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NEW EXISTENCE AND UNIQUENESS RESULTS FOR HIGH DIMENSIONAL FRACTIONAL DIFFERENTIAL SYSTEMS *

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Abstract. In this paper, using the Caputo fractional derivative approach, we present new results on the existence and uniqueness of solutions to $n-$dimensional nonlinear coupled systems for differential equations. We also discuss some examples to illustrate the main results.

1. Introduction and Preliminaries

The physical laws of dynamics are not always described by ordinary order differential equations. In some cases, their behavior is governed by fractional order differential equations. For more details, we refer the reader to the books of Hilfer in [14] and Podlubny in [28]. Other research works can be found in [2, 4, 7, 9, 20, 24, 25, 26, 27, 29, 31, 32]. Further, many authors have established the existence and uniqueness of solutions for some fractional systems. The reader may refer to the following research papers [1, 3, 5, 6, 8, 10, 11, 12, 15, 16, 17, 18, 19, 21, 33, 34]. These papers treat problems with only one nonlinear term depending on two unknown functions. Other cases, where we have more than one nonlinearity depending on some unknown functions, are more complicated and have not been discussed in the above cited works.

In this paper, we are concerned with the following nonlinear system:

(1.1)
$$\begin{cases} D^{\alpha_1}x_1(t)=\sum\limits_{i=1}^{m} f_i^1\left(t,x_1(t),x_2(t),...,x_n(t),D^{\beta_1}x_1(t),D^{\beta_2}x_2(t),...,D^{\beta_n}x_n(t)\right),\ t\in J,\\ D^{\alpha_2}x_2(t)=\sum\limits_{i=1}^{m} f_i^2\left(t,x_1(t),x_2(t),...,x_n(t),D^{\beta_1}x_1(t),D^{\beta_2}x_2(t),...,D^{\beta_n}x_n(t)\right),\ t\in J,\\ \vdots\\ D^{\alpha_n}x_n(t)=\sum\limits_{i=1}^{m} f_i^n\left(t,x_1(t),x_2(t),...,x_n(t),D^{\beta_1}x_1(t),D^{\beta_2}x_2(t),...,D^{\beta_n}x_n(t)\right),\ t\in J,\\ x_k(0)=x_k^{'}(0)=...=x_k^{(n-2)}(0)=x_k^{(n-1)}(1)=0,\ k=1,2,...,n, \end{cases}$$

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where $n-1<\alpha_k<n$, $n-2<\beta_k<n-1$, $k=1,2,...,n$, $n\in\mathbb{N}^*-\{1\}$, $J:=[0,1]$. The derivatives D^{α_k} and D^{β_k}, $k=1,2,...,n$, are in the sense of Caputo. The functions $\left(f_i^k\right)_{i=1,...,m}^{k=1,2,...,n}:J\times\mathbb{R}^{2n}\to\mathbb{R}$ will be specified later.

We present some basic definitions and properties which are used throughout this paper [13, 22, 28].

Definition 1.1. The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$ for a continuous function f on $[0,\infty[$ is defined as:

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds,\ \alpha>0,\ t\geq 0,$$

$$J^0f(t)=f(t),\ t\geq 0, \tag{1.2}$$

where $\Gamma(\alpha):=\int_0^\infty e^{-x}x^{\alpha-1}dx$.

Definition 1.2. The Caputo derivative of order α for a function $x:[0,\infty)\to\mathbb{R}$, which is at least $n-$times differentiable can be defined as:

$$D^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}x^{(n)}(s)\,ds=J^{n-\alpha}x^{(n)}(t), \tag{1.3}$$

for $n-1<\alpha<n$, $n\in\mathbb{N}^*$.

We recall the following lemmas [11, 12, 23, 30]:

Lemma 1.1. *Let $\alpha>0$. The solution to the differential equation $D^{\alpha}x(t)=0$, is given by*

$$x(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}, \tag{1.4}$$

such that $c_j\in\mathbb{R}$, $j=0,...,n-1$, $n=[\alpha]+1$.

Lemma 1.2. *Let $\alpha>0$. Then*

$$J^{\alpha}D^{\alpha}x(t)=x(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}, \tag{1.5}$$

where $c_j\in\mathbb{R}$, $j=0,1,...,n-1$, $n=[\alpha]+1$.

Lemma 1.3. *Let $q>p>0$, $f\in L^1([a,b])$. Then $D^pJ^qf(t)=J^{q-p}f(t)$, $t\in[a,b]$.*

Lemma 1.4. *Let E be a Banach space. Assume that $T:E\to E$ is completely continuous operator and the set $V:=\{x\in E, x=\mu Tx,\ 0<\mu<1\}$ is bounded. Then T has a fixed point in E.*

We prove the following auxiliary result:

Lemma 1.5. *Suppose that* $\left(Q_i^k\right)_{i=1,\dots,m}^{k=1,\dots,n} \in C\left([0,1],\mathbb{R}\right)$, *and consider the problem*

$$D^{\alpha_k}x_k(t) = \sum_{i=1}^{m} Q_i^k(t),\ t \in J,\ n-1 < \alpha_k < n;\ k = 1,2,\dots,n,\ m,n \in \mathbb{N}^*, \tag{1.6}$$

associated with the conditions:

$$x_k(0) = x_k'(0) = \dots = x_k^{(n-2)}(0) = x_k^{(n-1)}(1) = 0,\ k = 1,2,\dots,n. \tag{1.7}$$

Then, we have

$$\begin{aligned} x_k(t) &= \sum_{i=1}^{m}\int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} Q_i^k(s)\,ds \\ &- \tfrac{t^{n-1}}{(n-1)!\Gamma(\alpha_k-n+1)} \sum_{i=1}^{m}\int_0^1 (1-s)^{\alpha_k-n} Q_i^k(s)\,ds,\ k = 1,2,\dots,n. \end{aligned} \tag{1.8}$$

Proof. We use Lemma 1.1, Lemma 1.2 and (1.6). So, we can write

$$x_k(t) = \sum_{i=1}^{m}\int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} Q_i^k(s)\,ds - c_0 - c_1 t - c_2 t^2 - \dots - c_{n-1}t^{n-1}, \tag{1.9}$$

where $c_j \in \mathbb{R}$, $j = 0,1,2,\dots,n-1$ and $n-1 < \alpha_k < n$, $k \in \mathbb{N}^*$.
For all $k = 1,2,\dots,n$, $j = 0,1,\dots,n-2$, we have

$$x_k^{(j)}(0) = -j!c_j.$$

Using Lemma 1.3 and the relation (1.7), we obtain:

$$c_j = \begin{cases} 0, & j = 0,1,\dots,n-2, \\ \sum_{i=1}^{m}\int_0^1 \frac{(1-s)^{\alpha_k-n}}{(n-1)!\Gamma(\alpha_k-n+1)} Q_i^k(s)\,ds, & j = n-1. \end{cases} \tag{1.10}$$

Substituting the values of c_j into (1.9), we obtain (1.8). This ends the proof of the auxiliary result. □

Now, let us introduce the Banach space:

$S := \left\{(x_1,x_2,\dots,x_n) : x_k \in C\left([0,1],\mathbb{R}\right),\ D^{\beta_k}x_k \in C\left([0,1],\mathbb{R}\right),\ k = 1,2,\dots,n.\right\}$ equipped with the norm

$$\|(x_1,x_2,\dots,x_n)\|_S = \max\left(\|x_1\|,\|x_2\|,\dots,\|x_n\|,\left\|D^{\beta_1}x_1\right\|,\left\|D^{\beta_2}x_2\right\|,\dots,\left\|D^{\beta_n}x_n\right\|\right), \tag{1.11}$$

such that,

$$\|x_k\| = \sup_{t\in J}|x_k(t)|,\ \left\|D^{\beta_k}x_k\right\| = \sup_{t\in J}\left|D^{\beta_k}x_k(t)\right|,\ k = 1,2,\dots,n.$$

2. Main Results

We begin this section by introducing the following hypotheses:

(H_1) : There exist nonnegative constants $\left(\lambda_i^k\right)_j$, $i = 1,...,m, k = 1,2,..., n$, $j = 1,2,...,2n$, such that for all $t \in [0,1]$ and all $(x_1, x_2, ..., x_{2n})$, $(y_1, y_2, ..., y_{2n}) \in \mathbb{R}^{2n}$, we have

$$\left|f_i^k(t, x_1, x_2, ..., x_{2n}) - f_i^k(t, y_1, y_2, ..., y_{2n})\right| \leq \sum_{j=1}^{2n} \left(\lambda_i^k\right)_j \left|x_j - y_j\right|. \tag{2.1}$$

(H_2) : The functions $f_i^k : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ are continuous for each $i = 1,2,...,m, k = 1,2,...,n, m, n \in \mathbb{N}^*$.

(H_3) : There exist nonnegative constants $\left(L_i^k\right)_{i=1,...,m}^{k=1,2...,n}$, such that:
for each $t \in J$ and all $(x_1, x_2, ..., x_{2n}) \in \mathbb{R}^{2n}$

$$\left|f_i^k(t, x_1, x_2, ..., x_{2n})\right| \leq L_i^k,\ i = 1,...,m,\ k = 1,2,...,n. \tag{2.2}$$

Then, we consider the following quantities:

$$\begin{aligned} O_k &= \frac{\Sigma_k}{\Gamma(\alpha_k+1)},\ O_k^* = \frac{\Sigma_k}{\Gamma(\alpha_k - \beta_k + 1)}; \\ \Sigma_k &= \sum_{i=1}^{m} \left(\left(\lambda_i^k\right)_1 + \left(\lambda_i^k\right)_2 + ... + \left(\lambda_i^k\right)_{2n}\right),\ k = 1,2,...,n, \end{aligned} \tag{2.3}$$

$$\begin{aligned} A_k &= \frac{1}{\Gamma(\alpha_k+1)} + \frac{1}{(n-1)!\Gamma(\alpha_k+2-n)},\ k = 1,2,...,n, \\ A_k^* &= \frac{1}{\Gamma(\alpha_k - \beta_k + 1)} + \frac{1}{\Gamma(n-\beta_k)\Gamma(\alpha_k+2-n)},\ k = 1,2,...,n. \end{aligned} \tag{2.4}$$

Now, we are ready to prove the first main result:

Theorem 2.1. *If the hypothesis* (H_1) *and the inequality*

$$\max_{1\leq k\leq n}\left(O_k, O_k^*\right) < 1 \tag{2.5}$$

are satisfied, then (1.1) *has a unique solution on* J.

