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COMPARISON OF VARIOUS FRACTIONAL BASIS FUNCTIONS FOR SOLVING FRACTIONAL-ORDER LOGISTIC POPULATION MODEL

This paper is dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday

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Abstract. Three types of orthogonal polynomials (Chebyshev, Chelyshkov, and Legendre) are employed as basis functions in a collocation scheme to solve a nonlinear cubic initial value problem arising in population growth models. The method reduces the given problem to a set of algebraic equations consist of polynomial coefficients. Our main goal is to present a comparative study of these polynomials and to asses their performances and accuracies applied to the logistic population equation. Numerical applications are given to demonstrate the validity and applicability of the method. Comparisons are also made between the present method based on different basis functions and other existing approximation algorithms..

Keywords: Liouville-Caputo fractional derivative; Chebyshev and Chelyshkov polynomials; Collocation method; Logistic population model; Legendre polynomial.

1. Introduction

In the present work, we are aiming to find the approximate solutions of the fractional-order growth equation of single species with multiplicative Allee effect. This equation is governed by the following nonlinear ordinary differential equation [1]

$$D_*^{(\mu)} y(t) = r\, y(t) \left(1 - \frac{y(t)}{k}\right) (y(t) - m), \quad 0 < t \le R < \infty, \tag{1.1}$$

with the initial condition

$$y(0) = \lambda \ge 0. \tag{1.2}$$

Here, r, m, and k are positive constants denoting respectively per capita growth rate, Allee effect threshold and the carrying capacity of the environment. Here, $D_*^{(\mu)}$

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is the standard Liouville-Caputo fractional derivative operator and $0 < \mu \leq 1$. The fractional model (1.1) can be obtained by using the fractional derivative operator on the corresponding inter-order equation. The investigation of the stability of equilibrium points of (1.1) along with the sufficient conditions to ensure the existence and uniqueness of the coresponding solution are considered in [1]. To the best of our knowledge, the following approximative and numerical schemes are developed for the model problem (1.1)-(1.2). These include the Adams-type predictor-corrector method [1], Bessel-collocation method [27], and the spectral tau method based on shifted Jacobi polynomials [10].

The logistic population model is considered as an important type of nonlinear differential equations due to its ability to model several biological and social phenomena. Different variations of the population modelling are considered in the literature [19]. Among others, the following linear and nonlinear models can be mentioned, cf. [20, 10, 13, 26]

$$D_*^{(\mu)} y(t) = r^\mu\, y(t), \tag{1.3}$$

$$D_*^{(\mu)} y(t) = r\, y(t) \Big(1 - y(t)\Big), \tag{1.4}$$

$$D_*^{(\mu)} y(t) = r^\mu\, y(t) \Big(1 - y(t)\Big). \tag{1.5}$$

Historically, the origin of fractional differential equations traced back to Newton and Leibniz more than three centuries ago. To model many real world problems, it has turned out the use of fractional-order derivatives are more adequate rather than integer-order ones. That is due to the fact that fractional derivatives and integrals enable the description of the memory properties of various materials and processes [21, 15]. Therefore, one needs to extend the concept of ordinary differentiation as well as integration to an arbitrary non-integer order. The resulting fractional-order equations can be rarely solved exactly or analytically. Consequently, approximate and numerical techniques are playing an important role in identifying the solutions behaviour of such fractional equations. Indeed, the exact analytical solution of the aforementioned population models is not known except for the linear model (1.3) whose solution is written in terms of Mittag-Leffer infinite series, cf. [26].

Recently, considerable attention has been given to the establishment of techniques for the solution of the fractional differential equations using orthogonal functions. The main characteristic of this technique is that it reduces the solution of differential equations to the solution of a system of algebraic equations. Historically this approach originated from the use of Fourier [18], Walsh [7] and block-pulse functions [22] and was later extended to other classical orthogonal polynomials such as Chebyshev, Legendre, Hermite, and Laguerre polynomials [23]. In most of the presented works, the use of numerical techniques in conjunction with operational matrices for differentiation and integration operators of some orthogonal polynomials, for the solution of fractional differential equations on finite and infinite intervals, produced highly accurate solutions for such equations, see [3] for a recent review.

As already mentioned, the model problem (1.1)-(1.2) is known to possess no exact solutions in general. In this manuscript, we will propose approximation methods as extension of the previous works [17], [11, 12], [27], [14], and [25] for solving (1.1)-(1.2). We use the fractional-order polynomials including the Chebyshev, Chelyshkov, and Legendre functions to approximate the solution of (1.1) accurately on the interval $[0, R]$. The main idea of the proposed technique based on using these (orthogonal) functions along with collocation points is that it converts the differential or integral operator involved in (1.1)-(1.2) to an algebraic form, thus greatly reducing the computational effort.

Our manuscript is organized as follows. In the next section, some fundamental definitions of fractional calculus and relevant properties are presented. Then, in subsequent subsections a brief review of the properties of the Chebyshev, Chelyshkov, and Legendre polynomials is outlined. Section 3. is devoted to the presentation of the proposed collocation scheme applied to nonlinear logistic population initial value problem. Hence, the error estimation technique based on the residual function is developed for the present method. In computational Section 4., we apply the proposed method to the some test problems and report our numerical findings. We end the paper with few concluding remarks in Section 5.

2. Basic definitions

In this section, first some properties of the fractional calculus theory are presented. Afterwards, the definitions of fractional Chebyshev, Chelyshkov, and Legendre polynomials are recalled and some properties of them required for our subsequent sections are reviewed.

2.1. Fractional calculus

Definition 2.1. Suppose that $f(t)$ is n-times continuously differentiable. The fractional derivative $D_*^{(\mu)}$ of $f(t)$ of order $\mu > 0$ in the Liouville-Caputo's sense is defined as

$$D_*^{(\mu)} f(t) \quad = \quad \begin{cases} I^{n-\mu} f^{(n)}(t), & \text{if} \quad n-1 < \mu < n, \\ f^{(n)}(t), & \text{if} \quad \mu = n, \ n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where

$$I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(s)}{(t-s)^{1-\mu}}\, ds, \quad t > 0.$$

The properties of the operator $D_*^{(\mu)}$ can be found in [21, 15]. We make use of the followings

(2.2) $D_*^{(\mu)}(C) = 0 \quad (C \text{ is a constant}),$

(2.3) $D_*^{(\mu)}\, t^{\gamma} =$

$$\begin{cases} \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\mu)} t^{\gamma-\mu}, & \text{for} \quad \gamma \in \mathbb{N}_0 \text{ and } \gamma \geq \lceil \mu \rceil, \text{ or } \gamma \notin \mathbb{N}_0 \text{ and } \gamma > \lfloor \mu \rfloor, \\ 0, & \text{for} \quad \gamma \in \mathbb{N}_0 \text{ and } \gamma < \lceil \mu \rceil. \end{cases}$$

We have used the ceiling function $\lceil \mu \rceil$ to denote the smallest integer greater than or equal to μ, and the floor function $\lfloor \mu \rfloor$ to denote the largest integer less than or equal to μ.

