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ITERATIVE COMPUTATION FOR SOLVING CONVEX OPTIMIZATION PROBLEMS OVER THE SET OF COMMON FIXED POINTS OF QUASI-NONEXPANSIVE AND DEMICONTRACTIVE MAPPINGS

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Abstract. In this paper, a new iterative method for solving convex minimization problems over the set of common fixed points of quasi-nonexpansive and demicontractive mappings is constructed. Convergence theorems are also proved in Hilbert spaces without any compactness assumption. As an application, we shall utilize our results to solve quadratic optimization problems involving bounded linear operator. Our theorems are significant improvements on several important recent results.
Keywords:Fixed point algorithm, Convex minimization problem, Quasi-nonexpansive mapping, Demicontractive mappings.

1. Introduction

Let H be a real Hilbert space, K be a nonempty subset of H. A map $T : K \to K$ is said to be Lipschitz if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leqslant L\|x - y\|, \forall x, y \in K, \tag{1.1}$$

if $L < 1$, T is called *contraction* and if $L = 1$, T is called nonexpansive.
We denote by $Fix(T)$ the set of fixed points of the mapping T, that is $Fix(T) := \{x \in D(T) : x = Tx\}$. We assume that $Fix(T)$ is nonempty. If T is nonexpansive mapping, it is well known $Fix(T)$ is closed and convex. A map T is called quasi-nonexpansive if $\|Tx - p\| \leq \|x - p\|$ holds for all x in K and $p \in Fix(T)$. The

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mapping $T : K \to K$ is said to be firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \forall x, y \in K.$$

A mapping $T : K \to H$ is called k-strictly pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2, \forall x, y \in K.$$

A map T is called k-demi-contractive if $Fix(T) \neq \emptyset$ and for $k \in [0, 1)$, we have

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \forall x \in K, \ \ p \in Fix(T).$$

We note that the following inclusions hold for the classes of the mappings:

firmly nonexpansive $\subset$ nonexpansive $\subset$ quasi-nonexpansive $\subset$ k-strictly pseudo-contractive $\subset$ k-demi-contractive.

The function T in the following example is k-demi-contractive mapping but is not a k-strictly pseudo-contractive mapping.

Example 1.1. Let $H = \mathbb{R}$ and $K = [-1, 1]$. Define $T : K \to K$ by

$$Tx = \begin{cases} \frac{2}{3}x \sin(\frac{1}{x}), & x \neq 0 \\ 0 \quad x = 0. \end{cases} \tag{1.2}$$

Clearly $Fix(T) = \{0\}$. For $x \in K$, we have

$$\begin{aligned} |Tx - 0|^2 &= |\frac{2}{3}x \sin(\frac{1}{x})|^2 \\ &\leq |\frac{2}{3}x|^2 \\ &\leq |x|^2 \\ &\leq |x - 0|^2 + k|x - Tx|^2 \ \ \forall k \in [0, 1). \end{aligned}$$

Thus T is k demi-contractive for $k \in [0, 1)$. To see that T is not k strictly pseudo-contractive, choose $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$, then

$$|Tx - Ty|^2 > |x - y|^2 + k|x - y - (Tx - Ty)|^2.$$

Hence, T is not k strictly pseudo-contractive mapping for $k \in [0, 1)$.

The function T in the following example is k-demi-contractive mapping but is not not quasi-nonexpansive.

Example 1.2. Let f be a real function defined by $f(x) = -x^2 - x$; it can be seen that $f : [-2, 1] \to [-2, 1]$. This function is demicontractive on $[-2, 1]$ and continuous. It is not quasi-nonexpansive and is not pseudocontractive on $[-2, 1]$ (check for instance the condition of pseudocontractivity for $x = -1.5$ and $y = -0.6$).

For several years, the study of fixed point theory for nonlinear mappings has attracted, and continues to attract the interest of several well known mathematicians (see, [9, 10, 13, 4]).

Interest in the study of fixed point theory for nonlinear mappings stems, perhaps, mainly from its usefulness in real-world applications such as Game Theory and Market Economy and in other areas of mathematics, such as in Non-Smooth Differential Equations and Differential Inclusions, Optimization theory.

