

NEW INEQUALITIES OF OSTROWSKI TYPE FOR  
CO-ORDINATED CONVEX FUNCTIONS VIA GENERALIZED  
FRACTIONAL INTEGRALS \*

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**Abstract.** In this paper, we have established new inequalities of Ostrowski type for co-ordinated convex function by using generalized fractional integral. We have also discussed some special cases of our established results.

**Keywords:** inequalities of Ostrowski type; convex function; generalized fractional integral.

1. Introduction

In 1938, A. Ostrowski established the following fascinating integral inequality [11].

**Theorem 1.1.** [11] *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable function on  $(a, b)$  whose derivative is bounded on  $(a, b)$ , i.e.,  $\|f'(t)\|_{\infty} := \sup |f'(t)| < \infty$ , for all  $t \in (a, b)$ . Then we have the following integral inequality:*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})}{(b-a)^2} \right] (b-a) \|f'\|_{\infty},$$

for all  $x \in [a, b]$ . The  $\frac{1}{4}$  is the best possible.

The inequality (1.1) can be rewritten in equivalent form as:

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left[ \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \|f'\|_{\infty}.$$

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Since 1938 when A. Ostrowski proved his famous inequality, (see, [11]), many mathematicians have been working about and around it, in many different directions and with a lot of applications in Numerical Analysis and Probability, etc.

Several generalizations of the Ostrowski integral inequality for mappings of bounded variation, Lipschitzian, monotonic, absolutely continuous, convex mappings and  $n$ -times differentiable mappings with error estimates for some special means and for some numerical quadrature rules have been considered by many authors. For recent results and generalizations concerning Ostrowski's inequality see [1]-[4], [6]-[15] and the references therein.

Let us consider now a bidimensional interval  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$  and  $c < d$ , a mapping  $f : \Delta \rightarrow \mathbb{R}^2$  is said to be convex on  $\Delta$  if the following inequality holds:

$$(1.3) \quad f(tx+(1-t)z, ty+(1-t)w) \leq tf(x, y)+(1-t)f(z, w), \quad \forall (x, y), (z, w) \in \Delta \text{ and } t \in [0, 1].$$

The mapping  $f$  is said to be concave on co-ordinates  $\Delta$  if (1.3) holds in reversed direction.

A formal definition of co-ordinated convex (concave) functions may be expressed as:

**Definition 1.1.** [17] A function  $f : \Delta \rightarrow \mathbb{R}$  is called co-ordinated convex on  $\Delta$ , for all  $(x, u), (y, v) \in \Delta$  and  $t, s \in [0, 1]$ , if it satisfies the following inequality:

$$(1.4) \quad f(tx + (1-t)y, su + (1-s)v) \\ \leq ts f(x, u) + t(1-s)f(x, v) + s(1-t)f(y, u) + (1-t)(1-s)f(y, v).$$

The mapping  $f$  is a co-ordinated concave on  $\Delta$  if the inequality (1.4) holds in reversed direction for all  $t, s \in [0, 1]$  and  $(x, u), (y, v) \in \Delta$ .

In [5], Dragomir proved the following inequalities which is Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane  $\mathbb{R}^2$ .

**Theorem 1.2.** *Suppose that  $f : \Delta \rightarrow \mathbb{R}$  is co-ordinated convex, then we have the*

following inequalities:

$$\begin{aligned}
 (1.5) \quad & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
 & \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\
 & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\
 & \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\
 & \quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\
 & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
 \end{aligned}$$

The above inequalities are sharp. The inequalities in (1.5) holds in reversed direction if the mapping  $f$  is a co-ordinated concave.

In [10], Latif et al. established following Ostrowski type inequalities for co-ordinated convex functions:

**Theorem 1.3.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b, c < d, a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$  is convex on co-ordinates on  $\Delta$  and  $\left| \frac{\partial^2 f}{\partial s \partial t} \right| \leq M, (x, y) \in \Delta$ , then the following inequality holds:

$$\begin{aligned}
 (1.6) \quad & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A_1 \right| \\
 & \leq M \left[ \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \left[ \frac{(y-c)^2 + (d-y)^2}{2(d-c)} \right],
 \end{aligned}$$

where

$$A_1 = \frac{1}{d-c} \int_c^d f(x, v) dv + \frac{1}{b-a} \int_c^d f(u, y) dy.$$

**Theorem 1.4.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b, c < d, a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is convex

on co-ordinates on  $\Delta$ ,  $p, q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$  and  $\left| \frac{\partial^2 f}{\partial s \partial t}(x, y) \right| \leq M$ ,  $(x, y) \in \Delta$ , then the following inequality holds:

$$(1.7) \quad \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A_1 \right| \\ \leq \frac{M}{(1+p)^{\frac{2}{p}}} \left[ \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \left[ \frac{(y-c)^2 + (d-y)^2}{2(d-c)} \right],$$

where  $A_1$  is defined in Theorem 1.3.