2.2. Chebyshev functions

It is known that the classical Chebyshev polynomials are defined on $[-1,1]$. Starting with $T_0(z) = 1$ and $T_1(z) = z$, these polynomials satisfy the following recurrence relation [2]

$$T_{n+1}(z) = 2z\, T_n(z) - T_{n-1}(z), \quad n = 1, 2, \ldots.$$

By introducing the change of variable $z = 1 - 2(\frac{t}{R})^\alpha$, $\alpha > 0$, one obtains the shifted version of the polynomials defined on $[0, R]$ and will be denoted by $T_n^\alpha(t) = T_n(z)$. The explicit analytical form of $T_n^\alpha(t)$ of degree (αn) is given for $n = 0, 1, \ldots$

$$T_n^\alpha(t) = \sum_{k=0}^{n} c_{n,k}\, t^{\alpha k}, \quad c_{n,k} = (-1)^k \frac{n\, 2^{2k}\, (n+k-1)!}{(n-k)!\, R^{\alpha k}\, (2k)!}, \quad k = 0, 1, \ldots, n, \tag{2.4}$$

with $c_{0,k} = 1$ for all $k = 0, 1, \ldots, n$. It is proved in [17] that the set of fractional polynomial functions $\{T_0^\alpha, T_1^\alpha, \ldots\}$ is orthogonal on $[0, R]$ with respect to the weight function $w(t) = \frac{t^{\alpha/2-1}}{\sqrt{R^\alpha - t^\alpha}}$; i.e.

$$\int_0^R T_n^\alpha(t)\, T_m^\alpha(t) w(t) dt = \frac{\pi}{2\alpha} d_n \delta_{mn}, \quad n, m \geq 0.$$

Here, δ_{mn} is Kronecker delta function, $d_0 = 2$ while $d_n = 1$ for $n \geq 1$. Our aim is to find an approximate solution of model (1.1) expressed in the truncated Chebyshev series form (3.1)

$$y_{N,\alpha}(t) = \sum_{n=0}^{N} a_n\, T_n^\alpha(t), \quad 0 \leq t \leq R, \tag{2.5}$$

where the unknown coefficients a_n, $n = 0, 1, \ldots, N$ are sought. To proceed, we write $T_n^\alpha(t)$, $n = 0, 1, \ldots, N$ in the matrix form as follows

$$\mathbf{T}_\alpha(t) = \mathbf{B}_\alpha(t)\, \mathbf{D}_1 \Leftrightarrow \mathbf{T}_\alpha^t(t) = \mathbf{D}_1^t\, \mathbf{B}_\alpha^t(t), \tag{2.6}$$

here, a superscript t denotes the matrix transpose operation and

$$\mathbf{T}_\alpha(t) = [T_0^\alpha(t) \quad T_1^\alpha(t) \quad \ldots \quad T_N^\alpha(t)], \quad \mathbf{B}_\alpha(t) = \begin{bmatrix} 1 & t^\alpha & t^{2\alpha} & \ldots & t^{N\alpha} \end{bmatrix}.$$

The upper triangular $(N+1)\times(N+1)$ matrix $\mathbf{D}_1$ takes the form

$$\mathbf{D}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & c_{1,1} & c_{2,1} & c_{3,1} & \dots & c_{N-1,1} & c_{N,1} \\ 0 & 0 & c_{2,2} & c_{3,2} & \dots & c_{N-1,2} & c_{N,2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & c_{N-1,N-1} & c_{N,N-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & c_{N,N} \end{bmatrix}.$$

By means of (2.6) one can write the relation (2.5) in the matrix form

$$y_{N,\alpha}(t) = \mathbf{B}_\alpha(t)\,\mathbf{D}_1\,\mathbf{A}, \tag{2.7}$$

where the vector of unknown is $\mathbf{A} = [a_0 \quad a_1 \quad \dots \quad a_N]^t$.

2.3. Chelyshkov functions

The Chelyshkov polynomials were originally introduced by Chelyshkov [6, 5]. These polynomials are orthogonal over the interval $[0,1]$ with respect to the weight function $w(x) = 1$, and are explicitly defined by

$$C_{n,N}(t) = \sum_{k=0}^{N-n} (-1)^k \binom{N-n}{k} \binom{N+n+k+1}{N-n} t^{n+k}, \quad n = 0,1,\dots,N. \tag{2.8}$$

These polynomials satisfy the following orthogonality relation

$$\int_0^1 C_{n,N}(t)\,C_{m,N}(t)dt = \frac{\delta_{nm}}{n+m+1}.$$

Moreover, they can be obtained through the Jacobi polynomials $P_m^{\alpha,\beta}(t)$, where $\alpha,\beta > -1$, and $m \geq 0$ as

$$C_{n,N}(t) = (-1)^{N-n}\,t^n\,P_{N-n}^{0,2n+1}(t).$$

Now, we construct the fractional-order version of (2.8) by replacing $t \to t^\alpha$ as follows [25]

$$C_{n,N}^{\alpha}(t) = \sum_{k=n}^{N} (-1)^{k-n} \binom{N-n}{k-n} \binom{N+k+1}{N-n} \left(\frac{t^\alpha}{R}\right)^k, \quad n = 0,1,\dots,N. \tag{2.9}$$

It also is not a difficult task to show that the set of fractional polynomial functions $\{C_{0,N}^\alpha, C_{1,N}^\alpha, \dots\}$ is orthogonal on $[0,R]$ with respect to the weight function $w(t) \equiv t^{\alpha-1}$. This implies that

$$\int_0^R C_{n,N}^\alpha(t)\,C_{m,N}^\alpha(t)\,w(t)dt = \frac{R\delta_{nm}}{\alpha(2n+1)}, \quad n,m \geq 0.$$