Let K be a nonempty, closed convex subset of H. The nearest point projection from H to K, denoted by P_K assigns to each $x \in H$ the unique $P_K x$ with the property

$$\|x - P_K x\| \leq \|y - x\|$$

for all $y \in K$. It is well known that P_K satisfies

$$\langle x - y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2 \tag{1.3}$$

for all $y \in H$ and

$$\langle P_K z - y,\ z - P_K z\rangle \geq 0 \tag{1.4}$$

for all $z \in K$ and $y \in H$.

An operator $A : K \to H$ is called *monotone* if

$$\langle Ax - Ay,\, x - y\rangle_H \geq 0, \quad \forall\ x, y \in K,$$

A is called *k-strongly monotone* if there exists $k \in (0, 1)$ such that for each $x, y \in H$ such that

$$\langle Ax - Ay, x - y\rangle_H \geq k\|x - y\|^2.$$

An operator $A : H \to H$ is said to be *strongly positive bounded* if there exists a constant $c > 0$ such that

$$\langle Ax, x\rangle_H \geq c\|x\|^2, \quad \forall\ x \in H.$$

Remark 1.1. From the definion of A, we note that strongly positive bounded linear operator A is a $\|A\|$-Lipchitzian and c-strongly monotone operator.

Definition 1.1. Let H be a real Hilbert space. A function $g : H \to \mathbb{R}$ is said to be α-strongly convex if there exists $\alpha > 0$ such that for every $x, y \in H$ with $x \neq y$ and $\beta \in (0, 1)$, the following inequality holds:

$$g(\beta x + (1 - \beta)y) \leq \beta g(x) + (1 - \beta)g(y) - \alpha\|x - y\|^2. \tag{1.5}$$

Lemma 1.1. *Let H be a real Hilbert space and $g : H \to \mathbb{R}$ a real-valued differentiable convex function. Assume that g is strongly convex. Then the differential map $\nabla g : H \to H$ is strongly monotone, i.e., there exists a positive constant k such that*

$$\langle \nabla g(x) - \nabla g(y), x - y \rangle \geq k\|x - y\|^2 \ \forall\, x, y \in H. \tag{1.6}$$

Consider the following constrained optimization problem: Let H be a real Hilbert space. Given a convex objective function $g : H \to \mathbb{R}$ and $T : H \to H$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$, the problem can be expressed as

$$\begin{aligned} &\textit{Minimize } g(x) \\ &\textit{subject to } x \in Fix(T). \end{aligned} \tag{1.7}$$

Optimization problem for a convex objective function over the fixed points set of a nonexpansive mapping have been and will continue to be one of the central problems in nonlinear analysis and is one of the central issues in modern communication networks. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can be considered as solving optimization problems. Recently, many iterative algorithms for solving these problems have been proposed, see [6, 2, 5, 11, 8] and the references therein.

Very recently, H. Iiduka [7] motivated by the fact that convex optimization problem for a strictly convex objective function over the fixed point set of a nonexpansive mapping includes a network bandwidth allocation problem, which is one of the central issues in modern communication networks, he proposed a fixed point optimization algorithm for solving Problem (1.7).

Algorithm 1.1. Step 0. *Choose $x_0 \in H$ arbitrarily, set $\lambda_0 \subset (0, 1)$ $\alpha_0, \subset (0, 1]$, and $d_0 = -\nabla g(x_0)$ arbitrarily and let $n := 0$.* **Step 1.** *Given $x_n \in H$ and $d_n \in H$, choose $\lambda_n \subset (0, 1)$, $\alpha_n, \subset (0, 1]$ and compute $x_{n+1} \in K$ as*

$$\begin{cases} y_n = T(x_n + \lambda_n d_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) y_n. \end{cases} \tag{1.8}$$

Step 2. *Choose $\beta_{n+1} \in (0, 1]$ and compute the direction, $d_{n+1} \in H$, by*

$$d_{n+1} = -\nabla g(x_n) + \beta_{n+1} d_n.$$

Update $n := n + 1$ and go to Step 1.

