatrix}.$$

By (2.2), we have

$$y(w) = F(w,h(w)) \quad for \quad w\in\Omega. \tag{3.13}$$

Both (3.12) and (3.13) imply

$$y(w) = F(w,h(w)) = T(w,B(w,h(w))) \quad for \quad w\in\Omega. \tag{3.14}$$

Since the pair $\{F, TB\}$ is weakly compatible, then they commute at their coincidence point $h(w)$, i.e.

$$\begin{aligned} F(w,T(w,B(w,h(w)))) &= T(w,B(w,F(w,h(w)))) \\ \Rightarrow F(w,y(w)) &= T(w,B(w,y(w))) \end{aligned} \tag{3.15}$$

From condition **(A)**,

$$\|y(w)-F(w,y(w))\|^2 = \|E(w,y(w))-F(w,y(w))\|^2$$

$$\leq \phi\begin{pmatrix} \|S(w,A(w,y(w)))-T(w,B(w,y(w)))\|^2, \\ \{\|S(w,A(w,y(w)))-E(w,y(w))\|^2+\|T(w,B(w,y(w)))-F(w,y(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,y(w)))-F(w,y(w))\|^2+\|T(w,B(w,y(w)))-E(w,y(w))\|^2\} \end{pmatrix}.$$

By (3.11) and (3.15),

$$\|y(w)-F(w,y(w))\|^2 = \|E(w,y(w))-F(w,y(w))\|^2$$

$$\leq \phi\begin{pmatrix} \|y(w)-F(w,y(w))\|^2, \\ \{\|y(w)-y(w)\|^2+\|F(w,y(w))-F(w,y(w))\|\}, \\ \frac{1}{2}\{\|y(w)-F(w,y(w))\|^2+\|F(w,y(w))-y(w)\|^2\} \end{pmatrix}.$$

Applying (2.1) and using (3.15), we get

$$y(w) = F(w,y(w)) = T(w,B(w,y(w))) \quad for \quad w\in\Omega. \tag{3.16}$$

From (3.11) and (3.16), we have

$$y(w) = E(w,y(w)) = F(w,y(w)) = S(w,A(w,y(w))) = T(w,B(w,y(w))), w\in\Omega,$$

that is, $y(w)$ is a common random fixed point of E, F, SA and TB.
Now, we need to prove that

$$y(w) = S(w, y(w)) = A(w, y(w)) = T(w, y(w)) = B(w, y(w)), w \in \Omega.$$

By $AE = EA$ and condition (**A**) we have
$\|A(w, y(w)) - y(w)\|^2 = \|A(w, E(w, y(w))) - F(w, y(w))\|^2$
$= \|E(w, A(w, y(w))) - F(w, y(w))\|^2$

$$\leq \phi \begin{pmatrix} \|S(w,A(w,A(w,y(w)))-T(w,B(w,y(w)))\|^2, \\ \{\|S(w,A(w,A(w,y(w)))-E(w,A(w,y(w))\|^2+\|T(w,B(w,y(w)))-F(w,y(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,A(w,y(w)))-F(w,y(w))\|^2+\|T(w,B(w,y(w)))-E(w,A(w,y(w))\|^2\} \end{pmatrix}. \tag{3.17}$$

Since $AE = EA$ and $SA = AS$ we get, $E(w, A(w, y(w))) = A(w, E(w, y(w))) = A(w, y(w))$ and $S(w, A(w, A(w, y(w)))) = A(w, S(w, A(w, y(w)))) = A(w, y(w))$. Applying this in (3.17) we obtain

$$\|A(w, y(w)) - y(w)\|^2 \leq \phi \begin{pmatrix} \|A(w, y(w)) - y(w)\|^2, \\ \{\|A(w, y(w)) - A(w, y(w))\|^2 + \|y(w) - y(w)\|^2\}, \\ \frac{1}{2}\{\|A(w, y(w)) - y(w)\|^2 + \|y(w) - A(w, y(w))\|^2\} \end{pmatrix}.$$

By using (2.1), we have

$$A(w, y(w)) = y(w), w \in \Omega. \tag{3.18}$$

Since $S(w, A(w, y(w))) = y(w)$ and by (3.18), we have $S(w, y(w)) = y(w)$, i.e. $S(w, y(w)) = A(w, y(w)) = y(w)$.