**Theorem 1.5.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b$ ,  $c < d$ ,  $a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is convex on co-ordinates on  $\Delta$ ,  $q \geq 1$  and  $\left| \frac{\partial^2 f}{\partial s \partial t}(x, y) \right| \leq M$ ,  $(x, y) \in \Delta$ , then the following inequality holds:

$$(1.8) \quad \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A_1 \right| \\ \leq \frac{M}{4} \left[ \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \left[ \frac{(y-c)^2 + (d-y)^2}{2(d-c)} \right],$$

where  $A_1$  is defined in Theorem 1.3.

**Theorem 1.6.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b$ ,  $c < d$ ,  $a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is concave on co-ordinates on  $\Delta$ ,  $p, q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , then the following inequality holds:

$$(1.9) \quad \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A_1 \right| \\ \leq \frac{1}{(1+p)^{\frac{2}{p}}(b-a)(d-c)} \left[ (x-a)^2 \left\{ (y-c)^2 \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+a}{2}, \frac{y+c}{2} \right) \right| \right. \right. \\ \left. \left. + (d-y)^2 \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+a}{2}, \frac{y+d}{2} \right) \right| \right\} \right. \\ \left. + (b-x)^2 \left\{ (y-c)^2 \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{y+c}{2} \right) \right| \right. \right. \\ \left. \left. + (d-y)^2 \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{y+d}{2} \right) \right| \right\} \right],$$

where  $A_1$  is defined in Theorem 1.3.

In [9], Latif and Hussain established following Ostrowski type inequalities for co-ordinated convex function by using fractional integral:

**Theorem 1.7.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b, c < d, a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$  is convex on co-ordinates on  $\Delta$  and  $\left| \frac{\partial^2 f}{\partial s \partial t} \right| \leq M, (x, y) \in \Delta$ , then the following inequality holds for fractional integrals, with  $\alpha, \beta > 0$ :

$$(1.10) \quad \left| \frac{[(x-a)^\alpha + (b-x)^\alpha][(y-c)^\beta + (d-y)^\beta]}{(b-a)(d-c)} f(x, y) + A_2 \right| \leq \frac{(\alpha\beta + 2\alpha + 2\beta + 4)[(x-a)^\alpha + (b-x)^\alpha][(y-c)^\beta + (d-y)^\beta]}{(b-a)(d-c)(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)} M,$$

where

$$A_2 = \frac{\Gamma(\alpha+1)\Gamma(\beta+a)}{(b-a)(d-c)} \left[ J_{x^-, y^-}^{\alpha, \beta} f(a, c) + J_{x^-, y^+}^{\alpha, \beta} f(a, d) + J_{x^+, y^-}^{\alpha, \beta} f(b, c) + J_{x^+, y^+}^{\alpha, \beta} f(b, d) \right] - \frac{[(x-a)^\alpha + (b-x)^\alpha] \Gamma(\beta+1)}{(b-a)(d-c)} \left[ J_{y^-}^\beta f(x, c) + J_{y^+}^\beta f(x, d) \right] - \frac{[(y-c)^\beta + (d-y)^\beta] \Gamma(\alpha+1)}{(b-a)(d-c)} \left[ J_{x^-}^\alpha f(a, y) + J_{x^+}^\alpha f(b, y) \right].$$

**Theorem 1.8.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b, c < d, a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is convex on co-ordinates on  $\Delta, p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$  and  $\left| \frac{\partial^2 f}{\partial s \partial t}(x, y) \right| \leq M, (x, y) \in \Delta$ , then the following inequality holds for fractional integrals, with  $\alpha, \beta > 0$ :

$$(1.11) \quad \left| \frac{[(x-a)^\alpha + (b-x)^\alpha][(y-c)^\beta + (d-y)^\beta]}{(b-a)(d-c)} f(x, y) + A_2 \right| \leq \frac{1}{(\alpha p + 1)^{\frac{1}{p}} (\beta p + 1)^{\frac{1}{p}}} \frac{[(x-a)^\alpha + (b-x)^\alpha][(y-c)^\beta + (d-y)^\beta]}{(b-a)(d-c)} M,$$

where  $A_2$  is defined in Theorem 1.7.

**Theorem 1.9.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b, c < d, a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is convex on co-ordinates on  $\Delta, q \geq 1$  and  $\left| \frac{\partial^2 f}{\partial s \partial t}(x, y) \right| \leq M, (x, y) \in \Delta$ , then the following inequality holds for fractional integrals, with  $\alpha, \beta > 0$ :

$$(1.12) \quad \left| \frac{[(x-a)^\alpha + (b-x)^\alpha][(y-c)^\beta + (d-y)^\beta]}{(b-a)(d-c)} f(x, y) + A_2 \right| \leq \frac{1}{(\alpha p + 1)^{\frac{1}{p}} (\beta p + 1)^{\frac{1}{p}}} \frac{[(x-a)^\alpha + (b-x)^\alpha][(y-c)^\beta + (d-y)^\beta]}{(b-a)(d-c)} M,$$

where  $A_2$  is defined in Theorem 1.7.

