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THE EXISTENCE OF FIXED POINTS FOR HARDY-ROGERS CONTRACTIVE MAPPINGS WITH RESPECT TO A wt-DISTANCE IN b-METRIC SPACES

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Abstract. The aim of this paper is to prove some existence and uniqueness theorems of the fixed points for Hardy-Rogers type contraction with respect to a wt-distance in b-metric spaces endowed with a graph. These results prepare a more general statement, since we apply the condition of orbitally G-continuity of mappings instead of the condition of continuity, consider b-metric spaces endowed with a graph instead general b-metric spaces and use of control functions instead of constant numbers.
Key words: fixed-point, contractive mapping, metric space.

1. Introduction and preliminaries

In 1931, Wilson [16] defined the concept of symmetric space, as metric-like spaces without the condition of triangle inequality. Thereinafter, the concept of b-metric spaces as a generalization of symmetric and metric spaces were introduced by Bakhtin [2] and Czerwik [5]. On the other hand, in 1996, Kada et al. [11] defined the concept of w-distance in metric spaces and presented some fixed point theorems with respect to this distance. The concept of wt-distance on b-metric spaces as a generalization of w-distance was introduced by Hussain et al. [9]. Then they proved some fixed point theorems under a wt-distance in partially ordered b-metric spaces (also, see [7]).

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In this paper, we consider a wt-distance in b-metric spaces endowed with a directed graph and obtain some fixed point theorems of Hardy-Rogers type contraction [8] with respect to this distance, where all of the above works can be unified. In the following part, we will give some preliminary definitions, lemmas and notions which will be needed in the sequel.

Definition 1.1. [5] Let X be a nonempty set and $s \geq 1$ a given real number. Suppose that the mapping $d : X \times X \to [0, \infty)$ for all $x, y, z \in X$ satisfies the following conditions:

(d_1) $d(x, y) = 0$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$;

(d_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then d is called a b-metric and (X, d) is called a b-metric space.

Obviously, for $s = 1$, a b-metric space is a metric space. Also, for notions such as convergent and Cauchy sequences, completeness, continuity and etc in b-metric spaces, see [1, 4, 12, 14].

Definition 1.2. [9] Let (X, d) be a b-metric space and $s \geq 1$ be a given real number. A function $\rho : X \times X \to [0, +\infty)$ is called a wt-distance on X if for all $x, y, z \in X$, the following properties are satisfied:

(ρ_1) $\rho(x, z) \leq s[\rho(x, y) + \rho(y, z)]$;

(ρ_2) ρ is b-lower semi-continuous in its second variable; that is, if $y_n \to y$ in X, then $\rho(x, y) \leq s \liminf_n \rho(x, y_n)$;

(ρ_3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(z, x) \leq \delta$ and $\rho(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Obviously, for $s = 1$, every wt-distance is a w-distance. But, a w-distance is not necessary a wt-distance. Thus, each wt-distance is a generalization of w-distance.

Lemma 1.1. *[9] Let (X, d) be a b-metric space with the parameter $s \geq 1$ and ρ be a wt-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be two sequences in X, $\{a_n\}$ and $\{b_n\}$ be two sequences in $[0, +\infty)$ converging to zero and $x, y, z \in X$. Then the following conditions hold:*

(wt_1) *if $\rho(x_n, y) \leq a_n$ and $\rho(x_n, z) \leq b_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $\rho(x, y) = 0$ and $\rho(x, z) = 0$, then $y = z$;*

(wt_2) *if $\rho(x_n, x_m) \leq a_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

In 2008, Jachymski [10] provided a metric space with a graph and introduced Banach contraction principle in graph language (also, see [13]).

Let (X, d) be a b-metric space and G be a directed graph without parallel edges and with vertex set $V(G) = X$ and edge set $E(G)$ contains all loops. Then the graph G can be written by the ordered pair $(V(G), E(G))$ and (X, d) is named b-metric space endowed with the graph G. Also, The graph G is connected if there exists a path in G between every two vertices of G. For more details on graphs, see [3]. From here onwards, let (X, d) be a b-metric space endowed with a graph G, where $V(G) = X$ and $\Delta(X) \subseteq E(G)$ with $\Delta(X) = \{(x, x) \in X \times X : x \in X\}$. Also, let $\mathrm{Fix}(f)$ be the set of all fixed points of a self-map f on X and $X_f = \{x \in X : (x, fx) \in E(G)\}$.