Under suitable conditions, he proved that $\{x_n\}_{n\in\mathbb{N}}$ in Algorithm 1.1 weakly converges to a unique solution to Problem (1.7).

Motivated by above results and the fact that the class of demicontractive mappings properly includes that of quasi-nonexpansive, strictly pseudocontractive mappings, we consider the following convex minimization problem : Let K be a nonempty,

closed and convex subset a real Hilbert space H. Given a convex objective function $g : K \to \mathbb{R}$ be a differentiable, k-strongly convex real-valued function. Suppose the differential map $\nabla g : H \to H$ is L-Lipschitz. Let $T_1 : K \to K$ be a λ-demicontractive mapping and $T_2 : K \to K$ be a quasi-nonexpansive mapping such that $\Gamma := Fix(T_1) \cap Fix(T_2) \neq \emptyset$. In the present paper, our main purpose is to solve the minimization problem:

$$\text{find } x^* \in \Gamma \text{ such that } g(x^*) = \min_{x \in \Gamma} g(x). \tag{1.9}$$

We denote the set of solutions of Problem (1.9) by Ω.

2. Preliminaries

We start with the following demiclosedness principle for nonexpansive mappings.

Lemma 2.1. [1] *Let K be a closed convex subset of a real Hilbert space H. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then $I - T$ is demiclosed at origin.*

Lemma 2.2. [3] *Let H be a real Hilbert space. Then for any $x, y \in H$, the following inequalities hold:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - (1 - \lambda)\lambda\|x - y\|^2, \ \ \lambda \in (0, 1).$$

Lemma 2.3. [12] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, *(b)* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. *[14] Let K be a nonempty, closed convex subset of a real Hilbert space H. Let $A : K \to H$ be a k-strongly monotone and L-Lipschitzian operator with $k > 0$, $L > 0$. Assume that $0 < \eta < \frac{2k}{L^2}$ and $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$. Then for each $t \in \Big(0, min\{1, \frac{1}{\tau}\}\Big)$, we have*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq (1 - t\tau)\|x - y\|, \ \forall\, x, y \in K.$$

Lemma 2.5. [9] *Assume K is a closed convex subset of a Hilbert space H. Let $T : K \to K$ be a self-mapping of K. If T is a k-demicontractive mapping, then the fixed point set $Fix(T)$ is closed and convex.*

Lemma 2.6. *Let K be a nonempty, closed convex subset of a normed linear space E. Let $g : K \to \mathbb{R}$ a real valued differentiable convex function. Then x^* is a minimizer of g over K if and only if x^* solves the following variational inequality $\langle \nabla g(x^*), y - x^*\rangle \geq 0$ for all $y \in K$.*

Remark 2.1. By Lemma 2.6, $x^* \in \Omega$ if and only if x^* solves the following variational inequality problem :

$$\langle \nabla g(x^*), x^* - p\rangle \leq 0, \ \forall p \in \Gamma. \tag{2.1}$$

We denote the set of solutions of variational inequality problem (2.1) by $VI(\nabla g, \Gamma)$.

3. Main Results

In this section, we present our explicit iterative method for solving (1.9).

Lemma 3.1. *Let H be a real Hilbert space. Let K be a nonempty, closed convex subset of H and $g : K \to \mathbb{R}$ be a differentiable, k-strongly convex real-valued function. Suppose the differential map $\nabla g : K \to H$ is L-Lipschitz. Let $T_1 : K \to K$ be a λ-demicontractive mapping and $T_2 : K \to K$ be a quasi-nonexpansive mapping such that $\Gamma := Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Then, $VI(\nabla g, \Gamma)$ is nonempty.*

Proof. Set η and τ two real numbers such that $0 < \eta < \frac{2k}{L^2}$ and $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$. Let t_0 be a fixed real number such that $t_0 \in \Big(0, min\{1, \ \frac{1}{\tau}\}\Big)$. We observe that $P_\Gamma(I - t_0\eta\nabla g)$ is a contraction. Indeed, for all $x, y \in K$, by Lemma 2.4, we have

$$\begin{aligned} \|P_\Gamma(I - t_0\eta\nabla g)x - P_\Gamma(I - t_0\eta\nabla g)y\| &\leq \|(I - t_0\eta\nabla g)x - (I - t_0\eta\nabla g)y\| \\ &\leq (1 - t_0\tau)\|x - y\|. \end{aligned}$$