**Theorem 1.10.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b$ ,  $c < d$ ,  $a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is concave on co-ordinates on  $\Delta$ ,  $p, q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , then the following inequality holds for fractional integrals with  $\alpha, \beta > 0$ :

$$(1.13) \quad \left| \frac{[(x-a)^\alpha + (b-x)^\alpha][(y-c)^\beta + (d-y)^\beta]}{(b-a)(d-c)} f(x, y) + A_2 \right| \\ \leq \frac{1}{(1+\alpha p)^{\frac{1}{p}}(1+\beta p)^{\frac{1}{p}}(b-a)(d-c)} \\ \times \left[ (x-a)^{\alpha+1} \left\{ (y-c)^{\beta+1} \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+a}{2}, \frac{y+c}{2} \right) \right| \right. \right. \\ \left. \left. + (d-y)^{\beta+1} \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+a}{2}, \frac{y+d}{2} \right) \right| \right. \right. \\ \left. \left. + (b-x)^{\alpha+1} \left\{ (y-c)^{\beta+1} \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{y+c}{2} \right) \right| \right. \right. \right. \\ \left. \left. \left. + (d-y)^{\beta+1} \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{y+d}{2} \right) \right| \right\} \right] \right],$$

where  $A_2$  is defined in Theorem 1.7.

In [16], Sarikaya and Ertugral defined a new left-sided and right-sided generalized fractional integrals as follows:

$$(1.14) \quad {}_{a+}I_\varphi f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(x-t)}{x-t} f(t) dt, \quad x > a$$

$$(1.15) \quad {}_{b-}I_\varphi f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(t-x)}{t-x} f(t) dt, \quad x < b$$

respectively, where  $\varphi : [0, \infty) \rightarrow [0, \infty)$  a function which satisfies  $\int_0^1 \frac{\varphi(t)}{t} dt < \infty$ .

In [17], Yildirim et al. defined generalized fractional integrals for two variable functions as follows:

**Definition 1.2.** [17] Let  $f \in L_1([a, b] \times [c, d])$ . The generalized fractional integrals  ${}_{a+,c+}I_{\varphi,\psi}$ ,  ${}_{a+,d-}I_{\varphi,\psi}$ ,  ${}_{b-,c+}I_{\varphi,\psi}$  and  ${}_{b-,d-}I_{\varphi,\psi}$  are defined by

$$(1.16) \quad {}_{a+,c+}I_{\varphi,\psi} f(x, y) = \int_a^x \int_c^y \frac{\varphi(x-t)}{x-t} \frac{\psi(y-s)}{y-s} f(t, s) ds dt, \quad x > a, y > c,$$

$$(1.17) \quad {}_{a+,d-}I_{\varphi,\psi}f(x,y) = \int_a^x \int_y^d \frac{\varphi(x-t)}{x-t} \frac{\psi(s-y)}{s-y} f(t,s) ds dt, \quad x > a, \quad y < d,$$

$$(1.18) \quad {}_{b-,c+}I_{\varphi,\psi}f(x,y) = \int_x^b \int_c^y \frac{\varphi(t-x)}{t-x} \frac{\psi(y-s)}{y-s} f(t,s) ds dt, \quad x < b, \quad y > c,$$

and

$$(1.19) \quad {}_{b-,d-}I_{\varphi,\psi}f(x,y) = \int_x^b \int_y^d \frac{\varphi(t-x)}{t-x} \frac{\psi(s-y)}{s-y} f(t,s) ds dt, \quad x < b, \quad y < d.$$

Similar the above definitions, we can give the following integrals:

$$(1.20) \quad {}_{a+}I_{\varphi}f(x,c) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t,c) dt, \quad x > a$$

$$(1.21) \quad {}_{a+}I_{\varphi}f(x,d) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t,d) dt, \quad x > a,$$

$$(1.22) \quad {}_{c+}I_{\psi}f(a,y) = \int_c^y \frac{\psi(y-s)}{y-s} f(a,s) ds, \quad y > c,$$

and

$$(1.23) \quad {}_{d-}I_{\psi}f(b,y) = \int_y^d \frac{\psi(s-y)}{s-y} f(b,s) ds, \quad y < d.$$

The main objective of this paper is to establish new Ostrowski type inequalities for co-ordinated convex functions similar to [9, 10] by using generalized fractional integrals.