From the idea of Jachymski [10] and Petrusel and Rus [13], Fallahi et al. defined Picard operators in b-metric spaces and orbitally G-continuous mappings on X as follows:

Definition 1.3. [6, 7] Let (X, d) be a b-metric space. A self-map f on X is called a Picard operator if f has an unique fixed point x_* in X and $f^n x \to x_*$ for all $x \in X$.

Definition 1.4. [6, 7] Let (X, d) be a b-metric space endowed with a graph G. A mapping $f : X \to X$ is called orbitally G-continuous on X if for all $x, y \in X$ and all sequences $\{b_n\}$ of positive integers with $(f^{b_n} x, f^{b_{n+1}} x) \in E(G)$ for all $n \geq 1$, the convergence $f^{b_n} x \to y$ implies that $f(f^{b_n} x) \to fy$.

Note that a continuous mapping on b-metric spaces is orbitally G-continuous for all graphs G, but the converse is not generally true.

2. Main results

The following theorem is the principle result of this paper.

Theorem 2.1. *Let (X, d) be a complete b-metric space endowed with the graph G, $s \geq 1$ be a given real number and ρ be a wt-distance. Also, let $f : X \to X$ be an orbitally G-continuous mapping that preserves the edges of G; that is, $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$. Assume that there exist mappings $\mu_i : X \to [0, 1)$ for $i = 1, 2, \cdots, 5$ with $\mu_i(fx) \leq \mu_i(x)$ such that*

$$\begin{aligned} \rho(fx, fy) &\leq \mu_1(x)\rho(x, y) + \mu_2(x)\rho(x, fx) + \mu_3(x)\rho(y, fy) \\ &\quad + \mu_4(x)\rho(x, fy) + \mu_5(x)\rho(y, fx), \end{aligned} \tag{2.1}$$

$$\begin{aligned} \rho(fy, fx) &\leq \mu_1(x)\rho(y, x) + \mu_2(x)\rho(fx, x) + \mu_3(x)\rho(fy, y) \\ &\quad + \mu_4(x)\rho(fy, x) + \mu_5(x)\rho(fx, y) \end{aligned} \tag{2.2}$$

for all $x, y \in X$ with $(x, y) \in E(G)$, where

$$(s(\mu_1 + \mu_3 + 2\mu_4) + \mu_2 + (s^2 + s)\mu_5)(x) < 1. \tag{2.3}$$

Then $X_f \neq \emptyset$ if and only if f has a fixed point. Further, if $fv = v$, then $\rho(v, v) = 0$. Moreover, if the subgraph of G with the vertex set $\mathrm{Fix}(f)$ *is connected, then the restriction of f to X_f is a Picard operator.*

Proof. Since $\mathrm{Fix}(f) \subseteq X_f$, if f has a fixed point, then X_f is nonempty. Conversely, let $X_f \neq \emptyset$ and $x_0 \in X_f$. Since f preserves the edges of G, then $(x_{n-1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$, where $x_n = fx_{n-1} = f^n x_0$. Now, set $x = x_n$ and $y = x_{n-1}$ in (2.1) and apply (ρ_1). Then, by a simple calculation, we have

$$\begin{aligned} \rho(x_{n+1}, x_n) \leq\; & \mu_1(x_0)\rho(x_n, x_{n-1}) + (\mu_2 + s\mu_4 + s\mu_5)(x_0)\rho(x_n, x_{n+1}) \\ & +(\mu_3 + s\mu_5)(x_0)\rho(x_{n-1}, x_n) + s\mu_4(x_0)\rho(x_{n+1}, x_n). \end{aligned} \tag{2.4}$$

Similarly, set $x = x_n$ and $y = x_{n-1}$ in (2.2) and apply (ρ_1), we have

$$\begin{aligned} \rho(x_n, x_{n+1}) \leq\; & \mu_1(x_0)\rho(x_{n-1}, x_n) + (\mu_2 + s\mu_4 + s\mu_5)(x_0)\rho(x_{n+1}, x_n) \\ & +(\mu_3 + s\mu_5)(x_0)\rho(x_n, x_{n-1}) + s\mu_4(x_0)\rho(x_n, x_{n+1}). \end{aligned} \tag{2.5}$$