Proof. Let us define the nonlinear operator $T : S \to S$ by

$$T(x_1, x_2, ..., x_n)(t) := (T_1(x_1, x_2, ..., x_n)(t), T_2(x_1, x_2, ..., x_n)(t), ..., T_n(x_1, x_2, ..., x_n)(t)),\ t \in J,$$

such that

$$T_k(x_1, x_2, ..., x_n)(t) =$$

(2.6)

$$\sum_{i=1}^{m}\int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}\varphi_i^k(s)\,ds - \frac{t^{n-1}}{(n-1)!\Gamma(\alpha_k-n+1)}\sum_{i=1}^{m}\int_0^1 (1-s)^{\alpha_k-n}\varphi_i^k(s)\,ds,\ k=1,2,...,n,$$

where,

$$\varphi_i^k(s) = f_i^k\left(s, x_1(s), x_2(s), ..., x_n(s), D^{\beta_1}x_1(s), D^{\beta_2}x_2(s), ..., D^{\beta_n}x_n(s)\right).$$

We show that the operator T is contractive:
Let $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in S$. Then, for each $k = 1, 2, ..., n$ and $t \in J$, we have:

$$\left|T_k(x_1, x_2, ..., x_n)(t) - T_k(y_1, y_2, ..., y_n)(t)\right| \leq$$

(2.7)

$$\frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)}\sup_{s\in J}\sum_{i=1}^{m}\left|\begin{array}{l} f_i^k\left(s, x_1(s), x_2(s), ..., x_n(s), D^{\beta_1}x_1(s), D^{\beta_2}x_2(s), ..., D^{\beta_n}x_n(s)\right) \\ -f_i^k\left(s, y_1(s), y_2(s), ..., y_n(s), D^{\beta_1}y_1(s), D^{\beta_2}y_2(s), ..., D^{\beta_n}y_n(s)\right). \end{array}\right|$$

By (H_1), we can write

$$\left\|T_k(x_1, x_2, ..., x_n) - T_k(y_1, y_2, ..., y_n)\right\| \leq \frac{1}{\Gamma(\alpha_k+1)}\sum_{i=1}^{m}\left(\left(\lambda_i^k\right)_1 + \left(\lambda_i^k\right)_2 + ... + \left(\lambda_i^k\right)_{2n}\right)$$

(2.8)

$$\times \max\left(\left\|x_1-y_1\right\|, \left\|x_2-y_2\right\|, ..., \left\|x_n-y_n\right\|, \left\|D^{\beta_1}(x_1-y_1)\right\|, \left\|D^{\beta_2}(x_2-y_2)\right\|, ..., \left\|D^{\beta_n}(x_n-y_n)\right\|\right).$$

Therefore, for all $k = 1, 2, ..., n$,

$$\left\|T_k(x_1, x_2, ..., x_n) - T_k(y_1, y_2, ..., y_n)\right\| \leq$$

(2.9)

$$\frac{\Sigma_k}{\Gamma(\alpha_k+1)}\left\|\left(x_1-y_1, x_2-y_2, ..., x_n-y_n, D^{\beta_1}(x_1-y_1), D^{\beta_2}(x_2-y_2), ..., D^{\beta_n}(x_n-y_n)\right)\right\|_S.$$

On the other hand,

$$\left|D^{\beta_k}T_k(x_1, x_2, ..., x_n)(t) - D^{\beta_k}T_k(y_1, y_2, ..., y_n)(t)\right| \leq$$

(2.10)

$$\frac{t^{\alpha_k-\beta_k}}{\Gamma(\alpha_k-\beta_k+1)}\sup_{s\in J}\sum_{i=1}^{m}\left|\begin{array}{l} f_i^k\left(s, x_1(s), x_2(s), ..., x_n(s), D^{\beta_1}x_1(s), D^{\beta_2}x_2(s), ..., D^{\beta_n}x_n(s)\right) \\ -f_i^k\left(s, y_1(s), y_2(s), ..., y_n(s), D^{\beta_1}y_1(s), D^{\beta_2}y_2(s), ..., D^{\beta_n}y_n(s)\right) \end{array}\right|,$$

where $k = 1, 2, ..., n$.