## 2. Main Results

Throughout this section, for clarity, we have defined

$$\begin{aligned} \Lambda_1(g) &= \int_0^g \frac{\varphi((x-a)t)}{t} dt, & \Lambda_2(g) &= \int_0^g \frac{\varphi((b-x)t)}{t} dt \\ \Psi_1(h) &= \int_0^h \frac{\psi((y-c)s)}{s} ds, & \Psi_2(h) &= \int_0^h \frac{\psi((d-y)s)}{s} ds. \end{aligned}$$

**Lemma 2.1.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $\Delta^\circ$  with  $a < b$ ,  $c < d$ . If  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ ,  $a, c \geq 0$ , then following identity holds for all  $(x, y) \in \Delta$ :

$$\begin{aligned}
 (2.1) \quad & \frac{[\Lambda_2(1) + \Lambda_1(1)] [\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A \\
 &= \frac{(x-a)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t) \Psi_1(s) \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) ds dt \\
 &\quad - \frac{(x-a)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t) \Psi_2(s) \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) ds dt \\
 &\quad - \frac{(b-x)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t) \Psi_1(s) \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) ds dt \\
 &\quad + \frac{(b-x)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t) \Psi_2(s) \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) ds dt,
 \end{aligned}$$

where

$$\begin{aligned}
 A = & [x_-, y_- I_{\varphi, \psi} f(a, c) + x_-, y_+ I_{\varphi, \psi} f(a, d) + x_+, y_- I_{\varphi, \psi} f(b, c) + x_+, y_+ I_{\varphi, \psi} f(b, d)] \\
 & - \Psi_1(1) [x_- I_{\varphi} f(a, y) + x_+ I_{\varphi} f(b, y)] - \Psi_2(1) [x_- I_{\varphi} f(a, y) + x_+ I_{\varphi} f(b, y)] \\
 & - \Lambda_1(1) [y_- I_{\psi} f(x, c) + y_+ I_{\psi} f(x, d)] - \Lambda_2(1) [y_- I_{\psi} f(x, c) + y_+ I_{\psi} f(x, d)].
 \end{aligned}$$

*Proof.* Applying integration by parts and change of variables  $u = tx + (1-t)a$  and  $v = sy + (1-s)c$ , we get

$$\begin{aligned}
 (2.2) \quad & \int_0^1 \int_0^1 \Lambda_1(t) \Psi_1(s) \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) ds dt \\
 &= \int_0^1 \Lambda_1(t) \left\{ \int_0^1 \Psi_1(s) \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) ds \right\} dt \\
 &= \int_0^1 \Lambda_1(t) \left\{ \frac{\Psi_1(1)}{y-c} \frac{\partial}{\partial t} f(tx + (1-t)a, y) \right. \\
 &\quad \left. - \frac{1}{y-c} \int_0^1 \frac{\psi((y-c)s)}{s} \frac{\partial}{\partial t} f(tx + (1-t)a, sy + (1-s)c) ds dt \right\} \\
 &= \frac{\Psi_1(1)}{y-c} \int_0^1 \Lambda_1(t) \frac{\partial}{\partial t} f(tx + (1-t)a, y) dt \\
 &\quad - \frac{1}{y-c} \int_0^1 \frac{\psi((y-c)s)}{s} \left\{ \int_0^1 \Lambda_1(t) \frac{\partial}{\partial t} f(tx + (1-t)a, sy + (1-s)c) dt \right\} ds \\
 &= \frac{\Psi_1(1)}{y-c} \left\{ \frac{1}{x-a} \Lambda_1(1) f(x, y) - \frac{1}{x-a} \int_0^1 \frac{\varphi((x-a)t)}{t} f(tx + (1-t)a, y) dt \right\} \\
 &\quad - \frac{1}{y-c} \int_0^1 \frac{\psi((y-c)s)}{s} \left\{ \frac{1}{x-a} \Lambda_1(1) f(x, sy + (1-s)c) \right.
 \end{aligned}$$



$$\begin{aligned}
 & - \frac{1}{x-a} \int_0^1 \frac{\varphi((x-a)t)}{t} f(tx + (1-t)a, sy + (1-s)c) dt \Big\} ds \\
 = & \frac{\Psi_1(1)\Lambda_1(1)}{(y-c)(x-a)} f(x, y) - \frac{\Psi_1(1)}{(x-a)(y-c)} \int_0^1 \frac{\varphi((x-a)t)}{t} f(tx + (1-t)a, y) dt \\
 & - \frac{\Lambda_1(1)}{(x-a)(y-c)} \int_0^1 \frac{\psi((y-c)s)}{s} f(x, sy + (1-s)c) \\
 & + \frac{1}{(x-a)(y-c)} \int_0^1 \int_0^1 \frac{\varphi((x-a)t)}{t} \frac{\psi((y-c)s)}{s} f(tx + (1-t)a, sy + (1-s)c) ds dt \\
 = & \frac{\Psi_1(1)\Lambda_1(1)}{(y-c)(x-a)} f(x, y) - \frac{\Psi_1(1)}{(x-a)(y-c)} {}_{x-}I_{\varphi}f(a, y) \\
 & - \frac{\Lambda_1(1)}{(x-a)(y-c)} {}_{y-}I_{\psi}f(x, c) + \frac{1}{(x-a)(y-c)} {}_{x-,y-}I_{\varphi,\psi}f(a, c).
 \end{aligned}$$