Now, adding up (2.4) and (2.5), we obtain

$$\begin{aligned} \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_n) \leq\; & (\mu_1 + \mu_3 + s\mu_5)(x_0)[\rho(x_{n-1}, x_n) + \rho(x_n, x_{n-1})] \\ & +(\mu_2 + 2s\mu_4 + s\mu_5)(x_0)[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_n)]. \end{aligned}$$

Let $u_n = \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_n)$. Then

$$u_n \leq (\mu_1 + \mu_3 + s\mu_5)(x_0)u_{n-1} + (\mu_2 + 2s\mu_4 + s\mu_5)(x_0)u_n$$

for all $n \in \mathbb{N}$. Hence, we have $u_n \leq \alpha u_{n-1}$ for all $n \in \mathbb{N}$, where

$$0 \leq \alpha = \frac{(\mu_1 + \mu_3 + s\mu_5)(x_0)}{1 - (\mu_2 + 2s\mu_4 + s\mu_5)(x_0)} < \frac{1}{s}, \qquad \text{(by (2.3))}. \tag{2.6}$$

By repeating the procedure, we have $u_n \leq \alpha^n u_0$ for all $n \in \mathbb{N}$. Thus,

$$\rho(x_n, x_{n+1}) \leq u_n \leq \alpha^n[\rho(x_0, x_1) + \rho(x_1, x_0)]. \tag{2.7}$$

Now, let $m, n \in \mathbb{N}$ with $m > n$. It follows from (ρ_1), (2.6) and $s\alpha < 1$ (by (2.7)) that

$$\rho(x_n, x_m) \leq \frac{s\alpha^n}{1 - s\alpha}[\rho(x_1, x_0) + \rho(x_0, x_1)].$$

Clearly, $\frac{s\alpha^n}{1-s\alpha}[\rho(x_0, x_1) + \rho(x_1, x_0)]$ is a convergent sequence to zero. Hence, $\{x_n\}$ is a Cauchy sequence in X by Lemma 1.1.(wt_2). Since X is complete, there exists a point $v \in X$ such that $x_n = f^n x_0 \to v$ as $n \to \infty$. On the other hand, since $x_0 \in X_f$, we have $(f^n x_0, f^{n+1}x_0) \in E(G)$ for $n = 0, 1, \cdots$. Therefore, by orbital G-continuity of f, we obtain $f^{n+1}x_0 \to fv$. Since the limit of a sequence is unique, we conclude that $fv = v$; that is, v is a fixed point of the mapping f. Further, let $fv = v$ and consider $x = y = v$ in (2.1). Then, we have

$$\rho(v, v) = \rho(fv, fv) \leq (\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5)(v)\rho(v, v),$$

which implies that $\rho(v,v)=0$ (since $\sum_{i=1}^{5}\mu_i(v)<1$).

Next, assume that the subgraph of G with the vertex set $\mathrm{Fix}(f)$ is connected and $v_* \in X$ is another fixed point of f. Then there exists a path $(x_i)_{i=0}^N$ in G from v to v_* such that $x_1,\ldots,x_{N-1}\in \mathrm{Fix}(f)$ by setting $x_0=v$, $x_N=v_*$ and $(x_{i-1},x_i)\in E(G)$ for $i=1,\cdots,N$. Now, since $\rho(x_i,x_i)=0$ for each $i=1,2,\cdots N$, and by applying (2.1) and (2.2), we have

$$\rho(x_i,x_{i-1})=\rho(fx_i,fx_{i-1})\le(\mu_1+\mu_4)(x_i)\rho(x_i,x_{i-1})+\mu_5(x_i)\rho(x_{i-1},x_i) \tag{2.8}$$

$$\rho(x_{i-1},x_i)=\rho(fx_{i-1},fx_i)\le(\mu_1+\mu_4)(x_i)\rho(x_{i-1},x_i)+\mu_5(x_i)\rho(x_i,x_{i-1}). \tag{2.9}$$