The Chelyshkov basis polynomials given by equation (2.9) can be written in the matrix form [16, 25]

$$\mathbf{C}_\alpha(t) = \begin{bmatrix} C^\alpha_{0,N}(t) & C^\alpha_{1,N}(t) & \dots & C^\alpha_{N,N}(t) \end{bmatrix} = \mathbf{B}_\alpha(t)\,\mathbf{D}_2, \tag{2.10}$$

where $\mathbf{D}_2$ is an $(N+1)\times(N+1)$ matrix. If N is odd, the matrix $\mathbf{D}_2$ becomes

$$\mathbf{D}_2 = \begin{bmatrix} \binom{N}{0}\binom{N+1}{N} & 0 & \dots & 0 & 0 \\ -r\binom{N}{1}\binom{N+2}{N} & r\binom{N-1}{0}\binom{N+2}{N-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{N-1}\binom{N}{N-1}\binom{2N}{N} & -r^{N-1}\binom{N-1}{N-2}\binom{2N}{N-1} & \dots & r^{N-1}\binom{1}{0}\binom{2N}{1} & 0 \\ -r^N\binom{N}{N}\binom{2N+1}{N} & r^N\binom{N-1}{N-1}\binom{2N+1}{N-1} & \dots & r^N\binom{1}{1}\binom{2N+1}{1} & r^N \end{bmatrix},$$

where we have used $r = 1/R$. If N is even we have

$$\mathbf{D}_2 = \begin{bmatrix} \binom{N}{0}\binom{N+1}{N} & 0 & \dots & 0 & 0 \\ -r\binom{N}{1}\binom{N+2}{N} & r\binom{N-1}{0}\binom{N+2}{N-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -r^{N-1}\binom{N}{N-1}\binom{2N}{N} & r^{N-1}\binom{N-1}{N-2}\binom{2N}{N-1} & \dots & r^{N-1}\binom{1}{0}\binom{2N}{1} & 0 \\ r^N\binom{N}{N}\binom{2N+1}{N} & -r^N\binom{N-1}{N-1}\binom{2N+1}{N-1} & \dots & -r^N\binom{1}{1}\binom{2N+1}{1} & r^N \end{bmatrix}.$$

Analogously, we approximate $y(t)$ in terms of the truncated Chelyshkov series form as $y_{N,\alpha}(t) = \sum_{n=0}^{N} a_n\, C^\alpha_{n,N}(t)$. Using (2.10) one may rewrite $y_{N,\alpha}(t)$ as follows

$$y_{N,\alpha}(t) = \mathbf{B}_\alpha(t)\,\mathbf{D}_2\,\mathbf{A}. \tag{2.11}$$

2.4. Legendre functions

The orthogonal Legendre polynomials are originally defined on $[-1, 1]$. Utilizing the change of variable $x = (\frac{2t}{R} - 1)$ one can obtain the shifted Legendre polynomials defined in $[0, R]$ and satisfies in the following recurrence relation [2]

$$P_{n+1}(t) = \frac{2n+1}{n+1}(\frac{2t}{R} - 1)\,P_n(t) - \frac{n}{n+1}P_{n-1}(t), \quad n = 1, 2, \dots,$$

with $P_0(t) = 1$ and $P_1(t) = \frac{2t}{R} - 1$. The analytical form of $P_n(t)$ is explicitly defined for $n = 0, 1, \dots$

$$P_n(t) = \sum_{k=0}^{n} l_{n,k}\, t^k, \quad l_{n,k} = (-1)^{n+k}\frac{(n+k)!}{(n-k)!\,R^k\,(k!)^2},\ k = 0, 1, \dots, n. \tag{2.12}$$

Based on the shifted Legendre polynomials (2.12) one generates an orthogonal set of fractional-order Legendre functions by setting $t \to t^{\alpha}$ for $0 < \alpha \leq 1$, see [14]. They take the form

$$P_n^{\alpha}(t) = \sum_{k=0}^{n} l_{n,k}\, t^{k\alpha}, \quad n = 0, 1, \ldots. \tag{2.13}$$

It is proved in [14] that the set of fractional polynomial functions $\{P_0^{\alpha}, P_1^{\alpha}, \ldots\}$ is orthogonal on $[0, R]$ with respect to the weight function $w(t) \equiv t^{\alpha-1}$; i.e.

$$\int_0^R P_n^{\alpha}(t)\, P_m^{\alpha}(t) w(t) dt = \frac{R}{\alpha(2n+1)}\delta_{nm}, \quad n, m \geq 0.$$

The main important properties of the fractional-order Legendre functions can be found in [14] and [24].

Now, let us approximate the solution $y(t)$ of (1.1) in terms of fractional-order Legendre functions. Thus one gets $y_{N,\alpha}(t) = \sum_{n=0}^{N} a_n\, P_n^{\alpha}(t)$ or equivalently

$$y_{N,\alpha}(t) = \mathbf{P}_{\alpha}(t)\, \mathbf{A}, \quad \mathbf{P}_{\alpha}(t) = [P_0^{\alpha}(t) \quad P_1^{\alpha}(t) \quad \ldots \quad P_N^{\alpha}(t)]. \tag{2.14}$$

In a similar way as the Chebyshev and Chelyshkov functions, we write $P_n^{\alpha}(t)$ in the matrix form as follows

$$\mathbf{P}_{\alpha}(t) = \mathbf{B}_{\alpha}(t)\, \mathbf{D}_3^t \Leftrightarrow \mathbf{P}_{\alpha}^t(t) = \mathbf{D}_3\, \mathbf{B}_{\alpha}^t(t), \tag{2.15}$$

where the monomial basis vector $\mathbf{B}_{\alpha}(t)$ is previously defined in (2.6). Moreover, the matrix $\mathbf{D}_3$ in this case is a lower triangular matrix whose entries are obtained via (2.12) and has the form

$$\mathbf{D}_3 = \begin{bmatrix} l_{0,0} & l_{1,0} & l_{2,0} & \ldots & l_{N-1,0} & l_{N,0} \\ 0 & l_{1,1} & l_{2,1} & \ldots & l_{N-1,1} & l_{N,1} \\ 0 & 0 & l_{2,2} & \ldots & l_{N-1,2} & l_{N,2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & l_{N-1,N-1} & l_{N,N-1} \\ 0 & 0 & 0 & \ldots & 0 & l_{N,N} \end{bmatrix}.$$

Therefore, an equivalent form of (2.14) can be written as

$$y_{N,\alpha}(t) = \mathbf{B}_{\alpha}(t)\, \mathbf{D}_3\, \mathbf{A}. \tag{2.16}$$