Banach's Contraction Mapping Principle guarantees that $P_\Gamma(I - t_0\eta\nabla g)$ has a unique fixed point, say $x_1 \in H$. That is, $x_1 = P_\Gamma(I - t_0\eta\nabla g)x_1$. Thus, in view of inequality (1.3), it is equivalent to the following variational inequality problem

$$\langle \nabla g(x_1), x_1 - p\rangle \leq 0, \ \ \forall p \in \Gamma.$$

By using Remark 2.1, we have $x_1 \in \Omega$. This completes this proof. □

We show the main result of this paper, that is, the strong convergence analysis for Algorithm 3.1.

Algorithm 3.1. Step 0. *Take $\{\alpha_n\} \subset (0, 1)$, $\{\theta_n\}, \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\eta > 0$ arbitrarily choose $x_0 \in K$; and let $n := 0$.*
Step 1. *Given $x_n \in K$, compute $x_{n+1} \in K$ as*

$$\begin{cases} z_n = \theta_n x_n + (1 - \theta_n)T_1 x_n, \\ y_n = \beta_n z_n + (1 - \beta_n)T_2 z_n, \\ x_{n+1} = P_K(I - \eta\alpha_n\nabla g)y_n, \end{cases} \tag{3.1}$$

Update $n := n + 1$ and go to Step 1.

Now we perform the convergence analysis for Algorithm 3.1.

Theorem 3.1. *Assume that* $I-T_1$ *and* $I-T_2$ *are demiclosed at origin. Suppose that* $\{\alpha_n\}$, $\{\theta_n\}$ *and* $\{\beta_n\}$ *are the sequences such that:*

(i) $\lim_{n\to\infty}\alpha_n = 0, \quad \sum_{n=0}^{\infty}\alpha_n = \infty,$

(ii) $\theta_n \in]\lambda, 1[, \quad \liminf_{n\to\infty}(1-\theta_n)(\theta_n-\lambda) > 0$ *and* $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0.$

Assume that $0 < \eta < \frac{2k}{L^2}$, *then, the sequence* $\{x_n\}$ *defined by Algorithm 3.1 converges strongly to a unique solution of Problem (1.9).*

Proof. Firstly, we prove that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. From Lemma 3.1, we have $VI(\nabla g, \Gamma)$ is nonempty. In what follows, we denote x^* to be the unique solution of $VI(\nabla g, \Gamma)$. Without loss of generality, we can assume $\alpha_n \in \Big(0, min\{1\ , \frac{1}{\tau}\}\Big)$ where $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$. Let $p \in \Gamma$. By using (3.1) and Lemma 2.2, we have

$$\begin{aligned}\|z_n - p\|^2 &= \Big\|\theta_n(x_n - p) + (1-\theta_n)(T_1x_n - p)\Big\|^2 \\ &= \theta_n\|x_n - p\|^2 + (1-\theta_n)\|T_1x_n - p\|^2 \\ &\quad -\theta_n(1-\theta_n)\|T_1x_n - x_n\|^2.\end{aligned}$$

Using the fact that T_1 is λ-demi-contractive, we obtain

$$\begin{aligned}\|z_n - p\|^2 &\leq \theta_n\|x_n - p\|^2 + (1-\theta_n)\Big(\|x_n - p\|^2 + \lambda\|Tx_n - x_n\|^2\Big) \\ &\quad -\theta_n(1-\theta_n)\|T_1x_n - x_n\|^2 \\ &\leq \|x_n - p\|^2 - (1-\theta_n)(\theta_n - \lambda)\|T_1x_n - x_n\|^2.\end{aligned}$$