Now, adding up (2.8) and (2.9), we obtain

$$\rho(x_i,x_{i-1})+\rho(x_{i-1},x_i)\le(\mu_1+\mu_4+\mu_5)(x_i)[\rho(x_i,x_{i-1})+\rho(x_{i-1},x_i)],$$

which implies that $\rho(x_i,x_{i-1})+\rho(x_{i-1},x_i)=0$ (since $(\mu_1+\mu_4+\mu_5)(x_i)<1$). Hence, $\rho(x_i,x_{i-1})=\rho(x_{i-1},x_i)=0$. Now, by Lemma 1.1.($wt_1$), we have $d(x_i,x_{i-1})=0$; that is, $x_i=x_{i-1}$ for $i=1,2,\cdots,N$. Hence, $v=x_0=x_1=\cdots=x_{N-1}=v_*$. Therefore, the fixed point of f is unique and the restriction of f to X_f is a Picard operator. This completes the proof. $\square$

Example 2.1. Let $X=[0,1]$ and consider the mapping $d: X\times X\to\mathbb{R}^+$ by $d(x,y)=(x-y)^2$ for all $x,y\in X$. Then (X,d) is a b-metric space with $s=2$. Define the mapping $\rho: X\times X\to[0,\infty)$ by $\rho(x,y)=y^2$ for all $x,y\in X$. Then ρ is a wt-distance. Define $f: X\to X$ by $f1=\frac{1}{2}$ and $fx=\frac{x^2}{4}$ for $1\ne x\in X$. Clearly, f is not continuous on the whole X. Suppose that X is endowed with a graph $G=(V(G),E(G))$, where $V(G)=X$ and $E(G)=\{(x,x): x\in X\}$; that is, $E(G)$ contains nothing but all loops. Clearly, f is orbitally G-continuous on X. Consider mappings $\mu_1(x)=\frac{x^2}{4}$, $\mu_2(x)=\frac{x}{2}$ and $\mu_3(x)=\mu_4(x)=\mu_5(x)=0$ for all $x\in X$. Then

(i) if $x\ne1$, then $\mu_1(fx)=\mu_1(\frac{x^2}{4})=\frac{x^4}{64}\le\frac{x^2}{4}=\mu_1(x)$ and if $x=1$, then $\mu_1(f1)=\frac{1}{16}\le\frac{1}{4}=\mu_1(1)$;

(ii) if $x\ne1$, then $\mu_2(fx)=\mu_2(\frac{x^2}{4})=\frac{x^2}{8}\le\frac{x}{2}=\mu_2(x)$ and if $x=1$, then $\mu_2(f1)=\frac{1}{4}\le\frac{1}{2}=\mu_1(1)$;

(iii) $\mu_i(fx)\le\mu_i(x)$ for all $x\in X$ and $i=2,3,5$;

(iv) $(s(\mu_1+\mu_3+2\mu_4)+\mu_2+(s^2+s)\mu_5)(x)=2(\frac{x^2}{4})+\frac{x}{2}=\frac{x^2+x}{2}<1$ for all $x\in X$;

(v) let $x\in X$ with $(x,x)\in E(G)$. If $x\ne1$, then

$$\rho(fx,fx)=\frac{x^4}{16}\le\mu_1(x)\rho(x,x)+[\mu_2(x)+\mu_3(x)+\mu_4(x)+\mu_5(x)]\rho(x,fx)$$

and if $x=1$, then

$$\rho(f1,f1)=\frac{1}{4}\le\mu_1(1)\rho(1,1)+[\mu_2(1)+\mu_3(1)+\mu_4(1)+\mu_5(1)]\rho(1,f1).$$

Thus, (2.1) is established. Similarly, for the validity of (2.2), one can apply above approach with substitute first component with second component.

(vi) since $(0, f0) = (0, 0) \in E(G)$, we have $X_f \neq \emptyset$.

Hence, all of the conditions of Theorem 2.1 are true. Consequently, f has an unique fixed point $x = 0 \in [0, 1]$. Moreover, $\rho(0, 0) = 0^2 = 0$.

If we consider $\mu_i(x) = \mu_i$ for $i = 1, 2, \cdots, 5$, then we have the following theorem:

Theorem 2.2. *Let (X, d) be a complete b-metric space endowed with the graph G, $s \geq 1$ be a given real number and ρ be a wt-distance. Also, let $f : X \to X$ be an orbitally G-continuous mapping that preserves the edges of G. Assume that there exist constants $\mu_i \in [0, 1)$ for $i = 1, 2, \cdots, 5$ such that*

$$\begin{aligned}\rho(fx, fy) &\leq \mu_1\rho(x, y) + \mu_2\rho(x, fx) + \mu_3\rho(y, fy) + \mu_4\rho(x, fy) + \mu_5\rho(y, fx),\\ \rho(fy, fx) &\leq \mu_1\rho(y, x) + \mu_2\rho(fx, x) + \mu_3\rho(fy, y) + \mu_4\rho(fy, x) + \mu_5\rho(fx, y)\end{aligned}$$

for all $x, y \in X$ with $(x, y) \in E(G)$, where $s(\mu_1 + \mu_3 + 2\mu_4) + \mu_2 + (s^2 + s)\mu_5 < 1$. Then the assertions of the Theorem 2.1 are established.