Similarly, applying the integration by parts, we also get

$$\begin{aligned}
 (2.3) \quad & \int_0^1 \int_0^1 \Lambda_1(t)\Psi_2(s) \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) ds dt \\
 = & - \frac{\Lambda_1(1)\Psi_2(1)}{(x-a)(d-y)} f(x, y) + \frac{\Psi_2(1)}{(x-a)(d-y)} {}_{x-}I_{\varphi}f(a, y) \\
 & + \frac{\Lambda_1(1)}{(x-a)(d-y)} {}_{y+}I_{\psi}f(x, d) - \frac{1}{(x-a)(d-y)} {}_{x-,y+}I_{\varphi,\psi}f(a, d),
 \end{aligned}$$

$$\begin{aligned}
 (2.4) \quad & \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s) \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) ds dt \\
 = & - \frac{\Lambda_2(t)\Psi_1(s)}{(b-x)(y-c)} f(x, y) + \frac{\Psi_1(s)}{(b-x)(y-c)} {}_{x+}I_{\varphi}f(b, y) \\
 & + \frac{\Lambda_2(t)}{(b-x)(y-c)} {}_{y-}I_{\psi}f(x, c) - \frac{1}{(b-x)(y-c)} {}_{x+,y-}I_{\varphi,\psi}f(b, c),
 \end{aligned}$$

$$\begin{aligned}
 (2.5) \quad & \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s) \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) ds dt \\
 = & \frac{\Lambda_2(t)\Psi_2(s)}{(b-x)(d-y)} f(x, y) - \frac{\Psi_2(s)}{(b-x)(d-y)} {}_{x+}I_{\varphi}f(b, y) \\
 & - \frac{\Lambda_2(t)}{(b-x)(d-y)} {}_{y+}I_{\psi}f(x, d) - \frac{1}{(b-x)(y-c)} {}_{x+,y+}I_{\varphi,\psi}f(b, d).
 \end{aligned}$$

From (2.2)-(2.5) and dividing the resultant one by  $(b-a)(d-c)$ , we get our desired equality (2.1).  $\square$

**Theorem 2.1.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b$ ,  $c < d$ ,  $a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$  is convex on co-ordinates on  $\Delta$  and  $\left| \frac{\partial^2 f}{\partial s \partial t} \right| \leq M$ ,  $(x, y) \in \Delta$ , then the following inequality holds for generalized fractional integrals:

$$(2.6) \quad \left| \frac{[\Lambda_2(1) + \Lambda_1(1)][\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A \right| \\ \leq \frac{M}{(b-a)(d-c)} [(x-a)(y-c)I_1 + (x-a)(d-y)I_2 \\ + (b-x)(y-c)I_3 + (b-x)(d-y)I_4],$$

where

$$I_1 = \int_0^1 \int_0^1 \Lambda_1(t) \Psi_1(s) ds dt, \quad I_2 = \int_0^1 \int_0^1 \Lambda_1(t) \Psi_2(s) ds dt, \\ I_3 = \int_0^1 \int_0^1 \Lambda_2(t) \Psi_1(s) ds dt, \quad I_4 = \int_0^1 \int_0^1 \Lambda_2(t) \Psi_2(s) ds dt,$$

and  $A$  is defined in Lemma 2.1.

*Proof.* From Lemma 2.1, we get the following inequality that holds for all  $(x, y) \in \Delta$ :

$$(2.7) \quad \left| \frac{[\Lambda_2(1) + \Lambda_1(1)][\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A \right| \\ \leq \frac{(x-a)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t) \Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| ds dt \\ + \frac{(x-a)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t) \Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right| ds dt \\ + \frac{(b-x)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t) \Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right| ds dt \\ + \frac{(b-x)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t) \Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right| ds dt.$$

By the convexity of  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$  on co-ordinates on  $\Delta$  and  $\left| \frac{\partial^2 f}{\partial s \partial t} \right| \leq M$ ,  $(x, y) \in \Delta$ , we

have following inequalities:

$$\begin{aligned}
 (2.8) \quad & \int_0^1 \int_0^1 \Lambda_1(t)\Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| ds dt \\
 & \leq M \int_0^1 \int_0^1 ts\Lambda_1(t)\Psi_1(s) ds dt + M \int_0^1 \int_0^1 t(1-s)\Lambda_1(s)\Psi_1(s) ds dt \\
 & \quad + M \int_0^1 \int_0^1 (1-t)s\Lambda_1(t)\Psi_1(s) dt + M \int_0^1 \int_0^1 (1-t)(1-s)\Lambda_1(t)\Psi_1(s) ds dt \\
 & = M \int_0^1 \int_0^1 \Lambda_1(t)\Psi_1(s) ds dt.
 \end{aligned}$$

Similarly, we have following inequalities

$$\begin{aligned}
 (2.9) \quad & \int_0^1 \int_0^1 \Lambda_1(t)\Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right| ds dt \\
 & \leq M \int_0^1 \int_0^1 \Lambda_1(t)\Psi_2(s) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 (2.10) \quad & \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right| ds dt \\
 & \leq M \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 (2.11) \quad & \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right| \\
 & \leq M \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s) ds dt.
 \end{aligned}$$

Now using (2.8)-(2.11) in (2.7), then we have our required inequality (2.6).  $\square$

**Remark 2.1.** In Theorem 2.1, if we suppose  $\varphi(t) = t$  and  $\psi(s) = s$ , then the inequality (2.6) becomes inequality (1.6).