Again, since $BF = FB$ and using condition (**A**) we have
$\|y(w) - B(w, y(w))\|^2 = \|E(w, y(w)) - B(w, F(w, y(w)))\|^2$
$= \|E(w, y(w)) - F(w, B(w, y(w)))\|^2$

$$\leq \phi \begin{pmatrix} \|S(w,A(w,y(w)))-T(w,B(w,B(w,y(w)))\|^2, \\ \{\|S(w,A(w,y(w)))-E(w,y(w))\|^2+\|T(w,B(w,B(w,y(w)))-F(w,B(w,y(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,y(w)))-F(w,B(w,y(w)))\|^2+\|T(w,B(w,B(w,y(w)))-E(w,y(w))\|^2\} \end{pmatrix}. \tag{3.19}$$

Since $FB = BF$ and $TB = BT$ we have $F(w, B(w, y(w))) = B(w, F(w, y(w))) = B(w, y(w))$ and $T(w, B(w, B(w, y(w)))) = B(w, T(w, B(w, y(w)))) = B(w, y(w)$. Applying this in (3.19) we get

$$\|y(w) - B(w, y(w))\|^2 \leq \phi \begin{pmatrix} \|y(w) - B(w, y(w))\|^2, \\ \{\|y(w) - y(w)\|^2 + \|B(w, y(w)) - B(w, y(w))\|^2\}, \\ \frac{1}{2}\{\|y(w) - B(w, y(w))\|^2 + \|B(w, y(w)) - y(w)\|^2\} \end{pmatrix}.$$

By using (2.1), we have

$$B(w, y(w)) = y(w), \ w \in \Omega. \tag{3.20}$$

Since $T(w, B(w, y(w))) = y(w)$ and by (3.20) we have $T(w, y(w)) = y(w)$, i.e. $T(w, y(w)) = B(w, y(w)) = y(w)$.

For the uniqueness of the common random fixed point $y(w)$ of E, F, S, T, A and B , let $p(w) : \Omega \to C$ be another common random fixed point of E, F, S, T, A and B, using condition **(A)** we obtain
$\|y(w) - p(w)\|^2 = \|E(w, y(w)) - F(w, p(w))\|^2 \leq$

$$\phi\left(\begin{array}{c} \|S(w, A(w, y(w))) - T(w, B(w, p(w)))\|^2, \\ \{\|S(w, A(w, y(w))) - E(w, y(w))\|^2 + \|T(w, B(w, p(w))) - F(w, p(w))\|^2\}, \\ \frac{1}{2}\{\|S(w, A(w, y(w))) - F(w, p(w))\|^2 + \|T(w, B(w, p(w))) - E(w, y(w))\|^2\} \end{array}\right),$$

which yields

$$\|y(w) - p(w)\|^2 \quad \leq \quad \phi\left(\begin{array}{c} \|y(w) - p(w)\|^2, \\ \{\|y(w) - y(w)\|^2 + \|p(w) - p(w)\|^2\}, \\ \frac{1}{2}\{\|y(w) - p(w)\|^2 + \|p(w) - y(w)\|^2\} \end{array}\right)$$

(2.1) implies

$$y(w) = p(w) \quad for \quad w \in \Omega.$$

Now, suppose E is continuous, then $E(w, S(w, A(w, y_{2n}))) \to E(w, y(w))$. Since (E, SA) is compatible, then (3.7) implies that

$$S(w, A(w, E(w, g_{2n}(w)))) \to E(w, y(w)). \tag{3.21}$$

Using Condition A, we get
$\|E(w, E(w, g_{2n}(w))) - F(w, g_{2n-1}(w))\|^2 \leq$

$$\phi\left(\begin{array}{c} \|S(w,A(w,E(w,g_{2n}(w))))-T(w,B(w,g_{2n-1}(w)))\|^2, \\ \{\|S(w,A(w,E(w,g_{2n}(w))))-E(w,E(w,g_{2n}(w)))\|^2+\|T(w,B(w,g_{2n-1}(w)))-F(w,g_{2n-1}(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,E(w,g_{2n}(w))))-F(w,g_{2n-1}(w))\|^2+\|T(w,B(w,g_{2n-1}(w)))-E(w,E(w,g_{2n}(w)))\|^2\} \end{array}\right).$$