Now, several consequences of our main result follow for particular choices of the graph G. First, consider complete graph G_0 whose vertex set coincides with X; that is, $V(G_0) = X$ and $E(G_0) = X \times X$. Let $G = G_0$ in Theorem 2.1 and Theorem 2.2. It is clear that set X_f related to any self-map f on X coincides with the whole set X. Thus, we have two following corollaries:

Corollary 2.1. *Let (X, d) be a complete b-metric space endowed with the graph G, $s \geq 1$ be a given real number and ρ be a wt-distance. Also, let $f : X \to X$ be an orbitally G_0-continuous mapping. Assume that there exist mappings $\mu_i : X \to [0, 1)$ with $\mu_i(fx) \leq \mu_i(x)$ for $i = 1, 2, \cdots, 5$ such that*

$$\begin{aligned}\rho(fx, fy) &\leq \mu_1(x)\rho(x, y) + \mu_2(x)\rho(x, fx) + \mu_3(x)\rho(y, fy)\\ &\quad +\mu_4(x)\rho(x, fy) + \mu_5(x)\rho(y, fx),\\ \rho(fy, fx) &\leq \mu_1(x)\rho(y, x) + \mu_2(x)\rho(fx, x) + \mu_3(x)\rho(fy, y)\\ &\quad +\mu_4(x)\rho(fy, x) + \mu_5(x)\rho(fx, y)\end{aligned}$$

for all $x, y \in X$, where $(s(\mu_1 + \mu_3 + 2\mu_4) + \mu_2 + (s^2 + s)\mu_5)(x) < 1$. Then f is a Picard operator.

Corollary 2.2. *Let (X, d) be a complete b-metric space endowed with the graph G, $s \geq 1$ be a given real number and ρ be a wt-distance. Also, let $f : X \to X$ be an orbitally G_0-continuous mapping. Assume that there exist constants $\mu_i \in [0, 1)$ for $i = 1, 2, \cdots, 5$ such that*

$$\begin{aligned}\rho(fx, fy) &\leq \mu_1\rho(x, y) + \mu_2\rho(x, fx) + \mu_3\rho(y, fy) + \mu_4\rho(x, fy) + \mu_5\rho(y, fx),\\ \rho(fy, fx) &\leq \mu_1\rho(y, x) + \mu_2\rho(fx, x) + \mu_3\rho(fy, y) + \mu_4\rho(fy, x) + \mu_5\rho(fx, y)\end{aligned}$$

for all $x, y \in X$, where $s(\mu_1 + \mu_3 + 2\mu_4) + \mu_2 + (s^2 + s)\mu_5 < 1$. Then f is a Picard operator.

Now, let $(X, \sqsubseteq)$ be a poset (partially ordered set) and G_1 be the graph with $V(G_1) = X$ and $E(G_1) = \{(x, y) \in X \times X : x \sqsubseteq y\}$. Since $\sqsubseteq$ is reflexive, $E(G_1)$ contain all loops. By setting $G = G_1$ in Theorem 2.1 and Theorem 2.2, we obtain two following corollaries of our main fixed point theorems.