Then, for $k = 1, 2, ..., n$, we have

$$\left\| D^{\beta_k} T_k (x_1, x_2, ..., x_n) - D^{\beta_k} T_k (y_1, y_2, ..., y_n) \right\| \leq$$

(2.11)
$$\frac{\Sigma_k}{\Gamma(\alpha_k - \beta_k + 1)} \left\| \left(x_1 - y_1, x_2 - y_2, ..., x_n - y_n, D^{\beta_1}(x_1 - y_1), D^{\beta_2}(x_2 - y_2), ..., D^{\beta_n}(x_n - y_n) \right) \right\|_S .$$

Using (2.9) and (2.11), we get

$$\left\| T_k (x_1, x_2, ..., x_n) - T_k (y_1, y_2, ..., y_n) \right\|_S \leq$$

(2.12)
$$\max_{1 \leq k \leq n} \left(O_k, O_k^* \right) \left\| \left(x_1 - y_1, ..., x_n - y_n, D^{\beta_1}(x_1 - y_1), ..., D^{\beta_n}(x_n - y_n) \right) \right\|_S , \ k = 1, 2, ..., n.$$

Thus, by (2.5), we deduce that the operator T is contractive. Therefore, by the Banach fixed point theorem, T has a unique fixed point which is a solution of the system (1.1). □

We also prove the result:

Theorem 2.2. *Assume that* f_i^k, $i = 1, 2, ..., m$, $k = 1, 2, ..., n$ *satisfy* (H_2) *and* (H_3). *Then the nonlinear fractional system* (1.1) *has at least one solution on* J.

Proof. The proof will be given in two steps:

A : We show that T is completely continuous:

We begin by proving that T maps bounded sets into bounded sets in S : Let us consider the set $B_\delta := \{(x_1, x_2, ..., x_n) \in S;\ \|(x_1, x_2, ..., x_n)\|_S \leq \delta,\ \delta > 0\}$ and $(x_1, x_2, ..., x_n) \in B_\delta$. Then, for each $t \in J$, $k = 1, 2, ..., n$, and using (H_3), we can obtain

$$\|T_k (x_1, x_2, ..., x_n)\|$$

$$\begin{aligned}
&\leq \frac{t^{\alpha_k}}{\Gamma(\alpha_k + 1)} \times \sup_{s \in J} \sum_{i=1}^{m} \left| f_i^k \left(s, x_1(s), x_2(s), ..., x_n(s), D^{\beta_1} x_1(s), D^{\beta_2} x_2(s), ..., D^{\beta_n} x_n(s) \right) \right| \\
&\quad + \frac{1}{(n-1)!\Gamma(\alpha_k + 2 - n)} \sup_{s \in J} \sum_{i=1}^{m} \left| f_i^k \left(s, x_1(s), x_2(s), ..., x_n(s), D^{\beta_1} x_1(s), D^{\beta_2} x_2(s), ..., D^{\beta_n} x_n(s) \right) \right| \\
&\leq \left(\frac{1}{\Gamma(\alpha_k + 1)} + \frac{1}{(n-1)!\Gamma(\alpha_k + 2 - n)} \right) \sum_{i=1}^{m} L_i^k \\
&\leq A_k \sum_{i=1}^{m} L_i^k \qquad (2.13)
\end{aligned}$$

and

$$\left\| D^{\beta_k} T_k (x_1, x_2, ..., x_n) \right\|$$

$$\begin{aligned}
\leq \quad & \frac{t^{\alpha_k - \beta_k}}{\Gamma(\alpha_k - \beta_k + 1)} \times \sup_{s \in J} \sum_{i=1}^{m} \left| f_i^k \left(s, x_1(s), x_2(s), ..., x_n(s), D^{\beta_1} x_1(s), D^{\beta_2} x_2(s), ..., D^{\beta_n} x_n(s) \right) \right| \\
& + \frac{t^{n-\beta_k-1}}{\Gamma(n-\beta_k)\Gamma(\alpha_k+2-n)} \sup_{s \in J} \sum_{i=1}^{m} f_i^k \left(s, x_1(s), x_2(s), ..., x_n(s), D^{\beta_1} x_1(s), D^{\beta_2} x_2(s), ..., D^{\beta_n} x_n(s) \right) \\
\leq \quad & \left(\frac{1}{\Gamma(\alpha_k - \beta_k + 1)} + \frac{1}{\Gamma(n-\beta_k)\Gamma(\alpha_k+2-n)} \right) \sum_{i=1}^{m} L_i^k \\
\leq \quad & A_k^* \sum_{i=1}^{m} L_i^k \qquad (2.14)
\end{aligned}$$

Hence,

$$\|T(x_1, x_2, ..., x_n)\|_S \leq \max\left(A_k, A_k^*\right) \sum_{i=1}^{m} L_i^k < \infty.$$

This means that T maps bounded sets into bounded sets in S.