Ultimately, to obtain a solution in the form (2.11), (2.11), or (2.16) of the problem (1.1) on the interval $0 < t \leq R$, we will use the collocation points defined by

$$t_i = \frac{R}{N} i, \quad i = 0, 1, \ldots, N. \tag{2.17}$$

3. Description of the method

Now, suppose that we approximate the solution $y(t)$ of the nonlinear logistic population equation (1.1) in terms of $(N+1)$-terms Chebyshev, Chelyshkov or Legendre polynomials series denoted by $y_{N,\alpha}(t)$ on the interval $[0,R]$. As previously stated, in the vector form one may write

$$y(t) \approx y_{N,\alpha}(t) = \mathbf{B}_\alpha(t)\,\mathbf{U}\,\mathbf{A}. \tag{3.1}$$

Depending on which polynomial basis function we use in the approximation, the matrix $\mathbf{U}$ can be either $\mathbf{D}_1$, $\mathbf{D}_2$ or $\mathbf{D}_3$. These matrices are previously defined in (2.6), (2.10), and (2.15) respectively. Putting the collocation points (2.17) into (3.1), we arrive at a system of matrix equations

$$y_{N,\alpha}(t_i) = \mathbf{B}_\alpha(t_i)\,\mathbf{U}\,\mathbf{A}, \quad i = 0,1,\ldots,N.$$

These equations can be written in a single and compact representation as follows

$$\mathbf{Y} = \mathbf{B}\,\mathbf{U}\,\mathbf{A}, \tag{3.2}$$

where

$$\mathbf{Y} = \begin{bmatrix} y_{N,\alpha}(t_0) \\ y_{N,\alpha}(t_1) \\ \vdots \\ y_{N,\alpha}(t_N) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_\alpha(t_0) \\ \mathbf{B}_\alpha(t_1) \\ \vdots \\ \mathbf{B}_\alpha(t_N) \end{bmatrix}.$$

By taking the fractional derivative of order μ from the both sides of (3.1), we get

$$D_*^{(\mu)} y_{N,\alpha}(t) = D_*^{(\mu)}\,\mathbf{B}_\alpha(t)\,\mathbf{U}\,\mathbf{A}. \tag{3.3}$$

The calculation of $D_*^{(\mu)}\,\mathbf{T}_\alpha(t)$ can be easily obtained via the property (2.2) and (2.3) as follows

$$\mathbf{B}_\alpha^{(\mu)}(t) = D_*^{(\mu)}\,\mathbf{B}_\alpha(t) = [0 \quad D_*^{(\mu)}\,t^{\alpha} \quad \ldots \quad D_*^{(\mu)}\,t^{\alpha N}].$$

To obtain a system of matrix equations for the fractional derivative, we insert the collocation points (2.17) into (3.3) to get

$$D_*^{(\mu)} y_{N,\alpha}(t_i) = \mathbf{B}_\alpha^{(\mu)}(t_i)\,\mathbf{U}\,\mathbf{A}, \quad i = 0,1\ldots,N,$$

which can be written in the matrix form

$$\mathbf{Y}^{(\mu)} = \mathbf{B}^{(\mu)}\,\mathbf{U}\,\mathbf{A}, \tag{3.4}$$

where

$$\mathbf{Y}^{(\mu)} = \begin{bmatrix} D_*^{(\mu)} y_{N,\alpha}(t_0) \\ D_*^{(\mu)} y_{N,\alpha}(t_1) \\ \vdots \\ D_*^{(\mu)} y_{N,\alpha}(t_N) \end{bmatrix}, \quad \mathbf{B}^{(\mu)} = \begin{bmatrix} \mathbf{B}_\alpha^{(\mu)}(t_0) \\ \mathbf{B}_\alpha^{(\mu)}(t_1) \\ \vdots \\ \mathbf{B}_\alpha^{(\mu)}(t_N) \end{bmatrix}.$$

To continue, we approximate the nonlinear term $y^2(t)$. By substituting the collocation points into $y^2_{N,\alpha}(t)$ we arrive at the following matrix representation

$$\mathbf{Y}^2 = \begin{bmatrix} y^2_{N,\alpha}(t_0) \\ y^2_{N,\alpha}(t_1) \\ \vdots \\ y^2_{N,\alpha}(t_N) \end{bmatrix} = \begin{bmatrix} y_{N,\alpha}(t_0) & 0 & \dots & 0 \\ 0 & y_{N,\alpha}(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_{N,\alpha}(t_N) \end{bmatrix} \begin{bmatrix} y_{N,\alpha}(t_0) \\ y_{N,\alpha}(t_1) \\ \vdots \\ y_{N,\alpha}(t_N) \end{bmatrix},$$

which is equivalent to

$$\mathbf{Y}^2 = \widehat{\mathbf{Y}}\,\mathbf{Y}. \tag{3.5}$$

Also, the matrix $\widehat{\mathbf{Y}}$ can be written as a product of three block diagonal matrices as

$$\widehat{\mathbf{Y}} = \widehat{\mathbf{B}}\,\widehat{\mathbf{Q}}\,\widehat{\mathbf{A}}, \tag{3.6}$$

where

$$\widehat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_\alpha(t_0) & 0 & \dots & 0 \\ 0 & \mathbf{B}_\alpha(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{B}_\alpha(t_N) \end{bmatrix}, \quad \text{and}$$

$$\widehat{\mathbf{Q}} = \begin{bmatrix} \mathbf{U} & 0 & \dots & 0 \\ 0 & \mathbf{U} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{U} \end{bmatrix}, \quad \widehat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & 0 & \dots & 0 \\ 0 & \mathbf{A} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{A} \end{bmatrix}.$$

Similarly, by inserting the collocation points (2.17) into the $y^3(t)$ we arrive at the following matrix representation

$$\mathbf{Y}^3 = \begin{bmatrix} y^3_{N,\alpha}(t_0) \\ y^3_{N,\alpha}(t_1) \\ \vdots \\ y^3_{N,\alpha}(t_N) \end{bmatrix} = \begin{bmatrix} y^2_{N,\alpha}(t_0) & 0 & \dots & 0 \\ 0 & y^2_{N,\alpha}(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y^2_{N,\alpha}(t_N) \end{bmatrix} \begin{bmatrix} y_{N,\alpha}(t_0) \\ y_{N,\alpha}(t_1) \\ \vdots \\ y_{N,\alpha}(t_N) \end{bmatrix},$$

which implies that

$$\mathbf{Y}^3 = (\widehat{\mathbf{Y}})^2\,\mathbf{Y}, \tag{3.7}$$

where $\widehat{\mathbf{Y}}$ is defined in (3.6).

