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$\eta-$RICCI SOLITONS IN LORENTZIAN $\alpha-$SASAKIAN MANIFOLDS

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Abstract. In the present paper, we have studied η-Ricci solitons in Lorentzian $\alpha-$Sasakian manifolds satisfying certain curvature conditions. The existence of $\eta-$Ricci soliton in a Lorentzian $\alpha-$Sasakian manifold has been proved by a concrete example.
Keywords: $\eta-$Ricci solitons; Lorentzian $\alpha-$Sasakian manifolds; projective curvature tensor.

1. Introduction

In 1985, J. A. Oubina [14] defined and studied a new class of almost contact metric manifolds known as trans-Sasakian manifolds, which includes α-Sasakian, β-Kenmotsu and cosymplectic structures. In 2005, A. Yildiz and C. Murathan [5] studied conformally flat and quasi-conformally flat Lorentzian $\alpha-$Sasakian manifolds. Lorentzian $\alpha-$Sasakian manifolds have been studied by many authors such as [1, 3, 6]. Recently, U. C. De and P. Majhi have studied $\phi-$Weyl semisymmetric and $\phi-$projectively semisymmetric generalized Sasakian space-forms and obtained some intersesting results [21].

In 1982, R. S. Hamilton [20] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. G. Perelman [12, 13] used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group

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of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that [17, 18]

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.1}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively. Ricci solitons in the context of general relativity have been studied by M. Ali and Z. Ahsan [16].

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by J. T. Cho and M. Kimura [15]. They have studied Ricci solitons of real hypersurfaces in a non-flat complex space form and they defined η-Ricci soliton, which satisfies the equation

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{1.2}$$

where λ and μ are real number. In particular, if $\mu = 0$, then the notion η-Ricci soliton (g, V, λ, μ) is reduced to the notion of Ricci soliton (g, V, λ). Recenty, $\eta-$Ricci solitons have been studied by various authors such as A. Singh and S. Kishor [4], A. M. Blaga [9], D. G. Prakasha and B. S. Hadimani [11], S. Ghosh [19] and many others.

The paper is organized as follows: In Section 2, we give a brief introduction of Lorentzian $\alpha-$Sasakian manifolds. In Section 3, we discuss η-Ricci solitons in Lorentzian $\alpha-$Sasakian manifolds. Section 4 is devoted to study η-Ricci solitons in $\phi-$projectively semisymmetric Lorentzian $\alpha-$Sasakian manifolds. In Section 5, we study η-parallel $\phi-$tensor Lorentzian α-Sasakian manifolds admitting $\eta-$Ricci solitons. $\eta-$Ricci solitons in Lorentzian $\alpha-$Sasakian manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor have been studied in Section 6. In Section 7, we study η-Ricci solitons in recurrent Lorentzian α-Sasakian manifolds. Finally, we construct an example of 3-dimensional Lorentzian $\alpha-$Sasakian manifold which admits an $\eta-$Ricci soliton.

2. Preliminaries

A differentiable manifold of dimension n is called a Lorentzian $\alpha-$Sasakian manifold if it admits a $(1, 1)-$tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy [5]

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

$$g(X, \xi) = \eta(X) \tag{2.5}$$

for all vector fields X, Y on M.
Also Lorentzian $\alpha-$Sasakian manifolds satisfy

$$\nabla_X \xi = -\alpha\phi X, \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y), \tag{2.7}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and $\alpha \in \mathbb{R}$.

Furthermore, on a Lorentzian $\alpha-$Sasakian manifold M, the following relations hold [5, 6]:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.8}$$

$$R(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X], \tag{2.9}$$

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,X)\xi = \alpha^2[X + \eta(X)\xi], \tag{2.11}$$

$$S(X,\xi) = (n-1)\alpha^2\eta(X), \quad S(\xi,\xi) = -(n-1)\alpha^2, \tag{2.12}$$

$$Q\xi = (n-1)\alpha^2\xi, \tag{2.13}$$

$$(\nabla_X \phi)Y = \alpha g(X,Y)\xi - \alpha\eta(Y)X \tag{2.14}$$

for any vector fields X, Y and Z on M.