Thanks to (H_2), the operator T is continuous on S. On the other hand, for any $0 \leq t_1 < t_2 \leq 1$ and $(x_1, x_2, ..., x_n) \in B_\delta$, we have:

$$\|T_k(x_1, x_2, ..., x_n)(t_2) - T_k(x_1, x_2, ..., x_n)(t_1)\| \leq$$

(2.15)

$$\left(\frac{1}{\Gamma(\alpha_k+1)} \left(2(t_2 - t_1)^{\alpha_k} + \left(t_2^{\alpha_k} - t_1^{\alpha k} \right) \right) + \frac{1}{(n-1)!\Gamma(\alpha_k+2-n)} \left(t_2^{n-1} - t_1^{n-1} \right) \right) \sum_{i=1}^{m} L_i^k$$

and

$$\left\| D^{\beta_k} T_k(x_1, x_2, ..., x_n)(t_2) - D^{\beta_k} T_k(x_1, x_2, ..., x_n)(t_1) \right\| \leq$$

(2.16)

$$\left(\frac{1}{\Gamma(\alpha_k - \beta_k + 1)} \left(2(t_2 - t_1)^{\alpha_k - \beta_k} + \left(t_2^{\alpha_k - \beta_k} - t_1^{\alpha_k - \beta_k} \right) \right) + \frac{1}{\Gamma(n-\beta_k)\Gamma(\alpha_k+2-n)} \left(t_2^{n-\beta_k-1} - t_1^{n-\beta_k-1} \right) \right),$$

where, $k = 1, 2, ..., n$.

The right-hand sides of (2.15) and (2.16) are independent of $(x_1, x_2, ..., x_n) \in B_\delta$ and tend to zero as $t_2 - t_1 \to 0$. Thus T is equi-continuous. Finally, we can see by the above arguments that T is a completely continuous operator.

B : We consider the set $\Omega := \{(x_1, x_2, ..., x_n) \in S,\ (x_1, x_2, ..., x_n) = \mu T(x_1, x_2, ..., x_n),\ 0 < \mu < 1\}$ and show that is bounded:

Let $(x_1, x_2, ..., x_n) \in \Omega$, then $x_k(t) = \mu T_k(x_1, x_2, ..., x_n)(t)$. Thus, for each $t \in J$ and corresponding to (2.13) and (2.14), we have:

$$\|x_k\| \leq \mu A_k \sum_{i=1}^{m} L_i^k;\ \left\|D^{\beta_k} x_k\right\| \leq A_k^* \sum_{i=1}^{m} L_i^k,\ k = 1, 2, ..., n, \tag{2.17}$$

which implies

$$\begin{aligned} \|(x_1, x_2, ..., x_n)\|_S &\leq \mu \max\left(A_1 \sum_{i=1}^{m} L_i^1, A_2 \sum_{i=1}^{m} L_i^2, ..., A_n \sum_{i=1}^{m} L_i^n, A_1^* \sum_{i=1}^{m} L_i^1, A_2^* \sum_{i=1}^{m} L_i^2, ..., A_n^* \sum_{i=1}^{m} L_i^n\right). \\ (2.18) &< \infty. \end{aligned}$$

Therefore, Ω is bounded.
Consequently by the steps A, B and using lemma 2.4, we deduce that T has a fixed point which is a solution to (1.1). Theorem 2.2 is thus proved. □

3. Examples

We present two examples to illustrate our main results.

Example 3.1. We begin with the system:
(3.1)

$$\begin{cases} D^{\frac{7}{3}} x_1(t) = \dfrac{\left|x_1(t)+x_2(t)+x_3(t)+D^{\frac{4}{3}} x_1(t)+D^{\frac{3}{2}} x_2(t)+D^{\frac{5}{3}} x_3(t)\right|}{9\pi^3\left(1+\left|x_1(t)+x_2(t)+x_3(t)+D^{\frac{4}{3}} x_1(t)+D^{\frac{3}{2}} x_2(t)+D^{\frac{5}{3}} x_3(t)\right|\right)} \\ +\frac{1}{64\pi^2 e}\left(\frac{\sin(x_1(t))+\sin(x_2(t))+\sin(x_3(t))}{e^{t+1}} + \cos\left(D^{\frac{4}{3}} x_1(t)\right) + \cos\left(D^{\frac{3}{2}} x_2(t)\right) - \sin\left(D^{\frac{5}{3}} x_3(t)\right)\right),\ t \in]0, 1[, \\ D^{\frac{9}{4}} x_2(t) = \frac{1}{24\pi^3 e^{2t+1}}\left(\sin\left(D^{\frac{4}{3}} x_1(t)\right) + \sin\left(D^{\frac{3}{2}} x_2(t)\right) + \sin\left(D^{\frac{5}{3}} x_3(t)\right) + \frac{|x_1(t)+x_2(t)+x_3(t)|}{\pi+|x_1(t)+x_2(t)+x_3(t)|}\right) \\ +\frac{t^2}{16\pi^2 e^{t^2+1}}\left(\frac{\sin(x_1(t))+\cos(x_2(t))+\cos(x_3(t))-\cos\left(D^{\frac{4}{3}} x_1(t)\right)+\sin\left(D^{\frac{3}{2}} x_2(t)\right)+\sin\left(D^{\frac{5}{3}} x_3(t)\right)}{e+\sin(x_1(t))+\cos(x_2(t))+\cos(x_3(t))-\cos\left(D^{\frac{4}{3}} x_1(t)\right)+\sin\left(D^{\frac{3}{2}} x_2(t)\right)+\sin\left(D^{\frac{5}{3}} x_3(t)\right)}\right),\ t \in]0, 1[, \\ D^{\frac{8}{3}} x_3(t) = \frac{1}{32\pi}\left(\cos(x_1(t)) + \sin(x_2(t)) + \sin(x_3(t)) + \frac{\left|D^{\frac{4}{3}} x_1(t)+D^{\frac{3}{2}} x_2(t)+D^{\frac{5}{3}} x_3(t)\right|}{\left(1+\left|D^{\frac{4}{3}} x_1(t)+D^{\frac{3}{2}} x_2(t)+D^{\frac{5}{3}} x_3(t)\right|\right)}\right) \\ +\frac{1}{16\pi(t+e)^2}\left(\sin x_1(t) + \sin D^{\frac{4}{3}} x_1(t) + \sin D^{\frac{3}{2}} x_2(t)\ \frac{\left|x_2(t)+x_3(t)+D^{\frac{5}{3}} x_3(t)\right|}{3\pi e\left(1+\left|x_2(t)+x_3(t)+D^{\frac{5}{3}} x_3(t)\right|\right)}\right),\ t \in]0, 1[\\ x_k(0) = x_k'(0) = x_k''(1) = 0,\ k = 1, 2, 3. \end{cases}$$