Since $\theta_n \in]\lambda, 1[$, we have,

$$\|z_n - p\| \leq \|x_n - p\|. \tag{3.2}$$

Hence, we obtain

$$\begin{aligned}\|y_n - p\| &= \|\beta_nz_n + (1-\beta_n)T_2z_n - p\| \\ &\leq \beta_n\|z_n - p\| + (1-\beta_n)\|T_2z_n - p\| \\ &\leq \|z_n - p\|.\end{aligned}$$

By using inequality (3.2), we have

$$\|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \tag{3.3}$$

From (3.1), Lemma 2.4 and inequality (3.3), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|P_K(I - \alpha_n\eta\nabla g)y_n - p\| \\ &\leq (1-\tau\alpha_n)\|x_n - p\| + \alpha_n\|\eta\nabla g(p)\| \\ &\leq \max\{\|x_n - p\|, \frac{\|\eta\nabla g(p)\|}{\tau}\}.\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\eta\nabla g(p)\|}{\tau}\}, \ \ n \geq 1.$$

Hence $\{x_n\}$ is bounded also are $\{y_n\}$ and $\{\nabla g(x_n)\}$.
Consequently, by Lemma 2.4 and inequality (3.2), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|(I - \eta\alpha_n\nabla g)(y_n - p) - \alpha_n\eta\nabla g(p))\|^2 \\
&\leq \alpha_n^2\|\eta\nabla g(p)\|^2 + (1 - \tau\alpha_n)^2\|y_n - p\|^2 \\
&\quad +2\alpha_n(1 - \tau\alpha_n)\|\eta\nabla g(p)\|\|y_n - p\| \\
&\leq \alpha_n^2\|\eta\nabla g(p)\|^2 + (1 - \tau\alpha_n)^2\|x_n - p\|^2 \\
&\quad -(1 - \tau\alpha_n)^2(1 - \theta_n)(\theta_n - \lambda)\|T_1x_n - x_n\|^2 \\
&\quad +2\alpha_n(1 - \tau\alpha_n)\|\eta\nabla g(p)\|\|x_n - p\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
(1 - \tau\alpha_n)^2(1 - \theta_n)(\theta_n - \lambda)\|T_1x_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\|\eta\nabla g(p)\|^2 \\
+2\alpha_n(1 - \tau\alpha_n)\|\eta\nabla g(p)\|\|x_n - p\|.
\end{aligned}$$

Since $\{x_n\}$ is bounded, then there exists a constant $C > 0$ such that

$$\begin{aligned}
(3.4)\ (1 - \tau\alpha_n)^2(1 - \theta_n)(\theta_n - \lambda)\|T_1x_n - x_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad +\alpha_nC.
\end{aligned}$$

Now we prove that $\{x_n\}$ converges strongly to x^*. We divide the proof into two cases.
Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing sequence. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\lim_{n\to\infty}\left[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right] = 0. \tag{3.5}$$

It then implies from (3.4) that

$$\lim_{n\to\infty}(1 - \theta_n)(\theta_n - \lambda)\|T_1x_n - x_n\|^2 = 0. \tag{3.6}$$

Since $\lim\limits_{n\to\infty}\inf(1 - \theta_n)(\theta_n - \lambda) > 0$, we have

$$\lim_{n\to\infty}\left\|x_n - T_1x_n\right\| = 0. \tag{3.7}$$

Observing that,

$$\begin{aligned}
\|z_n - x_n\| &= \|\theta_nx_n + (1 - \theta_n)T_1x_n - x_n\| \\
&= \|\theta_nx_n + (1 - \theta_n)T_1x_n - \theta_nx_n - (1 - \theta_n)x_n\| \\
&= (1 - \theta_n)\|T_1x_n - x_n\| \\
&\leq \|T_1x_n - x_n\|.
\end{aligned}$$

Therefore, from (3.7) we get that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.8}$$

Next, we prove that $\limsup_{n\to+\infty}\langle \nabla g(x^*), x^* - x_n\rangle$. Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle \nabla g(x^*), x^* - x_n\rangle = \lim_{j\to+\infty}\langle \nabla g(x^*), x^* - x_{n_j})\rangle.$$