Definition 2.1. A Lorentzian $\alpha-$Sasakian manifold M is said to be a generalized η-Einstein manifold if its Ricci tensor S is of the form [7]

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\phi X,Y),$$

where a, b and c are smooth functions on M. If $c = 0$, $b = c = 0$ and $b = 0$, then the manifold is said to be an $\eta-$Einstein, Einstein and a special type of generalized η-Einstein manifold, respectively.

Definition 2.2. The projective curvature tensor C in an $n-$dimensional Lorentzian α-Sasakian manifold M is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{2.15}$$

where R is the Riemannian curvature tensor and r is the scalar curvature of the manifold.

3. η-Ricci solitions in Lorentzian $\alpha-$Sasakian manifolds

Suppose that a Lorentzian $\alpha-$Sasakian manifold admits an η-Ricci soliton (g, ξ, λ, μ). Then (1.2) holds and thus we have

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{3.1}$$

In a Lorentzian $\alpha-$Sasakian manifold, we find

$$(\pounds_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = -2\alpha g(X, \phi Y). \tag{3.2}$$

Combining (3.1) and (3.2), it follows that

$$S(X,Y) = -\lambda g(X,Y) + \alpha g(\phi X, Y) - \mu\eta(X)\eta(Y). \tag{3.3}$$

It yields

$$QX = -\lambda X + \alpha\phi X - \mu\eta(X)\xi. \tag{3.4}$$

By taking $Y = \xi$ in (3.3) and using (2.1), (2.3) and (2.5), we get

$$S(X,\xi) = (\mu - \lambda)\eta(X). \tag{3.5}$$

Thus from (2.12) and (3.5), we obtain

$$\mu - \lambda = (n-1)\alpha^2. \tag{3.6}$$

Hence in view of (3.3) and (3.6), we can state the following theorem:

Theorem 3.1. *If (g, ξ, λ, μ) is an η-Ricci soliton in a Lorentzian $\alpha-$Sasakian manifold, then the manifold is a generalized η-Einstein manifold of the form* (3.3) *and $\mu - \lambda = (n-1)\alpha^2$.*

In particular, if we take $\mu = 0$ in (3.3) and (3.6), then we obtain

$$S(X,Y) = -\lambda g(X,Y) + \alpha g(\phi X, Y), \tag{3.7}$$

$$\lambda = -(n-1)\alpha^2, \tag{3.8}$$

respectively. Thus we have

Corollary 3.1. *If (g, ξ, λ) is a Ricci soliton in a Lorentzian $\alpha-$Sasakian manifold, then the manifold is a special type of genralized η-Einstein manifold and its Ricci solition is always shrinking.*

Now, let (g, V, λ, μ) be a Ricci soliton in a Lorentzian $\alpha-$Sasakian manifold such that V is pointwise collinear with ξ, i.e., $V = b\xi$, where b is a function. Then (1.2) holds and we have

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(X, \nabla_Y \xi) + (Yb)\eta(X)$$

$$+2S(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)=0$$

which in view of (2.6) takes the form

$$-2b\alpha g(\phi X,Y)+(Xb)\eta(Y)+(Yb)\eta(X) \tag{3.9}$$

$$+2S(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)=0.$$

Putting $Y=\xi$ in (3.9) and using (2.1), (2.3), (2.5) and (2.12), we find

$$-(Xb)+[(\xi b)+2(n-1)\alpha^2+2\lambda-2\mu]\eta(X)=0. \tag{3.10}$$

Again putting $X=\xi$ in (3.10) and using (2.1), we get

$$(\xi b)+(n-1)\alpha^2+\lambda-\mu=0. \tag{3.11}$$

Combining the equations (3.10) and (3.11), it follows that

$$db=[(n-1)\alpha^2+\lambda-\mu]\eta. \tag{3.12}$$

Now applying d on (3.12), we get

$$[(n-1)\alpha^2+\lambda-\mu]\eta=0 \quad\Rightarrow\quad \mu-\lambda=(n-1)\alpha^2, \quad d\eta\neq 0. \tag{3.13}$$

Thus from (3.12) and (3.13), we obtain $db=0$, i.e., b is a constant. Therefore, (3.9) takes form

$$S(X,Y)=-\lambda g(X,Y)+b\alpha g(\phi X,Y)-\mu\eta(X)\eta(Y). \tag{3.14}$$

Hence in view of (3.13) and (3.14), we can state the following theorem:

Theorem 3.2. *If (g,ξ,λ,μ) is an η-Ricci soliton in an $n-$dimensional Lorentzian $\alpha-$Sasakian manifold, such that V is pointwise collinear with ξ, then V is a constant multiple of ξ and the manifold is a generalized η-Einstein manifold of the form* (3.14) *and $\mu-\lambda=(n-1)\alpha^2$.*

4. η-Ricci solitions in ϕ-projectively semisymmetric Lorentzian $\alpha-$Sasakian manifolds

Definition 4.1. A Lorentzian $\alpha-$Sasakian manifold is said to be $\phi-$projectively semisymmetric if [20]

$$P(X,Y)\cdot\phi=0$$

for all X,Y on M.

Let M be an n-dimensional $\phi-$projectively semisymmetric Lorentzian $\alpha-$Sasakian manifold admits η-Ricci soliton. Therefore $P(X,Y)\cdot\phi = 0$ turns into

$$(P(X,Y)\cdot\phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0 \tag{4.1}$$

for any vector fields $X, Y, Z \in \chi(M)$. From (2.15), it follows that

$$P(X,Y)\phi Z = R(X,Y)\phi Z - \frac{1}{n-1}[S(Y,\phi Z)X - S(X,\phi Z)Y], \tag{4.2}$$

$$\phi P(X,Y)Z = \phi R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)\phi X - S(X,Z)\phi Y]. \tag{4.3}$$

Combining the equations (4.1), (4.2) and (4.3), we have

$$R(X,Y)\phi Z - \phi R(X,Y)Z - \frac{1}{n-1}[S(Y,\phi Z)X - S(X,\phi Z)Y] \tag{4.4}$$

$$+\frac{1}{n-1}[S(Y,Z)\phi X - S(X,Z)\phi Y] = 0$$

which by taking $Y = \xi$ and using (2.3), (2.9) and (2.12) is reduced to

$$S(X,\phi Z) = (n-1)\alpha^2 g(X,\phi Z). \tag{4.5}$$

In view of (3.3), (4.5) takes the form

$$[\lambda + (n-1)\alpha^2]g(X,\phi Z) - \alpha g(\phi X,\phi Z) = 0. \tag{4.6}$$

By replacing X by ϕX in (4.6) and using (2.2), we get

$$[\lambda + (n-1)\alpha^2]g(\phi X,\phi Z) - \alpha g(X,\phi Z) = 0. \tag{4.7}$$

By adding (4.6) and (4.7), we obtain

$$[\lambda + (n-1)\alpha^2 - \alpha](g(\phi X,\phi Z) + g(X,\phi Z)) = 0$$

from which it follows that $\lambda = -(n-1)\alpha^2 + \alpha$ and hence from (3.6), we get $\mu = \alpha$. Thus we can state the following theorem:

Theorem 4.1. *If (g,ξ,λ,μ) is an η-Ricci soliton in an n-dimensional $\phi-$projectively semisymmetric Lorentzian $\alpha-$Sasakian manifold, then $\lambda = -(n-1)\alpha^2 + \alpha$ and $\mu = \alpha$.*

Now from the relations (3.3), (3.6) and (4.7), we obtain

$$S(X,Y) = (n-1)\alpha^2 g(X,Y). \tag{4.8}$$

Thus we have

Corollary 4.1. *An n-dimensional $\phi-$projectively semisymmetric Lorentzian $\alpha-$Sasakian manifold admitting an η-Ricci soliton (g,ξ,λ,μ) is an Einstein manifold.*

5. η-parallel $\phi-$tensor Lorentzian α-Sasakian manifolds admitting $\eta-$Ricci solitons

In this section, we study the $\eta-$parallel $\phi-$tensor in Lorentzian α-Sasakian manifolds. If the $(1,1)$ tensor ϕ is $\eta-$parallel, then we have [10]

$$g((\nabla_X \phi)Y, Z) = 0 \tag{5.1}$$

for all $X, Y, Z \in \chi(M)$. From (2.14) and (5.1), we get

$$g(X,Y)\eta(Z) - \eta(Y)g(X,Z) = 0, \quad \text{where} \quad \alpha \neq 0. \tag{5.2}$$