Now, we are able to compute the Chebyshev, Chelyshkov, and Legendre solutions of (1.1). The collocation procedure is based on calculating these polynomial coefficients by means of collocation points defined in (2.17). To proceed, inserting the collocation points into the fractional logistic population differential equation to get the system

$$D_*^{(\mu)}\,y(t_i) = -rm\,y(t_i) + r(1+\frac{m}{k})\,y^2(t_i) - \frac{r}{k}\,y^3(t_i), \quad i = 0,1,\dots,N.$$

In the matrix form we may write the above equations as

$$\mathbf{Y}^{(\mu)} + \mathbf{M}\,\mathbf{Y} - \mathbf{N}\,\mathbf{Y}^2 + \mathbf{K}\,\mathbf{Y}^3 = \mathbf{Z}, \tag{3.8}$$

where the coefficient matrices $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{K}$ of size $(N+1)\times(N+1)$ and the vector $\mathbf{Z}$ of size $(N+1)\times 1$ have the following forms

$$\mathbf{M} = \begin{bmatrix} rm & 0 & \dots & 0 \\ 0 & rm & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & rm \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} r(1+\frac{m}{k}) & 0 & \dots & 0 \\ 0 & r(1+\frac{m}{k}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r(1+\frac{m}{k}) \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} \frac{r}{k} & 0 & \dots & 0 \\ 0 & \frac{r}{k} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{r}{k} \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By putting the relations (3.2), (3.4), and (3.5), (3.7) into (3.8), the fundamental matrix equation is obtained

$$\mathbf{W}\,\mathbf{A} = \mathbf{Z}, \tag{3.9}$$

where

$$\mathbf{W} := \mathbf{B}^{(\mu)}\,\mathbf{U} + \mathbf{M}\,\mathbf{B}\,\mathbf{U} - \mathbf{N}\,\widehat{\mathbf{B}}\,\widehat{\mathbf{Q}}\,\widehat{\mathbf{A}}\,\mathbf{B}\,\mathbf{U} + \mathbf{K}\,(\widehat{\mathbf{B}}\,\widehat{\mathbf{Q}}\,\widehat{\mathbf{A}})^2\,\mathbf{B}\,\mathbf{U}.$$

Obviously, (3.9) is a nonlinear matrix equation with a_n, $n = 0, 1, \dots, N$, being the unknowns Chebyshev, Chelyshkov, or Legendre coefficients. To take into account the initial condition $y(0) = \lambda$, we tend $t \to 0$ in (3.1) to get the following matrix representation

$$\widetilde{\mathbf{Y}}_0\,\mathbf{A} = \lambda, \qquad \widetilde{\mathbf{Y}}_0 := \mathbf{B}_\alpha(0)\,\mathbf{U} = [y_{00} \quad y_{01} \quad \dots \quad y_{0N}]^t.$$

Consequently, by replacing the first row of the augmented matrix $[\mathbf{W}; \mathbf{Z}]$ by the row matrix $[\widetilde{\mathbf{Y}}_0; \lambda]$, we arrive at the nonlinear algebraic system

$$\widetilde{\mathbf{W}}\,\mathbf{A} = \widetilde{\mathbf{Z}}.$$

Thus, the unknown Chebyshev, Chelyshkov, or Legendre coefficients in (3.1) will be calculated via solving this nonlinear system of equations. This task can be performed using for instance the Newton's iterative method.

3.1. Accuracy of solutions

Since the exact solution of the fractional logistic population differential equation is not known, we need to measure the accuracy of the proposed collocation scheme.

Due to the fact that the truncated Chebyshev, Chelyshkov, and Legendre series (2.5), (2.8), and (2.12) are approximate solutions of (1.1), we expect that the residual obtained by inserting the computed approximated solutions $y_{N,\alpha}(t)$ into the differential equation becomes approximately small. This implies that for $t = t_s \in [0, R]$, $s = 0, 1, \ldots$

$$E_{N,\alpha}(t_s) = D_*^{(\mu)}\, y_{N,\alpha}(t_s) + C_0\, y_{N,\alpha}(t_s) - C_1\, y^2_{N,\alpha}(t_s) + C_2\, y^3_{N,\alpha}(t_s) \cong 0, \tag{3.10}$$

where $C_0 = rm$, $C_1 = r + rm/k$, $C_2 = r/k$, and $E_{N,\alpha}(t_s) \leq 10^{-\ell_s}$ (ℓ_s is positive integer). If $\max 10^{-\ell_s} \leq 10^{-\ell}$ (ℓ positive integer) is prescribed, then the truncation limit N is increased until the difference $E_{N,\alpha}(t_s)$ at each of the points becomes smaller than the prescribed $10^{-\ell}$, see [4, 27]. Here, we note that the μth-order fractional derivative of the approximate solution (3.10) is computed by using the property (2.3). As the error function is clearly zero at the collocation points (2.17), one expect that $E_{N,\alpha}(t)$ tend to zero as N increased. This says that the smallness of the residual error function means that the approximate solutions are close to the exact solution.

4. Numerical Applications

To illustrate the accuracy and effectiveness of the proposed polynomials collocation methods, two test examples are solved in this section. For comparison, we also implement the collocation spectral method based on the Bessel functions of the first kind in [27].

To start, we take $\mu = 1/3$ in (1.1) and set $\alpha = 10/21$ as the order of basis functions. The parameters are considered as $\lambda = 0.8$, $r = 1/2$, $m = 1$, and $k = 10$. The approximate solutions $y_{N,\alpha}(t)$ of this model problem using Chebyshev, Chelyshkov, and Legendre basis functions for $N = 6$ in the interval $0 \leq t \leq 5$ are obtained as follows, respectively:

$$y^{Cheb}_{6,\frac{10}{21}}(t) = 0.000403175741883\, t^{\frac{20}{7}} - 0.0437836398275\, t^{\frac{10}{7}} - 0.129582980375\, t^{\frac{10}{21}}$$
$$+0.0581143648443\, t^{\frac{20}{21}} + 0.0188356028419\, t^{\frac{40}{21}} - 0.00426036069079\, t^{\frac{50}{21}} + 0.8,$$

$$y^{Chel}_{6,\frac{10}{21}} = 0.000431604305758\, t^{\frac{20}{7}} - 0.0459657062667\, t^{\frac{10}{7}} - 0.137940445153\, t^{\frac{10}{21}}$$
$$+0.0610146767185\, t^{\frac{20}{21}} + 0.0200210715840\, t^{\frac{40}{21}} - 0.00455595754387\, t^{\frac{50}{21}} + 0.8,$$

$$y^{Leg}_{6,\frac{10}{21}} = 0.000403170590320\, t^{\frac{20}{7}} - 0.04378450703\, t^{\frac{10}{7}} - 0.129583270558\, t^{\frac{10}{21}}$$
$$+0.0581151779775\, t^{\frac{20}{21}} + 0.0188359825223\, t^{\frac{40}{21}} - 0.00426039646904\, t^{\frac{50}{21}} + 0.8.$$