Corollary 2.3. *Let $(X, \sqsubseteq)$ be a poset, (X, d) be a complete b-metric space and $s \geq 1$ be a given real number. Also, ρ be a wt-distance and $f : X \to X$ be a nondecreasing and orbitally G_1-continuous mapping. Assume that there exist mappings $\mu_i : X \to [0, 1)$ with $\mu_i(fx) \leq \mu_i(x)$ for $i = 1, 2, \cdots, 5$ such that*

$$\begin{aligned} \rho(fx, fy) &\leq \mu_1(x)\rho(x, y) + \mu_2(x)\rho(x, fx) + \mu_3(x)\rho(y, fy) \\ &\quad + \mu_4(x)\rho(x, fy) + \mu_5(x)\rho(y, fx), \\ \rho(fy, fx) &\leq \mu_1(x)\rho(y, x) + \mu_2(x)\rho(fx, x) + \mu_3(x)\rho(fy, y) \\ &\quad + \mu_4(x)\rho(fy, x) + \mu_5(x)\rho(fx, y) \end{aligned}$$

for all $x, y \in X$ with $x \sqsubseteq y$, where $(s(\mu_1 + \mu_3 + 2\mu_4) + \mu_2 + (s^2 + s)\mu_5)(x) < 1$. Then f has a fixed point if and only if there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$. Further, if $fv = v$, then $\rho(v, v) = 0$. Moreover, if the subgraph of G_1 with the vertex set $\mathrm{Fix}(f)$ *is connected, then the restriction of f to the set of all points in $x \in X$ such $x \sqsubseteq fx$ is a Picard operator.*

Corollary 2.4. *Let $(X, \sqsubseteq)$ be a poset, (X, d) be a complete b-metric space and $s \geq 1$ be a given real number. Also, ρ be a wt-distance and $f : X \to X$ be a nondecreasing and orbitally G_1-continuous mapping. Assume that there exist constants $\mu_i \in [0, 1)$ for $i = 1, 2, \cdots, 5$ such that*

$$\begin{aligned} \rho(fx, fy) &\leq \mu_1\rho(x, y) + \mu_2\rho(x, fx) + \mu_3\rho(y, fy) + \mu_4\rho(x, fy) + \mu_5\rho(y, fx), \\ \rho(fy, fx) &\leq \mu_1\rho(y, x) + \mu_2\rho(fx, x) + \mu_3\rho(fy, y) + \mu_4\rho(fy, x) + \mu_5\rho(fx, y) \end{aligned}$$

for all $x, y \in X$ with $x \sqsubseteq y$, where $s(\mu_1 + \mu_3 + 2\mu_4) + \mu_2 + (s^2 + s)\mu_5 < 1$. Then the assertions of the Corollary 2.3 are established.

Now, let X be a poset endowed with the graph G_2 given by $V(G_2) = X$ and $E(G_2) = \{(x, y) \in X \times X : x \sqsubseteq y \vee y \sqsubseteq x\}$; that is, an ordered pair $(x, y) \in X \times X$ is an edge of G_2 if and only if x and y are comparable elements of $(X, \sqsubseteq)$. Consider $G = G_2$ in Theorem 2.1 and Theorem 2.2. Then we have other fixed point corollaries as follows.

Corollary 2.5. *Let $(X, \sqsubseteq)$ be a poset, (X, d) be a complete b-metric space and $s \geq 1$ be a given real number. Also, let ρ be a wt-distance and $f : X \to X$ be a nondecreasing and orbitally G_2-continuous mapping which maps comparable elements of X onto comparable elements. Assume that there exist mappings $\mu_i : X \to [0, 1)$ with $\mu_i(fx) \leq \mu_i(x)$ for $i = 1, 2, \cdots, 5$ such that*

$$\begin{aligned} \rho(fx, fy) &\leq \mu_1(x)\rho(x, y) + \mu_2(x)\rho(x, fx) + \mu_3(x)\rho(y, fy) \\ &\quad + \mu_4(x)\rho(x, fy) + \mu_5(x)\rho(y, fx), \\ \rho(fy, fx) &\leq \mu_1(x)\rho(y, x) + \mu_2(x)\rho(fx, x) + \mu_3(x)\rho(fy, y) \\ &\quad + \mu_4(x)\rho(fy, x) + \mu_5(x)\rho(fx, y) \end{aligned}$$

for all comparable $x, y \in X$, where $(s(\mu_1 + \mu_3 + 2\mu_4) + \mu_2 + (s^2 + s)\mu_5)(x) < 1$. Then f has a fixed point in X if and only if there exists $x_0 \in X$ such that x_0 and fx_0 are comparable. Moreover if $fv = v$, then $\rho(v, v) = 0$. Also, if the subgraph of G_2 with the vertex set $\mathrm{Fix}(f)$ *is connected, then the restriction of f to the set of all points in $x \in X$ such that x and fx are comparable is a Picard operator.*