For this example, we have:
$n = 3, m = 2, \alpha_1 = \frac{7}{3}, \alpha_2 = \frac{9}{4}, \alpha_3 = \frac{8}{3}, \beta_1 = \frac{4}{3}, \beta_2 = \frac{3}{2}, \beta_3 = \frac{5}{3}, J = [0, 1]$.
On the other hand,

$$f_1^1(t, x_1, x_2, x_3, x_4, x_5, x_6) = \frac{|x_1 + x_2 + x_3 + x_4 + x_5 + x_6|}{9\pi^3(1 + |x_1 + x_2 + x_3 + x_4 + x_5 + x_6|)}, \tag{3.2}$$

$$f_2^1(t, x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{64\pi^2 e}\left(\frac{\sin x_1 + \sin x_2 + \sin x_3}{e^{t+1}} + \cos x_4 + \cos x_5 - \sin x_6\right), \tag{3.3}$$

(3.4) $$f_1^2(t,x_1,x_2,x_3,x_4,x_5,x_6)=\frac{1}{24\pi^3e^{2t+1}}\left(\sin x_4+\sin x_5+\sin x_6+\frac{|x_1+x_2+x_3|}{\pi+|x_1+x_2+x_3|}\right),$$

(3.5)
$$f_2^2(t,x_1,x_2,x_3,x_4,x_5,x_6)=\frac{t^2}{16\pi^2e^{t^2+1}}\left(\frac{\sin x_1+\cos x_2+\cos x_3-\cos x_4+\sin x_5+\sin x_6}{e+\sin x_1+\cos x_2+\cos x_3-\cos x_4+\sin x_5+\sin x_6}\right),$$

(3.6) $$f_1^3(t,x_1,x_2,x_3,x_4,x_5,x_6)=\frac{1}{32\pi}\left(\cos x_1+\sin x_2+\sin x_3+\frac{|x_4+x_5+x_6|}{(1+|x_4+x_5+x_6|)}\right)$$

and
(3.7)
$$f_2^3(t,x_1,x_2,x_3,x_4,x_5,x_6)=\frac{1}{16\pi(t+e)^2}\left(\sin x_1+\sin x_4+\sin x_5+\frac{|x_2+x_3+x_6|}{3\pi e(1+|x_2+x_3+x_6|)}\right).$$

So, for $t\in[0,1]$ and $(x_1,x_2,x_3,x_4,x_5,x_6),(y_1,y_2,y_3,y_4,y_5,y_6)\in\mathbb{R}^6$, we have:

$$\left|f_1^1(t,x_1,x_2,x_3,x_4,x_5,x_6)-f_1^1(t,y_1,y_2,y_3,y_4,y_5,y_6)\right|\le$$

(3.8) $$\frac{1}{9\pi^3}|x_1-y_1|+\frac{1}{9\pi^3}|x_2-y_2|+\frac{1}{9\pi^3}|x_3-y_3|+\frac{1}{9\pi^3}|x_4-y_4|+\frac{1}{9\pi^3}|x_5-y_5|+\frac{1}{9\pi^3}|x_6-y_6|,$$

$$\left|f_2^1(t,x_1,x_2,x_3,x_4,x_5,x_6)-f_2^1(t,y_1,y_2,y_3,y_4,y_5,y_6)\right|\le$$

(3.9)
$$\frac{1}{64\pi^2e^2}|x_1-y_1|+\frac{1}{64\pi^2e^2}|x_2-y_2|+\frac{1}{64\pi^2e^2}|x_3-y_3|+\frac{1}{64\pi^2e}|x_4-y_4|+\frac{1}{64\pi^2e}|x_5-y_5|+\frac{1}{64\pi^2e}|x_6-y_6|,$$

$$\left|f_1^2(t,x_1,x_2,x_3,x_4,x_5,x_6)-f_1^2(t,y_1,y_2,y_3,y_4,y_5,y_6)\right|\le$$