The corresponding approximation by means of Bessel function of the first kind takes the form [27]

$$\begin{aligned} y^{Bes}_{6,\frac{10}{21}} &= 0.000431603553833\, t^{\frac{20}{7}} - 0.0459656939040\, t^{\frac{10}{7}} - 0.137940444198\, t^{\frac{10}{21}} \\ &+ 0.0610146708563\, t^{\frac{20}{21}} + 0.020021055753\, t^{\frac{40}{21}} - 0.0045559512912\, t^{\frac{50}{21}} + 0.8. \end{aligned}$$

The above results show clearly a similarity between the solutions obtained by the Chebyshev and Legendre collocation schemes. The same conclusion can be made from the two others polynomials obtained via Chelyshkov and Bessel functions. To further justify this fact, we plot the above approximations in Fig. 4.1. To validate our results, we also employ the predictor-corrector PECE method of Adams-Bashforth-Moulton type described in [8] using $\mu = 1/3$ and step size $h = 1/100$.

Furthermore, we calculate the error function defined in (3.10) for the above approximations. The results are depicted in Fig. 4.2, left plot, in which we used $\mu = 1/3$ and $\alpha = 10/21$. If one uses the same μ as α, a slightly better result is obtained; the right plot in Fig. 4.2 shows the corresponding error functions.

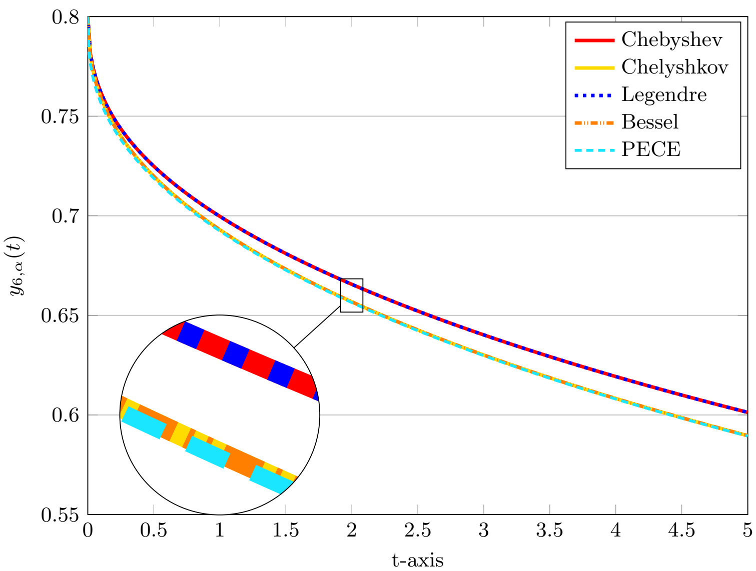


FIG. 4.1: The approximated Chebyshev/Chelyshkov/Legendre/Bessel series solutions $y_{6,\alpha}(t)$ using $\mu = 1/3$, $\alpha = 10/21$ for $r = 1/2$, $m = 1$, and $k = 10$.

Indeed, using μ equals to α give rises to the following approximations

$$y^{Cheb}_{6,\frac{1}{3}}(t) = 0.00145592739178\,t - 0.000408618142451\,t^2 - 0.0770406217645\,t^{\frac{1}{3}}$$
$$-0.0235193760879\,t^{\frac{2}{3}} - 0.00399643402926\,t^{\frac{4}{3}} + 0.00264600147359\,t^{\frac{5}{3}} + 0.8,$$

$$y^{Chel}_{6,\frac{1}{3}} = -0.0008400274797318\,t - 0.000452218849990\,t^2 - 0.08228036902327\,t^{\frac{1}{3}}$$
$$-0.02412078745183\,t^{\frac{2}{3}} - 0.002486450422357\,t^{\frac{4}{3}} + 0.002555336195047\,t^{\frac{5}{3}} + 0.8,$$

$$y^{Leg}_{6,\frac{1}{3}} = 0.0007238288842966\,t - 0.000383633654034\,t^2 - 0.0772002548625\,t^{\frac{1}{3}}$$
$$-0.02298707225931\,t^{\frac{2}{3}} - 0.00348653436761\,t^{\frac{4}{3}} + 0.002467593603573\,t^{\frac{5}{3}} + 0.8,$$

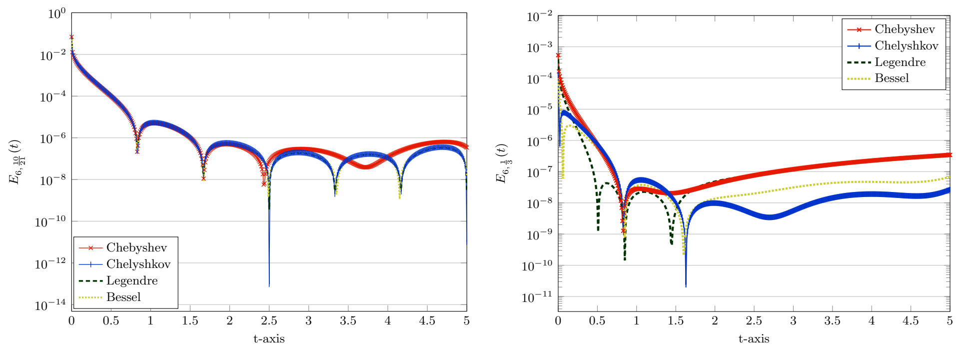


FIG. 4.2: Comparison of the error functions using Chebyshev, Chelyshkov, Legendre, and Bessel functions with $\mu = 1/3$, $\alpha = 10/21$ (left) and μ, $\alpha = 1/3$ (right) for $r = 1/2$, $m = 1$, $k = 10$ and $N = 6$.

$$y^{Bes}_{6,\frac{1}{3}} = -0.0005578958339751\, t - 0.0004622063664707\, t^2 - 0.08222082885753\, t^{\frac{1}{3}}$$
$$-0.02432274380530\, t^{\frac{2}{3}} - 0.002685716064483\, t^{\frac{4}{3}} + 0.002625907960449\, t^{\frac{5}{3}} + 0.8.$$

In Table 4.1, we report the numerical results correspond to $N = 11$ obtained by the Chebyshev, Chelyshkov, and Legendre-collocation procedures using using $\mu = 1/3$ and $\alpha = 10/21$ at some points $t \in [0, 5]$. A comparison in this table is made with the Bessel polynomials approach from [27].