Corollary 2.6. *Let $(X, \sqsubseteq)$ be a poset, (X, d) be a complete b metric space and $s \geq 1$ be a given real number, ρ be a wt-distance and $f : X \to X$ be a nondecreasing and orbitally G_2-continuous mapping which maps comparable elements of X onto comparable elements. Suppose that there exist constants $\mu_i \in [0, 1)$ for $i = 1, 2, \cdots, 5$ such that*

$$\begin{aligned} \rho(fx, fy) &\leq \mu_1\rho(x, y) + \mu_2\rho(x, fx) + \mu_3\rho(y, fy) + \mu_4\rho(x, fy) + \mu_5\rho(y, fx), \\ \rho(fy, fx) &\leq \mu_1\rho(y, x) + \mu_2\rho(fx, x) + \mu_3\rho(fy, y) + \mu_4\rho(fy, x) + \mu_5\rho(fx, y) \end{aligned}$$

for all comparable $x, y \in X$, where $s(\mu_1 + \mu_3 + 2\mu_4) + \mu_2 + (s^2 + s)\mu_5 < 1$. Then the assertions of Corollary 2.5 are established.

For our next consequence, let $\varepsilon > 0$ be a fixed number. Note that two elements $x, y \in X$ are said to be ε-closed if $d(x, y) < \varepsilon$. Consider the ε-graph G_3 with $V(G_3) = X$ and $E(G_3) = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$. Note that $E(G_3)$ contains all loops. Now, let $G = G_3$ in Theorem 2.1 and Theorem 2.2. Then we have the following consequences of our main fixed point theorems as follow.

Corollary 2.7. *Let (X, d) be a complete b-metric space endowed with the graph G_3, $s \geq 1$ be a given real number and $\varepsilon > 0$. Also, let ρ be a wt-distance and $f : X \to X$ be an orbitally G_3-continuous mapping which maps ε-close elements of X onto ε-close elements. Assume that there exist mappings $\mu_i : X \to [0, 1)$ with $\mu_i(fx) \leq \mu_i(x)$ for $i = 1, 2, \cdots, 5$ such that*

$$\begin{aligned} \rho(fx, fy) &\leq \mu_1(x)\rho(x, y) + \mu_2(x)\rho(x, fx) + \mu_3(x)\rho(y, fy) \\ &\quad + \mu_4(x)\rho(x, fy) + \mu_5(x)\rho(y, fx), \\ \rho(fy, fx) &\leq \mu_1(x)\rho(y, x) + \mu_2(x)\rho(fx, x) + \mu_3(x)\rho(fy, y) \\ &\quad + \mu_4(x)\rho(fy, x) + \mu_5(x)\rho(fx, y) \end{aligned}$$

for all $x, y \in X$ such that x and y are ε-close elements, where

$$(s(\mu_1 + \mu_3 + 2\mu_4) + \mu_2 + (s^2 + s)\mu_5)(x) < 1.$$

Then T has a fixed point on X if and only if there exists $x_0 \in X$ such that x_0 and fx_0 are ε-close. Moreover, if $fv = v$, then $\rho(v, v) = 0$. Also, if the subgraph of G_3 with the vertex set $\mathrm{Fix}(f)$ *is connected, then the restriction of f to the set of all points in $x \in X$ such x and fx are ε-close is a Picard operator.*

Corollary 2.8. *Let (X, d) be a complete b-metric space endowed with the graph G_3, $s \geq 1$ be a given real number and $\varepsilon > 0$. Also, ρ be a wt-distance and $f : X \to X$*

be an orbitally G_3-continuous mapping which maps ε-close elements of X onto ε-close elements. Suppose that there exist constants $\mu_i \in [0,1)$ for $i = 1, 2, \cdots, 5$ such that

$$\begin{aligned}
\rho(fx, fy) &\leq \mu_1\rho(x, y) + \mu_2\rho(x, fx) + \mu_3\rho(y, fy) + \mu_4\rho(x, fy) + \mu_5\rho(y, fx),\\
\rho(fy, fx) &\leq \mu_1\rho(y, x) + \mu_2\rho(fx, x) + \mu_3\rho(fy, y) + \mu_4\rho(fy, x) + \mu_5\rho(fx, y)
\end{aligned}$$

for all $x, y \in X$ such that x and y are ε-close elements, where

$$s(\mu_1 + \mu_3 + 2\mu_4) + \mu_2 + (s^2 + s)\mu_5 < 1.$$

Then the assertions of Corollary 2.7 are established.