(3.10)
$$\frac{1}{24\pi^2e}|x_1-y_1|+\frac{1}{24\pi^2e}|x_2-y_2|+\frac{1}{24\pi^2e}|x_3-y_3|+\frac{1}{24\pi^3e}|x_4-y_4|+\frac{1}{24\pi^3e}|x_5-y_5|+\frac{1}{24\pi^3e}|x_6-y_6|,$$

$$\left|f_2^2(t,x_1,x_2,x_3,x_4,x_5,x_6)-f_2^2(t,y_1,y_2,y_3,y_4,y_5,y_6)\right|\le$$

(3.11)
$$\frac{1}{16\pi^2}|x_1-y_1|+\frac{1}{16\pi^2}|x_2-y_2|+\frac{1}{16\pi^2}|x_3-y_3|+\frac{1}{16\pi^2}|x_4-y_4|+\frac{1}{16\pi^2}|x_5-y_5|+\frac{1}{16\pi^2}|x_6-y_6|,$$

$$\left|f_1^3(t,x_1,x_2,x_3,x_4,x_5,x_6)-f_1^3(t,y_1,y_2,y_3,y_4,y_5,y_6)\right|\le$$

(3.12)
$$\frac{1}{32\pi}|x_1-y_1|+\frac{1}{32\pi}|x_2-y_2|+\frac{1}{32\pi}|x_3-y_3|+\frac{1}{32\pi}|x_4-y_4|+\frac{1}{32\pi}|x_5-y_5|+\frac{1}{32\pi}|x_6-y_6|,$$

and

$$\left|f_2^3(t,x_1,x_2,x_3,x_4,x_5,x_6)-f_2^3(t,y_1,y_2,y_3,y_4,y_5,y_6)\right|\le$$

(3.13)
$$\frac{1}{16\pi e^2}|x_1-y_1|+\frac{1}{48\pi^2e^3}|x_2-y_2|+\frac{1}{48\pi^2e^3}|x_3-y_3|+\frac{1}{16\pi e^2}|x_4-y_4|+\frac{1}{16\pi e^2}|x_5-y_5|+\frac{1}{48\pi^2e^3}|x_6-y_6|.$$

We can take:

(3.14) $$\left(\lambda_1^1\right)_1=\left(\lambda_1^1\right)_2=\left(\lambda_1^1\right)_3=\left(\lambda_1^1\right)_4=\left(\lambda_1^1\right)_5=\left(\lambda_1^1\right)_6=\frac{1}{9\pi^3},$$

$$\left(\lambda_2^1\right)_1 = \left(\lambda_2^1\right)_2 = \left(\lambda_2^1\right)_3 = \frac{1}{64\pi^2 e^2},\ \left(\lambda_2^1\right)_4 = \left(\lambda_2^1\right)_5 = \left(\lambda_2^1\right)_6 = \frac{1}{64\pi^2 e}, \tag{3.15}$$

$$\left(\lambda_1^2\right)_1 = \left(\lambda_1^2\right)_2 = \left(\lambda_1^2\right)_3 = \frac{1}{24\pi^2 e},\ \left(\lambda_1^2\right)_4 = \left(\lambda_1^2\right)_5 = \left(\lambda_1^2\right)_6 = \frac{1}{24\pi^3 e}, \tag{3.16}$$

$$\left(\lambda_2^2\right)_1 = \left(\lambda_2^2\right)_2 = \left(\lambda_2^2\right)_3 = \left(\lambda_2^2\right)_4 = \left(\lambda_2^2\right)_5 = \left(\lambda_2^2\right)_6 = \frac{1}{16\pi^2}, \tag{3.17}$$

$$\left(\lambda_1^3\right)_1 = \left(\lambda_1^3\right)_2 = \left(\lambda_1^3\right)_3 = \left(\lambda_1^3\right)_4 = \left(\lambda_1^3\right)_5 = \left(\lambda_1^3\right)_6 = \frac{1}{32\pi}, \tag{3.18}$$

and

$$\left(\lambda_2^3\right)_1 = \left(\lambda_2^3\right)_4 = \left(\lambda_2^3\right)_5 = \frac{1}{16\pi e^2},\ \left(\lambda_2^3\right)_2 = \left(\lambda_2^3\right)_3 = \left(\lambda_2^3\right)_6 = \frac{1}{48\pi^2 e^3}. \tag{3.19}$$

It follows that:

$$\Sigma_1 = 0.023891,\ \Sigma_2 = 0.044138,\ \Sigma_3 = 0.068076. \tag{3.20}$$

Since

$$\Gamma(\alpha_1 + 1) = 2.778062,\ \Gamma(\alpha_2 + 1) = 2.549257,\ \Gamma(\alpha_3 + 1) = 4.012356, \tag{3.21}$$

$$\Gamma(\alpha_1 - \beta_1 + 1) = 1,\ \Gamma(\alpha_2 - \beta_2 + 1) = 0.919062,\ \Gamma(\alpha_3 - \beta_3 + 1) = 1, \tag{3.22}$$

then it yields that:

$$\begin{aligned} O_1 &= 0.008599,\ O_2 = 0.009372,\ O_3 = 0.005954, \\ O_1^* &= 0.023891,\ O_2^* = 0.048025,\ O_3^* = 0.068076, \end{aligned} \tag{3.23}$$

$$\max\left(O_1,\ O_2, O_3, O_1^*,\ O_2^*, O_3^*\right) < 1.$$

The condition (2.5) is satisfied. So by Theorem 2.1, we deduce that the system (3.1) has a unique solution on $[0, 1]$.