**Remark 2.2.** In Theorem 2.1, if we take  $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$  and  $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ , then the inequality (2.6) is reduced to the inequality (1.10).

**Theorem 2.2.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b, c < d, a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is convex on co-ordinates on  $\Delta, p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$  and  $\left| \frac{\partial^2 f}{\partial s \partial t}(x, y) \right| \leq M, (x, y) \in \Delta$ , then the following inequality holds for generalized fractional integrals:

$$\begin{aligned}
 (2.12) \quad & \left| \frac{[\Lambda_2(1) + \Lambda_1(1)] [\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A \right| \\
 & \leq \frac{M}{(b-a)(d-c)} [(x-a)(y-c)J_1 + (x-a)(d-y)J_2 \\
 & \quad + (b-x)(y-c)J_3 + (b-x)(d-y)J_4],
 \end{aligned}$$

where

$$J_1 = \left( \int_0^1 \int_0^1 (\Lambda_1(t)\Psi_1(s))^p dsdt \right)^{\frac{1}{p}}, \quad J_2 = \left( \int_0^1 \int_0^1 (\Lambda_1(t)\Psi_2(s))^p dsdt \right)^{\frac{1}{p}},$$

$$J_3 = \left( \int_0^1 \int_0^1 (\Lambda_2(t)\Psi_1(s))^p dsdt \right)^{\frac{1}{p}}, \quad J_4 = \left( \int_0^1 \int_0^1 (\Lambda_2(t)\Psi_2(s))^p dsdt \right)^{\frac{1}{p}},$$

and  $A$  is defined as in Lemma 2.1.

*Proof.* From Lemma 2.1 and the Hölder inequality, we have the following inequality that holds for all  $(x, y) \in \Delta$  :

$$(2.13) \left| \frac{[\Lambda_2(1) + \Lambda_1(1)][\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A \right|$$

$$\leq \frac{(x-a)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t)\Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| dsdt$$

$$+ \frac{(x-a)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t)\Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right| dsdt$$

$$+ \frac{(b-x)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right| dsdt$$

$$+ \frac{(b-x)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right| dsdt$$

$$\leq \frac{(x-a)(y-c)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 (\Lambda_1(t)\Psi_1(s))^p dsdt \right)^{\frac{1}{p}}$$

$$\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q dsdt \right)^{\frac{1}{q}}$$

$$+ \frac{(x-a)(d-y)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 (\Lambda_1(t)\Psi_2(s))^p dsdt \right)^{\frac{1}{p}}$$

$$\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q dsdt \right)^{\frac{1}{q}}$$

$$+ \frac{(b-x)(y-c)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 (\Lambda_2(t)\Psi_1(s))^p dsdt \right)^{\frac{1}{p}}$$

$$\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q dsdt \right)^{\frac{1}{q}}$$

$$+ \frac{(b-x)(d-y)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 (\Lambda_2(t)\Psi_2(s))^p dsdt \right)^{\frac{1}{p}}$$

$$\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q dsdt \right)^{\frac{1}{q}}.$$

As we know that  $\left| \frac{\partial^2 f}{\partial s \partial t}(x, y) \right|^q$  is co-ordinated convex and  $\left| \frac{\partial^2 f}{\partial s \partial t}(x, y) \right|^q \leq M$ , for all  $(x, y) \in \Delta$ , then we have the following inequality:

$$\begin{aligned}
 (2.14) \quad & \left( \int_0^1 \int_0^1 (\Lambda_1(t)\Psi_1(s))^p dsdt \right)^{\frac{1}{p}} \\
 & \times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q dsdt \right)^{\frac{1}{q}} \\
 & \leq M \left( \int_0^1 \int_0^1 (\Lambda_1(t)\Psi_1(s))^p dsdt \right)^{\frac{1}{p}}.
 \end{aligned}$$