In the second experiment, we set $\mu = 8/10$, $\alpha = 6/7$ and use the parameters $r = 1/2$, $m = 1$, $k = 10$ as for the first case. In this case, we first consider the approximate solutions $y_{3,\alpha}(t)$ obtained via (3.9) of the model (1.1) for different polynomials in the interval $[0, 5]$. These polynomials of fractional order $\alpha = 6/7$ are obtained as follows

$$\begin{aligned} y^{Cheb}_{3,\frac{6}{7}}(t) &= 0.00135308957515058\, t^{18/7} - 0.0141863508549924\, t^{12/7} \\ &- 0.0530254117658248\, t^{6/7} + 0.8, \end{aligned}$$

$$\begin{aligned} y^{Leg}_{3,\frac{6}{7}}(t) &= 0.00135308957514853\, t^{18/7} - 0.0141863508549652\, t^{12/7} \\ &- 0.0530254117658408\, t^{6/7} + 0.8, \end{aligned}$$

$$\begin{aligned} y^{Chel}_{3,\frac{6}{7}}(t) &= 0.00239567782739856\, t^{18/7} - 0.0181886989840546\, t^{12/7} \\ &- 0.0713242913743184\, t^{6/7} + 0.8, \end{aligned}$$

$$\begin{aligned} y^{Bes}_{3,\frac{6}{7}}(t) &= 0.00239567440428439\, t^{18/7} - 0.0181886759461094\, t^{12/7} \\ &- 0.0713243387363622\, t^{6/7} + 0.8. \end{aligned}$$

Table 4.1: Comparison of numerical approximations in fractional Chebyshev, Chelyshkov, and Legendre-collocation methods for $N = 11$, $\mu = 1/3$, and $\alpha = 10/21$ with $r = 1/2$, $m = 1$, $k = 10$.

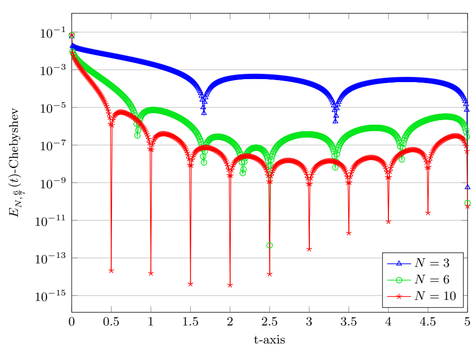
t	Chebyshev	Chelyshkov	Legendre	Bessel [27]
0.0	0.800000000000000	0.799999278128293	0.800000002346789	0.8
0.1	0.756938514122078	0.750720355971572	0.756800106479977	0.757299929343
0.5	0.718356585092500	0.717065283660215	0.718315644610171	0.719053533865
0.8	0.701146284741409	0.700526184555178	0.701115558982678	0.701988430676
1.1	0.687275920912118	0.687016466289264	0.687250477599117	0.688230723198
1.5	0.671748232619617	0.671810217070104	0.671726923772954	0.672823131171
1.8	0.661581827887138	0.661820098256081	0.661562638423935	0.662731306189
2.1	0.652336198269439	0.652716988678142	0.652318667387550	0.653550389695
2.5	0.641119373021142	0.641655992776889	0.641103504792762	0.642407766286
2.8	0.633377305676302	0.634012066595954	0.633362385608193	0.634713985075
3.1	0.626111073800506	0.626830874127975	0.626096980389975	0.627490786996
3.5	0.617051206659663	0.617867674288841	0.617038187117911	0.618481357894
3.8	0.610663124253371	0.611542868110921	0.610650880180289	0.612126624513
4.1	0.604580099793391	0.605518095678698	0.604568481573332	0.606073610313
4.5	0.596890007487173	0.597899498980620	0.596878825634452	0.598418968647
4.8	0.591404597688423	0.592459282464779	0.591393696691324	0.592957119829
5.0	0.587869951320244	0.588945877644788	0.587859669371108	0.589436884397

In the next experiments, we fix $N = 3$ and $\mu = 8/10$, $\alpha = 6/7$. We employ the error function (3.10) and compare the results obtained by different polynomial functions. Table 4.2 demonstrates the numerical values of these error functions at some points $t \in [0, 5]$. As the above approximations show, the errors $E_{3,\frac{6}{7}}(t)$ for the Chebyshev and Legendre as well as Chelyshkov and Bessel (our implementation) are approximately similar. Note, in the last column we reports the results from [27]. To see whether the error function $E_{N,\alpha}(t)$ is a decreasing function of N or not, we fix $\mu = 8/10$ and $\alpha = 6/7$ as above but use various $N = 3, 6$ and $N = 10$ in simulation. We select the Chebyshev and Chelyshkov as the basis functions among others. The results are visualized in Fig. 4.3. While the left picture illustrates the Chebyshev error functions, the right one is obtained via Chelyshkov collocation procedure.

Next, to see the effect of using various values of $\alpha \geq \mu$, we fix $N = 7$ and $\mu = 8/10$. Hence, we exploit several values of $\alpha = \mu$, 58/70, 6/7 and compute the numerical solutions at some points in $[0, 5]$. The results are shown in Table. 4.3 while using the Chelyshkov basis functions. To justify our results we compare the computed solutions in this table with Bessel collocation approach [27]. The last two columns are obtained using $\mu = 8/10, \alpha = 6/7$ and $n = 6, 11$ respectively. Looking at Table 4.3 reveals that using Chelyshkov collocation method with $N = 7$ but

Table 4.2: Comparison of error functions in fractional Chebyshev/Legendre and Chelyshkov/Bessel collocation methods for $N = 3$, $\mu = 8/10$, and $\alpha = 6/7$ with $r = 1/2$, $m = 1$, $k = 10$.