From (3.7) and $I - T_1$ is demiclosed, we obtain $a \in Fix(T_1)$. From Lemma 2.2, the fact that T_2 is nonexpansive and (3.3), we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\beta_n z_n + (1-\beta_n)T_2 z_n - p\|^2 \\ &= \beta_n\|z_n - p\|^2 + (1-\beta_n)\|T_2 z_n - p\|^2 - (1-\beta_n)\beta_n\|T_2 z_n - z_n\|^2 \\ &\leq \|x_n - p\|^2 - (1-\beta_n)\beta_n\|T_2 z_n - z_n\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|(I - \alpha_n\eta\nabla g)y_n - p\|^2 \\ &\leq \|(I - \alpha_n\eta\nabla g)(y_n - p) - \alpha_n\eta\nabla g(p)\|^2 \\ &\leq \alpha_n^2\|\eta\nabla g(p)\|^2 + (1-\alpha_n\tau)^2\|y_n - p\|^2 \\ &\quad +2\alpha_n(1-\alpha_n\tau)\|\eta\nabla g(p)\|\|y_n - p\| \\ &\leq \alpha_n^2\|\eta\nabla g(p)\|^2 + (1-\alpha_n\tau)^2\|x_n - p\|^2 \\ &\quad -(1-\alpha_n\tau)^2(1-\beta_n)\beta_n\|T_2 z_n - z_n\|^2 \\ &\quad +2\alpha_n(1-\alpha_n\tau)\|\eta\nabla g(p)\|\|x_n - p\|. \end{aligned}$$

Thus, we get

$$\begin{aligned} (1-\alpha_n\tau)^2\beta_n(1-\beta_n)\|T_2 z_n - z_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\|\eta\nabla g(p)\|^2 \\ &\quad +2\alpha_n(1-\alpha_n\tau)\|\eta\nabla g(p)\|\|x_n - p\|. \end{aligned} \tag{3.9}$$

Since $\{x_n\}$ is bounded, then there exists a constant $B > 0$ sucht that

$$(1-\alpha_n\tau)^2\beta_n(1-\beta_n)\|T_2 z_n - z_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n B. \tag{3.10}$$

It then implies from (3.10) and (3.5), that

$$\lim_{n\to\infty} \beta_n(1-\beta_n)\|T_2 z_n - z_n\| = 0. \tag{3.11}$$

Since $\lim_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty} \|z_n - T_2 z_n\| = 0. \tag{3.12}$$

Since $z_{n_j} \rightharpoonup a$, it follows from (3.12) and Lemma 2.1, we have $a \in Fix(T_2)$. Therefore, $a \in \Gamma$. On the other hand, by using x^* solves $VI(\nabla g, \Gamma)$, we then have

$$\begin{aligned}\limsup_{n\to+\infty}\langle \nabla g(x^*), x^* - x_n\rangle &= \lim_{j\to+\infty}\langle \nabla g(x^*), x^* - x_{n_j}\rangle \\ &= \langle \nabla g(x^*), x^* - a\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$.

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \langle (I - \eta\alpha_n\nabla g)y_n - x^*, x_{n+1} - x^*\rangle \\ &\leq \langle (I - \eta\alpha_n\nabla g)y_n - x^* + \eta\alpha_n\nabla g(x^*) - \eta\alpha_n\nabla g(x^*), x_{n+1} - x^*\rangle \\ &\leq \|(I - \alpha_n\eta\nabla g)(y_n - x^*)\|\|x_{n+1} - x^*\| + \alpha_n\langle \eta\nabla g(x^*), x^* - x_{n+1}\rangle \\ &\leq (1 - \alpha_n\tau)\|x_n - x^*\|^2 + 2\alpha_n\eta\langle \nabla g(x^*), x^* - x_{n+1}\rangle.\end{aligned}$$

From Lemma 2.3, its follows that $x_n \to x^*$.

Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing sequence. Set $B_n = \|x_n - x^*\|^2$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ B_k \leq B_{k+1}\}$.
We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ *as* $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. From (3.4), we have

$$(1 - \theta_{\tau(n)})(\theta_{\tau(n)} - \lambda)\left\|x_{\tau(n)} - T_1x_{\tau(n)}\right\|^2 \leq \alpha_{\tau(n)}C.$$

Since $\theta_{\tau(n)} \in]\lambda, 1[$, we have

$$\lim_{n\to\infty}\left\|x_{\tau(n)} - T_1x_{\tau(n)}\right\|^2 = 0. \tag{3.13}$$

By same argument as in case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ are bounded in H and $\limsup_{\tau(n)\to+\infty}\langle \nabla gx^*, x^* - x_{\tau(n)})\rangle \leq 0$. We have for all $n \geq n_0$,

$$0 \leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \leq \alpha_{\tau(n)}[-\tau\|x_{\tau(n)} - x^*\|^2 + 2\eta\langle \nabla gx^*, x^* - x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \leq \frac{2\eta}{\tau}\langle \nabla gx^*, x^* - x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Furthermore, for all $n \geq n_0$, we have $B_{\tau(n)} \leq B_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $n > \tau(n)$); because $B_j > B_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As consequence, we have for all $n \geq n_0$,

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

We now apply Theorem 3.1 for solving convex optimization problems over the set of common fixed point of two nonexpansive mappings without demiclosedness assumption.

Theorem 3.2. *Let H be a real Hilbert space and K be a nonempty, closed convex subset of H. Let $g : K \to \mathbb{R}$ be a differentiable, k-strongly convex real-valued function and suppose the differential map $\nabla g : K \to H$ is L-Lipschitz. Let $T_1 : K \to K$ and $T_2 : K \to K$ two nonexpansive mappings such that $\Gamma := Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Assume that $0 < \eta < \frac{2k}{L^2}$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} z_n = \theta_n x_n + (1-\theta_n)T_1 x_n, \\ y_n = \beta_n z_n + (1-\beta_n)T_2 z_n, \\ x_{n+1} = P_K(I - \eta\alpha_n \nabla g)y_n, \end{cases} \tag{3.14}$$

Suppose that $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\beta_n\}$ are the sequences such that:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$

(ii) $\theta_n \in]0,1[, \quad$ *and* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. *Then, the sequence $\{x_n\}$ defined by (3.14) converges strongly to a minimizer of g over $Fix(T_1) \cap Fix(T_2)$.*

Proof. Since every nonexpansive mapping is quasi-nonexpansive and 0-demicontractive. The proof follows Lemma 2.1 and Theorem 3.1. □

We apply Theorem 3.1 for solving the following quadratic optimization problem:

$$\text{find } x^* \in \Gamma \text{ such that } g(x^*) = \min_{x\in\Gamma} g(x), \text{ where } g(x) = \frac{1}{2}\langle Ax, x\rangle. \tag{3.15}$$

Theorem 3.3. *Let H be a real Hilbert space and K be a nonempty, closed convex subset of H. Let $A : K \to H$ be strongly bounded linear operator with coefficient $k > 0$. Let $T_1 : K \to K$ be a λ-demicontractive mapping and $T_2 : K \to K$ be a quasi-nonexpansive mapping such that $\Gamma := Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Assume that $0 < \eta < \frac{2k}{\|A\|^2}$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} z_n = \theta_n x_n + (1-\theta_n)T_1 x_n, \\ y_n = \beta_n z_n + (1-\beta_n)T_2 z_n, \\ x_{n+1} = P_K(I - \eta\alpha_n A)y_n, \end{cases} \tag{3.16}$$

Assume that $I - T_1$ and $I - T_2$ are demiclosed at origin. Suppose that $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\beta_n\}$ are the sequences such that:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$
(ii) $\theta_n \in]\lambda, 1[, \quad$ *and* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. *Then, the sequence* $\{x_n\}$ *defined by (3.16) converges strongly to a solution of problem (3.15).*

Proof. The proof follows Theorem 3.1 and Remark 1.1 with $\nabla gx = Ax$. □

Now, we give some remarks on our results as follows:

(1) Our results improve many recent results using fixed point optimization algorithm to approximate minimizers of convex functions over the set of common fixed points of nonlinear mappings.

(2) Our results are applicable for solving variational inequality problems involving strongly monotone and Lipschitzian operator and fixed point problems involving quasi-nonexpansive and demicontractive mappings.

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