Letting $(n \to \infty)$ and using (3.21), we obtain
$\|E(w, y(w)) - y(w)\|^2 \leq$

$$\phi\left(\begin{array}{c} \|E(w, y(w)) - y(w)\|^2, \\ \{\|E(w, y(w)) - E(w, y(w))\|^2 + \|y(w) - y(w)\|^2\}, \\ \frac{1}{2}\{\|E(w, y(w)) - y(w)\|^2 + \|y(w) - E(w, y(w))\|^2\} \end{array}\right),$$

which implies

$$E(w, y(w)) = y(w), w \in \Omega. \tag{3.22}$$

Since $y(w) = E(w, y(w)) \in E(w, H) \subset T(w, B(w, H))$, there exists $h(w) \in C$ such that

$$y(w) = T(w, B(w, h(w))) \quad for \quad w \in \Omega. \tag{3.23}$$

Using condition (**A**) we obtain
$\|E(w, E(w, g_{2n}(w))) - F(w, h(w))\|^2 \leq$

$$\phi\left(\begin{array}{c} \|S(w,A(w,E(w,g_{2n}(w))))-T(w,B(w,h(w)))\|^2, \\ \{\|S(w,A(w,E(w,g_{2n}(w))))-E(w,E(w,g_{2n}(w)))\|^2+\|T(w,B(w,h(w)))-F(w,h(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,E(w,g_{2n}(w))))-F(w,h(w))\|^2+\|T(w,B(w,h(w)))-E(w,E(w,g_{2n}(w)))\|^2\} \end{array}\right).$$

Letting $(n \to \infty)$ and using (3.21),(3.22) and (3.23), we obtain

$$\|y(w) - F(w,h(w))\|^2 \leq \phi\left(\begin{array}{c} \|y(w) - y(w)\|^2, \\ \{\|y(w) - y(w)\|^2 + \|y(w) - F(w,y(w))\|^2\}, \\ \frac{1}{2}\{\|y(w) - F(w,h(w))\|^2 + \|y(w) - y(w)\|^2\}. \end{array}\right),$$

hence by (2.2) and (3.23), we have

$$y(w) = F(w,h(w)) = T(w,B(w,h(w))) \quad for \quad w \in \Omega. \tag{3.24}$$

Since the pair $\{F, TB\}$ is weakly compatible, then they commute at their coincidence point $h(w)$, i.e.

$$\begin{aligned} F(w,T(w,B(w,h(w)))) &= T(w,B(w,F(w,h(w)))) \\ \Rightarrow F(w,y(w)) &= T(w,B(w,y(w))). \end{aligned} \tag{3.25}$$

From condition **(A)**,
$\|E(w,g_{2n}(w)) - F(w,y(w))\|^2 \leq$

$$\phi\left(\begin{array}{c} \|S(w,A(w,g_{2n}(w)))-T(w,B(w,y(w)))\|^2, \\ \{\|S(w,A(w,g_{2n}(w)))-E(w,g_{2n}(w))\|^2+\|T(w,B(w,y(w)))-F(w,y(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,g_{2n}(w)))-F(w,y(w))\|^2+\|T(w,B(w,y(w)))-E(w,g_{2n}(w))\|^2\} \end{array}\right).$$

Letting $(n \to \infty)$ and using (3.25), we obtain

$$\|y(w) - F(w,y(w))\|^2 \leq \phi\left(\begin{array}{c} \|y(w) - F(w,y(w))\|^2, \\ \{\|y(w) - y(w)\|^2 + \|F(w,y(w)) - F(w,y(w))\|^2\}, \\ \frac{1}{2}\{\|y(w) - F(w,h(w))\|^2 + \|F(w,y(w)) - y(w)\|^2\} \end{array}\right),$$

from (2.2), we obtain

$$y(w) = F(w,y(w)) \quad for \quad w \in \Omega. \tag{3.26}$$

From (3.25) and (3.26), we have

$$y(w) = F(w,y(w)) = T(w,B(w,y(w))) \quad for \quad w \in \Omega. \tag{3.27}$$

Since $y(w) = F(w,y(w)) \in F(w,H) \subset S(w,A(w,H))$, there exists $f(w) \in C$ such that

$$S(w,A(w,f(w))) = y(w) \quad for \quad w \in \Omega. \tag{3.28}$$

Again using condition (**A**) and (3.27), we have
$\|E(w,f(w)) - y(w)\|^2 = \|E(w,f(w)) - F(w,y(w))\|^2 \leq$