t	Chebyshev	Chelyshkov	Legendre	Bessel	Bessel [27]
0.0	5.8666667_{-02}	7.3600000_{-02}	5.8666667_{-02}	7.3600000_{-02}	7.36000000000_{-02}
0.1	1.2372352_{-02}	1.1880958_{-02}	1.2372352_{-02}	1.1880924_{-02}	8.91567095017_{-03}
0.5	5.5046468_{-03}	4.3507888_{-03}	5.5046468_{-03}	4.3507667_{-03}	1.99296107666_{-03}
0.8	3.0589969_{-03}	2.1210125_{-03}	3.0589969_{-03}	2.1209978_{-03}	8.46185079361_{-04}
1.1	1.5039416_{-03}	9.0326860_{-04}	1.5039416_{-03}	9.0325952_{-04}	3.41262782597_{-04}
1.5	2.9615144_{-04}	1.4128415_{-04}	2.9615144_{-04}	1.4128047_{-04}	5.49335472238_{-05}
1.8	1.6982381_{-04}	6.4923105_{-05}	1.6982381_{-04}	6.4924075_{-05}	2.79352687459_{-05}
2.1	3.7838526_{-04}	1.0764412_{-04}	3.7838526_{-04}	1.0764336_{-04}	5.62760899174_{-05}
2.5	3.9162970_{-04}	5.7880301_{-05}	3.9162970_{-04}	5.7878565_{-05}	5.21372970094_{-05}
2.8	2.8292735_{-04}	1.1166955_{-05}	2.8292735_{-04}	1.1165393_{-05}	3.51686446321_{-05}
3.1	1.2750623_{-04}	9.5410734_{-06}	1.2750623_{-04}	9.5417305_{-06}	1.49559607016_{-05}
3.5	8.4963558_{-05}	2.0244083_{-05}	8.4963558_{-05}	2.0245750_{-05}	9.35893830234_{-06}
3.8	2.0838829_{-04}	7.7239067_{-05}	2.0838829_{-04}	7.7243331_{-05}	2.21148978971_{-05}
4.1	2.7510980_{-04}	1.4112847_{-04}	2.7510980_{-04}	1.4113610_{-04}	2.83498750927_{-05}
4.5	2.4829257_{-04}	1.7908045_{-04}	2.4829257_{-04}	1.7909384_{-04}	2.48914327358_{-05}
4.8	1.2896874_{-04}	1.1530618_{-04}	1.2896874_{-04}	1.1532493_{-04}	1.27732345915_{-05}
5.0	5.6388995_{-10}	9.6146640_{-11}	5.6379307_{-10}	2.2949767_{-08}	0



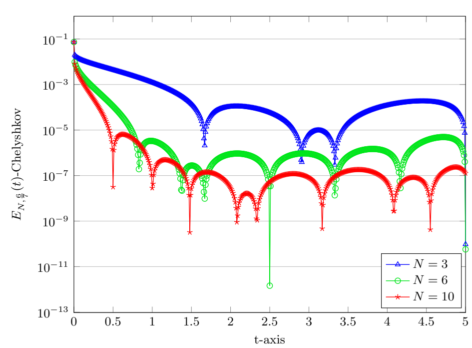


FIG. 4.3: comparison of error functions using Chebyshev (left) and Chelyshkov functions (right) with $\mu = 8/10$, $\alpha = 6/7$, and different $N = 3, 6, 10$.

$\alpha = 58/70$ one gets a comparable result while using Bessel basis functions with $N = 11$.

Table 4.3: Comparison of numerical solutions in Chelyshkov collocation method for $N = 7$, $\mu = 8/10$, and different $\alpha = 8/10$, $58/70$, $6/7$ with $r = 1/2$, $m = 1$, $k = 10$.

	Chelyshkov $(\mu = \frac{8}{10}, N = 7)$			Bessel [27] $(\mu = \frac{8}{10}, \alpha = \frac{6}{7})$	
t	$\alpha = \frac{8}{10}$	$\alpha = \frac{58}{70}$	$\alpha = \frac{6}{7}$	$N = 6$	$N = 11$
0.0	0.80000000000	**0.8000000000**	0.80000000000	0.8	**0.8**
0.1	0.78718882912	**0.787**58463172	0.78802089791	0.788007903475	**0.787**696000559
0.5	0.74982396960	**0.7502**5364375	0.75075459981	0.750739706221	**0.7502**42595254
0.8	0.72356379677	**0.7239**8067700	0.72446611403	0.724456716829	**0.7239**72085247
1.1	0.69746594429	**0.69787**925083	0.69835946985	0.698352878145	**0.69787**5832476
1.5	0.66253413522	**0.66294**840744	0.66342949839	0.663422970008	**0.66294**6544268
1.8	0.63620080581	**0.63661**604251	0.63709835065	0.637090911256	**0.63661**4170560
2.1	0.60981684229	**0.61023**179673	0.61071381759	0.610705745410	**0.61023**0091937
2.5	0.57473447564	**0.57514**638100	0.57562483235	0.575616794783	**0.57514**5107178
2.8	0.54865002906	**0.54905**746525	0.54953072965	0.549523095432	**0.54905**6408163
3.1	0.52290039893	**0.52330**146175	0.52376737213	0.523760058308	**0.52330**0477415
3.5	0.48930161944	**0.48969**130186	0.49014406249	0.490136706629	**0.48969**0377445
3.8	0.46480191614	**0.46518**106852	0.46562160232	0.465614022991	**0.46518**0289715
4.1	0.44101520794	**0.44138**236008	0.44180893517	0.441801414133	**0.44138**1760632
4.5	0.41055630328	**0.41090**564596	0.41131160135	0.411305250793	**0.410905**049742
4.8	0.38874292431	**0.389077**89139	0.38946730819	0.389462357675	**0.389077**139783
5.0	0.37472016558	**0.37504**515130	0.37542301726	0.375418410434	**0.37504**4417235

5. Conclusions

In this manuscript, an approximation algorithm based on different polynomials is developed for solving the nonlinear fractional-order logistic population equation modelling the single species multiplicative Allee effect. Exploiting the fractional Chebyshev, Chelyshkov, and Legendre functions along with the collocation points we convert the differential equation into an algebraic system of nonlinear equations. Numerical test problems are given to demonstrate efficiency and accuracy of the proposed method. Moreover, the performance of these three basis functions has assessed and a comparison between them and other existing schemes is made. Furthermore, the reliability of the proposed technique is checked through defining the residual error functions. Referring to graphs and tables we conclude that using the fractional Chelyshkov function produces a more accurate result compared to Chebyshev and Legendre basis functions. The proposed technique can be easily applied to other logistic population models (1.3)-(1.5) and other problems in science and engineering.

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