Remark 2.1. (i) For Banach contraction principle with respect to a wt-distance on b-metric spaces endowed with the graph G and with the parameter $s \geq 1$, we must consider the condition $\rho(fx, fy) \leq \mu\rho(x, y)$ for all $x, y \in X$, where $\mu \in [0, \frac{1}{s})$.

(ii) Sometimes the constant numbers which satisfy Theorem 2.2 and Corollaries 2.2, 2.4, 2.6 and 2.8 are difficult to find. Thus, it is better to define such mappings $\mu_i(x)$ as another auxiliary tool of the b-metric such as Theorem 2.1 and Corollaries 2.1, 2.3, 2.5 and 2.7.

3. Conclusion

In this paper, we applied the condition of orbitally G-continuity of mapping instead the condition of continuity of mapping, b-metric spaces endowed with graph instead of metric spaces and control functions instead of constants, under which can be unified some theorems of existing literature such as Kada et al. [11], Fallahi et al. [6, 7], Hussain et al. [9], Petrusel and Rus [13], and Soleimani Rad et al. [15]. Also, one can apply this method for other results in fixed point theory. We finish this paper with a question. Can one prove the same results by considering some another conditions instead of the continuity of the mapping f and by considering one contractive relation instead two contractive relations?

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REFERENCES

1. S. Aleksić, H. Huang, Z. D. Mitrović and S. Radenović: *Remarks on some fixed point results in b-metric spaces.* J. Fixed Point Theory Appl. **20** (2018), 4:147.
2. I. A. Bakhtin: *The contraction mapping principle in almost metric space.* Functional Analysis. **30** (1989), 26–37.
3. J. A. Bondy and U. S. R. Murty: *Graph Theory.* Springer, 2008.

4. M. Bota, A. Molnar and C. Varga: *On Ekeland's variational principle in b-metric spaces.* Fixed Point Theory. **12** (2) (2011), 21–28.
5. S. Czerwik: *Contraction mappings in b-metric spaces.* Acta. Math. Inform. Univ. Ostrav. **1** (1) (1993), 5–11.
6. K. Fallahi, A. Petrusel and G. Soleimani Rad: *Fixed point results for pointwise Chatterjea type mappings with respect to a c-distance in cone metric spaces endowed with a graph.* U.P.B. Sci. Bull. (Series A). **80** (1) (2018), 47–54.
7. K. Fallahi, D. Savić and G. Soleimani Rad: *The existence theorem for contractive mappings on wt-distance spaces endowed with a graph and its application.* SCMA. **13** (1) (2019), 1–15.
8. G. E. Hardy and T. D. Rogers: *A generalization of a fixed point theorem of Reich.* Canad. Math. Bull. **16** (1973), 201–206.
9. N. Hussain, R. Saadati and R. P. Agrawal: *On the topology and wt-distance on metric type spaces.* Fixed Point Theory Appl. **2014**, 2014:88.
10. J. Jachymski: *The contraction principle for mappings on a metric space with a graph.* Proc. Amer. Math. Soc. **136** (2008), 1359–1373.
11. O. Kada, T. Suzuki and W. Takahashi: *Nonconvex minimization theorems and fixed point theorems in complete metric spaces.* Math. Japon. **44** (1996), 381–391.
12. Z. D. Mitrović and N. Hussain: *On weak quasicontractions in b-metric spaces.* Publ. Math. Debrecen. **94** (3-4) (2019), 289–298.
13. A. Petrusel and I. A. Rus: *Fixed point theorems in ordered L-spaces.* Proc. Amer. Math. Soc. **134** (2) (2006), 411–418.
14. D. Rakić, A. Mukheimer, T. Dosenović, Z. D. Mitrović and S. Radenović: *On some new fixed point results in fuzzy b-metric spaces.* J Inequal Appl. **2020**, 2020:99.
15. G. Soleimani Rad, H. Rahimi and C. Vetro: *Fixed point results under generalized c-distance with application to nonlinear fourth-order differential equation.* Fixed Point Theory. **20** (2) (2019), 635–648.
16. W. A. Wilson: *On semi-metric spaces.* Amer. Jour. Math. **53** (2) (1931), 361–373.