Analogously, we also have following inequalities

$$\begin{aligned}
 (2.15) \quad & \left( \int_0^1 \int_0^1 (\Lambda_1(t)\Psi_2(s))^p dsdt \right)^{\frac{1}{p}} \\
 & \times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)cd) \right|^q dsdt \right)^{\frac{1}{q}} \\
 & \leq M \left( \int_0^1 \int_0^1 (\Lambda_1(t)\Psi_2(s))^p dsdt \right)^{\frac{1}{p}},
 \end{aligned}$$

$$\begin{aligned}
 (2.16) \quad & \left( \int_0^1 \int_0^1 (\Lambda_2(t)\Psi_1(s))^p dsdt \right)^{\frac{1}{p}} \\
 & \times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q dsdt \right)^{\frac{1}{q}} \\
 & \leq M \left( \int_0^1 \int_0^1 (\Lambda_2(t)\Psi_1(s))^p dsdt \right)^{\frac{1}{p}},
 \end{aligned}$$

$$\begin{aligned}
 (2.17) \quad & \left( \int_0^1 \int_0^1 (\Lambda_2(t)\Psi_2(s))^p dsdt \right)^{\frac{1}{p}} \\
 & \times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q dsdt \right)^{\frac{1}{q}} \\
 & \leq M \left( \int_0^1 \int_0^1 (\Lambda_2(t)\Psi_2(s))^p dsdt \right)^{\frac{1}{p}}.
 \end{aligned}$$

By using (2.14)-(2.17) in (2.13), then we have our desired inequality (2.12).  $\square$

**Remark 2.3.** In Theorem 2.2, if we suppose  $\varphi(t) = t$  and  $\psi(s) = s$ , then the inequality (2.12) becomes inequality (1.7).

**Remark 2.4.** In Theorem 2.2, if we take  $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$  and  $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ , then the inequality (2.12) is reduced to the inequality (1.11).

**Theorem 2.3.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b$ ,  $c < d$ ,  $a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is convex on co-ordinates on  $\Delta$ ,  $q \geq 1$  and  $\left| \frac{\partial^2 f}{\partial s \partial t}(x, y) \right| \leq M$ ,  $(x, y) \in \Delta$ , then the following inequality holds for generalized fractional integrals:

$$(2.18) \quad \left| \frac{[\Lambda_2(1) + \Lambda_1(1)] [\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A \right| \\ \leq \frac{M}{(b-a)(d-c)} [(x-a)(y-c)I_1 + (x-a)(d-y)I_2 \\ + (b-x)(y-c)I_3 + (b-x)(d-y)I_4],$$

where  $I_1, I_2, I_3$  and  $I_4$  are same as defined in Theorem 2.1 and  $A$  is defined as in Lemma 2.1.

*Proof.* From Lemma 2.1 and the power mean inequality, we get the following inequality that holds for all  $(x, y) \in \Delta$  :

$$(2.19) \quad \left| \frac{[\Lambda_2(1) + \Lambda_1(1)] [\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A \right| \\ \leq \frac{(x-a)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t) \Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| ds dt \\ + \frac{(x-a)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t) \Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right| ds dt \\ + \frac{(b-x)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t) \Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right| ds dt \\ + \frac{(b-x)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t) \Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right| ds dt. \\ \leq \frac{(x-a)(y-c)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \Lambda_1(t) \Psi_1(s) ds dt \right)^{1-\frac{1}{q}} \\ \times \left( \int_0^1 \int_0^1 \Lambda_1(t) \Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}} \\ + \frac{(x-a)(d-y)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \Lambda_1(t) \Psi_2(s) ds dt \right)^{1-\frac{1}{q}} \\ \times \left( \int_0^1 \int_0^1 \Lambda_1(t) \Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}} \\ + \frac{(b-x)(y-c)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \Lambda_2(t) \Psi_1(s) ds dt \right)^{1-\frac{1}{q}} \\ \times \left( \int_0^1 \int_0^1 \Lambda_2(t) \Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}} \\ + \frac{(b-x)(d-y)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \Lambda_2(t) \Psi_2(s) ds dt \right)^{1-\frac{1}{q}} \\ \times \left( \int_0^1 \int_0^1 \Lambda_2(t) \Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}}$$

$$\begin{aligned}
 & + \frac{(b-x)(y-c)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s) dsdt \right)^{1-\frac{1}{q}} \\
 & \times \left( \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q dsdt \right)^{\frac{1}{q}} \\
 & + \frac{(b-x)(d-y)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s) dsdt \right)^{1-\frac{1}{q}} \\
 & \times \left( \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q dsdt \right)^{\frac{1}{q}}.
 \end{aligned}$$

As we know that  $\left| \frac{\partial^2 f}{\partial s \partial t}(x, y) \right|^q$  is co-ordinated convex and  $\left| \frac{\partial^2 f}{\partial s \partial t}(x, y) \right| \leq M$ , for all  $(x, y) \in \Delta$ , then we have the following inequality:

$$\begin{aligned}
 (2.20) \quad & \int_0^1 \int_0^1 \Lambda_1(t)\Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q dsdt \\
 & \leq M^q \int_0^1 \int_0^1 \Lambda_1(t)\Psi_1(s).
 \end{aligned}$$

Similarly, we have following inequalities:

$$\begin{aligned}
 (2.21) \quad & \int_0^1 \int_0^1 \Lambda_1(t)\Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q dsdt \\
 & \leq M^q \int_0^1 \int_0^1 \Lambda_1(t)\Psi_2(s)
 \end{aligned}$$

$$\begin{aligned}
 (2.22) \quad & \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q dsdt \\
 & \leq M^q \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s),
 \end{aligned}$$

$$\begin{aligned}
 (2.23) \quad & \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q dsdt \\
 & \leq M^q \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s).
 \end{aligned}$$

By using (2.20)-(2.23) in (2.19), we have our desired inequality (2.18).  $\square$

**Remark 2.5.** In Theorem 2.3, if we suppose  $\varphi(t) = t$  and  $\psi(s) = s$ , then the inequality (2.18) becomes inequality (1.8).