Example 3.2. To illustrate the second main result, let us consider the system:

$$\left\{ \begin{array}{c} D^{\frac{9}{4}} x_1(t) = \frac{\pi(t+1)\sin\left(D^{\frac{3}{2}} x_1(t) + D^{\frac{4}{3}} x_2(t) + D^{\frac{5}{4}} x_3(t)\right)}{2 - \cos(x_1(t) + x_2(t) + x_3(t))} \\ + \frac{e^t\left(t^2+1\right)}{2\pi + \cos\left(x_2(t) + D^{\frac{4}{3}} x_2(t)\right)\cos\left(x_3(t) + D^{\frac{5}{4}} x_3(t)\right) + \sin^2\left(x_1(t) D^{\frac{3}{2}} x_1(t)\right)},\ t \in\ ]0,1[\,, \\ D^{\frac{7}{3}} x_2(t) = \frac{e^2 \sin(x_1(t) + x_2(t) + x_3(t))}{2\pi + \cos\left(D^{\frac{3}{2}} x_1(t) + D^{\frac{4}{3}} x_2(t) + D^{\frac{5}{4}} x_3(t)\right)} \\ + \frac{3t^2\cos(x_2(t) + x_3(t))}{e^{t^2+1} - \cos\left(x_1(t) - D^{\frac{3}{2}} x_1(t) + D^{\frac{4}{3}} x_2(t) + D^{\frac{5}{4}} x_3(t)\right)},\ t \in\ ]0,1[\,, \\ D^{\frac{5}{2}} x_3(t) = \frac{\sin(x_3(t))}{2e + cox\left(x_1(t) + x_2(t) + D^{\frac{3}{2}} x_1(t) + D^{\frac{4}{3}} x_2(t) + D^{\frac{5}{4}} x_3(t)\right)} \\ + \cos(x_1(t) + x_2(t) + x_3(t))\sin\left(D^{\frac{3}{2}} x_1(t) + D^{\frac{4}{3}} x_2(t) + D^{\frac{5}{4}} x_3(t)\right),\ t \in\ ]0,1[\,, \\ x_k(0) = x_k'(0) = x_k''(1) = 0,\ k = 1,2,3. \end{array} \right. \tag{3.24}$$

We have: $n=3, m=2, \alpha_1=\frac{9}{4}, \beta_1=\frac{3}{2}; \alpha_2=\frac{7}{3}, \beta_2=\frac{4}{3}, \alpha_3=\frac{5}{2}, \beta_3=\frac{5}{4}, J=[0,1]$.

Since

$$\left|f_1^1(t,x_1,x_2,x_3,x_4,x_5,x_6)\right|=\left|\frac{\pi(t+1)\sin(x+x_5+x_6)}{2-\cos(x_1+x_2+x_3)}\right|\leq 2\pi, \tag{3.25}$$

$$\left|f_2^1(t,x_1,x_2,x_3,x_4,x_5,x_6)\right|=\left|\frac{e^t(t^2+1)}{2\pi+\cos(x_2+x_5)\cos(x_3+x_6)+\sin^2(x_1x_4)}\right|\leq\frac{e}{\pi}, \tag{3.26}$$

$$\left|f_1^2(t,x_1,x_2,x_3,x_4,x_5,x_6)\right|=\left|\frac{e^2\sin(x_1+x_2+x_3)}{2\pi+\cos(x_4+x_5+x_6)}\right|\leq\frac{e^2}{2\pi-1}, \tag{3.27}$$

$$\left|f_2^2(t,x_1,x_2,x_3,x_4,x_5,x_6)\right|=\left|\frac{3t^2\cos(x_2+x_3)}{e^{t^2+1}-\cos(x_1-x_4+x_5+x_6)}\right|\leq\frac{3}{e-1}, \tag{3.28}$$

$$\left|f_1^3(t,x_1,x_2,x_3,x_4,x_5,x_6)\right|=\left|\frac{\sin x_3}{2e+cox(x_1+x_2+x_4+x_5+x_6)}\right|\leq\frac{1}{2e-1}$$

and

$$\left|f_2^3(t,x_1,x_2,x_3,x_4,x_5,x_6)\right|=\left|\cos(x_1+x_2+x_3)\sin(x_4+x_5+x_6)\right|\leq 1,$$

then, we can see that the functions $\left(f_i^k\right)_{i=1,2}^{k=1,2,3}$ are continuous and bounded on $[0,1]\times\mathbb{R}^6$. So, by Theorem 2.2, the system (3.24) has at least one solution on $[0,1]$.

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