Putting $Z = \xi$ in (5.2), we find

$$g(X,Y) = -\eta(X)\eta(Y)$$

which by replacing Y by QY and using (2.12) yields

$$S(X,Y) = -\alpha^2(n-1)\eta(X)\eta(Y). \tag{5.3}$$

From (3.3) and (5.3), it follows that

$$\lambda g(X,Y) - \alpha g(\phi X, Y) + (\mu - (n-1)\alpha^2)\eta(X)\eta(Y) = 0$$

which by replacing Y by ϕY becomes

$$\lambda g(X, \phi Y) - \alpha g(\phi X, \phi Y) = 0. \tag{5.4}$$

Now by replacing X by ϕX in (5.4) and using (2.2), we find

$$\lambda g(\phi X, \phi Y) - \alpha g(X, \phi Y) = 0. \tag{5.5}$$

By adding (5.4) and (5.5), we obtain $\lambda = \alpha$ and hence from (3.6) we get $\mu = \alpha + (n-1)\alpha^2$. Thus we have the following theorem:

Theorem 5.1. *If (g, ξ, λ, μ) is an η-Ricci soliton in an n-dimensional Lorentzian $\alpha-$Sasakian manifold and if the tensor ϕ is $\eta-$parallel , then $\lambda = \alpha$ and $\mu = \alpha + (n-1)\alpha^2$.*

Now from the relations (3.3), (3.6) and (5.5), we obtain

$$S(X,Y) = -(n-1)\alpha^2\eta(X)\eta(Y). \tag{5.6}$$

Thus we have

Corollary 5.1. *If (g, ξ, λ, μ) is an η-Ricci soliton in an n-dimensional Lorentzian $\alpha-$Sasakian manifold and if the tensor ϕ is $\eta-$parallel, then the manifold is a special type of $\eta-$Einstein manifold.*

6. $\eta-$Ricci solitons in Lorentzian $\alpha-$Sasakian manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor

In this section, we consider $\eta-$Ricci solitons in Lorentzian $\alpha-$Sasakian manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor. A. Gray [2] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor.

Definition 6.1. A Lorentzian $\alpha-$Sasakian manifold is said to have Codazzi type of Ricci tensor if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$$

for all $X, Y, Z \in \chi(M)$,

Taking covariant derivative of (3.3) and making use of (2.7) and (2.14), we find

$$(\nabla_X S)(Y,Z) = \alpha^2[g(X,Y)\eta(Z) - g(X,Z)\eta(Y)] \tag{6.1}$$

$$+\alpha\mu[g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y)].$$

If the Ricci tensor S is of Codazzi type, then

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{6.2}$$

In view of (6.1), (6.2) takes the form

$$\alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + \alpha\mu[g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)] = 0$$

which by putting $X = \xi$ and using (2.1), (2.3)-(2.5) gives

$$\alpha g(\phi Y,\phi Z) - \mu g(\phi Y,Z) = 0, \quad \alpha \neq 0. \tag{6.3}$$

Now by replacing Z by ϕZ in (6.3) and using (2.2), we find

$$\alpha g(\phi Y,Z) - \mu g(\phi Y,\phi Z) = 0. \tag{6.4}$$

By adding (6.3) and (6.4), we obtain $\mu = \alpha$ and hence from (3.6) we get $\lambda = \alpha - (n-1)\alpha^2$. Thus we have the following:

Theorem 6.1. *Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton in an n-dimensional Lorentzian $\alpha-$Sasakian manifold and if the manifold has Ricci tensor of Codazzi type, then $\lambda = \alpha - (n-1)\alpha^2$ and $\mu = \alpha$.*

Definition 6.2. A Lorentzian $\alpha-$Sasakian manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \tag{6.5}$$

for all $X, Y, Z \in \chi(M)$.