**Remark 2.6.** In Theorem 2.3, if we take  $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$  and  $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ , then the inequality (2.18) reduces to the inequality (1.12).

**Theorem 2.4.** Let  $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$  be twice partial differentiable mapping on  $\Delta^\circ$  with  $a < b$ ,  $c < d$ ,  $a, c \geq 0$  such that  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is concave on co-ordinates on  $\Delta$ ,  $p, q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , then the following inequality hold for generalized fractional integrals:

$$\begin{aligned}
 (2.24) \quad & \left| \frac{[\Lambda_2(1) + \Lambda_1(1)][\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A \right| \\
 & \leq \frac{1}{(b-a)(d-c)} \left[ (x-a)(y-c) \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+a}{2}, \frac{y+c}{2} \right) \right| J_1 \right. \\
 & \quad + (x-a)(d-y) \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+a}{2}, \frac{y+d}{2} \right) \right| J_2 \\
 & \quad + (b-x)(y-c) \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{y+c}{2} \right) \right| J_3 \\
 & \quad \left. + (b-x)(d-y) \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{y+d}{2} \right) \right| J_4 \right],
 \end{aligned}$$

where  $J_1, J_2, J_3$  and  $J_4$  are same as defined in Theorem 2.2.

*Proof.* From Lemma 2.1 and the Hölder inequality, we have the following inequality that holds for all  $(x, y) \in \Delta$  :

$$\begin{aligned}
 (2.25) \quad & \left| \frac{[\Lambda_2(1) + \Lambda_1(1)][\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A \right| \\
 & \leq \frac{(x-a)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t) \Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right| ds dt \\
 & \quad + \frac{(x-a)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t) \Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right| ds dt \\
 & \quad + \frac{(b-x)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t) \Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right| ds dt \\
 & \quad + \frac{(b-x)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t) \Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right| ds dt \\
 & \leq \frac{(x-a)(y-c)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 (\Lambda_1(t) \Psi_1(s))^p ds dt \right)^{\frac{1}{p}} \\
 & \quad \times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q ds dt \right)^{\frac{1}{q}}
 \end{aligned}$$



$$\begin{aligned}
 & + \frac{(x-a)(d-y)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 (\Lambda_1(t)\Psi_2(s))^p dsdt \right)^{\frac{1}{p}} \\
 & \times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q dsdt \right)^{\frac{1}{q}} \\
 & + \frac{(b-x)(y-c)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 (\Lambda_2(t)\Psi_1(s))^p dsdt \right)^{\frac{1}{p}} \\
 & \times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q dsdt \right)^{\frac{1}{q}} \\
 & + \frac{(b-x)(d-y)}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 (\Lambda_2(t)\Psi_2(s))^p dsdt \right)^{\frac{1}{p}} \\
 & \times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q dsdt \right)^{\frac{1}{q}}.
 \end{aligned}$$

Since  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$  is concave on co-ordinates on  $\Delta$ , so an application of (1.5) with inequalities in reversed direction, we have following inequalities:

$$\begin{aligned}
 (2.26) \quad & \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)c) \right|^q dsdt \\
 & \leq \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+a}{2}, \frac{y+c}{2} \right) \right|^q,
 \end{aligned}$$

$$\begin{aligned}
 (2.27) \quad & \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, sy + (1-s)d) \right|^q dsdt \\
 & \leq \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+a}{2}, \frac{y+d}{2} \right) \right|^q,
 \end{aligned}$$

$$\begin{aligned}
 (2.28) \quad & \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q dsdt \\
 & \leq \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{y+c}{2} \right) \right|^q,
 \end{aligned}$$

$$\begin{aligned}
 (2.29) \quad & \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)b, sy + (1-s)d) \right|^q dsdt \\
 & \leq \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{x+b}{2}, \frac{y+d}{2} \right) \right|^q.
 \end{aligned}$$

By using (2.26)-(2.29) in (2.25), then we have our desired inequality (2.24).  $\square$

**Remark 2.7.** In Theorem 2.4, if we suppose  $\varphi(t) = t$  and  $\psi(s) = s$ , then the inequality (2.24) becomes inequality (1.9).

**Remark 2.8.** In Theorem 2.4, if we take  $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$  and  $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ , then the inequality (2.24) is reduced to the inequality (1.13).

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