$$\phi\left(\begin{array}{c} \|S(w,A(w,f(w)))-T(w,B(w,y(w)))\|^2, \\ \{\|S(w,A(w,f(w)))-E(w,f(w))\|^2+\|T(w,B(w,y(w)))-F(w,y(w))\|^2\}, \\ \frac{1}{2}\{\|S(w,A(w,f(w)))-F(w,y(w))\|^2+\|T(w,B(w,y(w)))-E(w,f(w))\|^2\} \end{array}\right),$$

using (3.27) and (3.28), we get

$$\|E(w, f(w)) - y(w)\|^2 \leq \phi\begin{pmatrix} \|y(w) - y(w)\|^2, \\ \{\|y(w) - E(w, f(w))\|^2 + \|y(w) - y(w)\|^2\}, \\ \frac{1}{2}\{\|y(w) - y(w)\|^2 + \|y(w) - E(w, f(w))\|^2\} \end{pmatrix},$$

(2.2) implies

$$E(w, f(w)) = y(w) \quad for \quad w \in \Omega. \tag{3.29}$$

Both (3.28) and (3.29) imply that

$$S(w, A(w, f(w))) = E(w, f(w)) = y(w) \quad for \quad w \in \Omega. \tag{3.30}$$

Since E and SA are compatible, then

$$\begin{aligned} S(w, A(w, E(w, f(w)))) &= E(w, S(w, A(w, f(w)))) \\ \Rightarrow S(w, A(w, y(w))) &= E(w, y(w)) = y(w). \end{aligned} \tag{3.31}$$

From (3.27) and (3.31), we obtain $S(w, A(w, y(w))) = E(w, y(w)) = F(w, y(w)) = T(w, B(w, y(w))) = y(w)$ for $w \in \Omega$, that is $y(w)$ is the common random fixed point of E, F, SA, TB.
Similarly, we can prove that $y(w)$ is the unique common random fixed point of E, F, A, S, T, B.
Now, if the second condition of the Theorem 3.1 is satisfied that is, either F or TB is continuous, the pair (F, TB) is compatible and the pair (E, SA) is weakly compatible. Then the proof is similar to (1). □

If we put $A = B = I$ (the identity random mapping) in Theorem 3.1, we have the following corollary:

Corollary 3.1. *Let C be a nonempty closed subset of a separable Hilbert space H. Let E, F, S and $T : \Omega \times C \to C$ be four random mappings defined on C such that for $w \in \Omega$, E, F, S and T satisfy condition the following conditions:*
$\|E(w, x(w)) - F(w, y(w))\|^2 \leq$

$$\phi\begin{pmatrix} \|S(w, x(w)) - T(w, y(w))\|^2, \\ \{\|S(w, x(w)) - E(w, x(w))\|^2 + \|T(w, y(w)) - F(w, y(w))\|^2\}, \\ \frac{1}{2}\{\|S(w, x(w)) - F(w, y(w))\|^2 + \|T(w, y(w)) - E(w, x(w))\|^2\} \end{pmatrix},$$

for $x, y \in H$ and $w \in \Omega$.

$$E(w, H) \subset T(w, H) \ \text{ and } \ F(w, H) \subset S(w, H).$$

then E, F, S and T have a unique common random fixed point if one of the following conditions is satisfied:

1. *Either E or S is continuous, the pair (E, S) is compatible and the pair (F, T) is weakly compatible.*

2. *Either F or T is continuous, the pair (F, T) is compatible and the pair (E, S) is weakly compatible.*

Also, if we put $S = A = T = B = I$ (the identity random mapping) in Theorem 3.1, we obtain the following result, which is proved in [8].

Corollary 3.2. *Let C be a nonempty closed subset of a separable Hilbert space H. Let E and $F : \Omega \times C \to C$ be two random mappings defined on C satisfying the following condition:*
$||E(w, x(w)) - F(w, y(w))||^2 \leq$

$$\phi\begin{pmatrix} ||x(w) - y(w)||^2, \\ \{||x(w) - E(w, x(w))||^2 + ||y(w) - F(w, y(w))||^2\}, \\ \frac{1}{2}\{||x(w) - F(w, y(w))||^2 + ||y(w) - E(w, x(w))||^2\} \end{pmatrix},$$

for $x, y \in H$ and $w \in \Omega$. Then E and F have a unique common random fixed point.

4. Conclusion

A unique common random fixed point for six random mappings is obtained by using the implicit relation defined in Definition (2.2) and some certain conditions on a closed subset C of a separable Hilbert space. Also we obtain some corollaries of Theorem 3.1.

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