Let (g, ξ, λ, μ) be an $\eta-$Ricci soliton in an n-dimensional Lorentzian $\alpha-$Sasakian manifold and the manifold has cyclic parallel Ricci tensor, then (6.5) holds. Taking covariant derivative of (3.3) and making use of (2.7) and (2.14), we find

$$(\nabla_X S)(Y,Z) = \alpha^2[g(X,Y)\eta(Z) - g(X,Z)\eta(Y)] \tag{6.6}$$

$$+\alpha\mu[g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y)].$$

Similarly, we have

$$(\nabla_Y S)(Z,X) = \alpha^2[g(Y,Z)\eta(X) - g(Y,X)\eta(Z)] \tag{6.7}$$

$$+\alpha\mu[g(\phi Y,Z)\eta(X) + g(\phi Y,X)\eta(Z)],$$

and

$$(\nabla_Z S)(X,Y) = \alpha^2[g(Z,X)\eta(Y) - g(Z,Y)\eta(X)] \tag{6.8}$$

$$+\alpha\mu[g(\phi Z,X)\eta(Y) + g(\phi Z,Y)\eta(X)].$$

By using (6.6)-(6.8) in (6.5), we obtain

$$\alpha\mu[g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y) + g(\phi Y,Z)\eta(X)] = 0$$

which by taking $Z = \xi$ reduces to

$$\alpha\mu g(\phi X,Y) = 0. \tag{6.9}$$

Since the manifold under consideration is non-cosymplectic and $g(\phi X, Y) \neq 0$, in general, therefore (6.9) yields $\mu = 0$. Therefore the $\eta-$Ricci soliton becomes Ricci soliton. Thus we have the following:

Theorem 6.2. *An $\eta-$Ricci soliton in a non-cosymplectic Lorentzian $\alpha-$Sasakian manifold whose Ricci tensor is of Codazzi-type becomes a Ricci soliton.*

7. η-Ricci solitons on recurrent Lorentzian α-Sasakian manifolds

Definition 7.1. An n-dimensional Lorentzian α-Sasakian manifold is said to be recurrent if there exists a non-zero 1-form A such that [8]

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W \tag{7.1}$$

for all vector fields X, Y, Z and W on M. If the 1-form A vanishes, then the manifold reduces to a symmetric manifold.

Assume that M is a recurrent Lorentzian α-Sasakian manifold. Therefore the curvature tensor of the manifold satisfies (7.1). By a suitable contraction of (7.1), we get

$$(\nabla_X S)(Z,W) = A(X)S(Z,W). \tag{7.2}$$

This implies that

$$\nabla_X S(Z,W) - S(\nabla_X Z,W) - S(Z,\nabla_X W) = A(X)S(Z,W) \tag{7.3}$$

which by taking $W=\xi$ and using (2.6) and (2.12) yields

$$S(Z,\phi X) = (n-1)\alpha^2 g(\phi X,Z) + (n-1)\alpha A(X)\eta(Z), \quad \alpha \neq 0. \tag{7.4}$$

In view of (3.3), (7.4) takes the form

$$\alpha g(X,Z) + \alpha\eta(X)\eta(Z) = [\lambda + (n-1)\alpha^2]g(\phi X,Z) + (n-1)\alpha A(X)\eta(Z). \tag{7.5}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from (7.5), we have

$$\alpha g(X,Z) = [\lambda + (n-1)\alpha^2]g(\phi X,Z) + (n-2)\alpha\eta(X)\eta(Z). \tag{7.6}$$

By replacing Z by ϕZ in (7.6), we get

$$\alpha g(X,\phi Z) = [\lambda + (n-1)\alpha^2]g(\phi X,\phi Z) \tag{7.7}$$

which by replacing X by ϕX and using (2.2), becomes

$$\alpha g(\phi X,\phi Z) = [\lambda + (n-1)\alpha^2]g(X,\phi Z). \tag{7.8}$$

By adding (7.7) and (7.8), we obtain $\lambda = -(n-1)\alpha^2 - \alpha$ and hence from (3.6) we get $\mu = -\alpha$. Thus we can state the following:

Theorem 7.1. *If (g,ξ,λ,μ) is an $\eta-$Ricci soliton in an n-dimensional recurrent Lorentzian α-Sasakian manifold, then $\lambda = -(n-1)\alpha^2 - \alpha$ and $\mu = -\alpha$.*

Now from the relations (3.3), (3.6) and (7.7), we obtain

$$S(X,Y) = (n-1)\alpha^2 g(X,Y). \tag{7.9}$$

Thus we have

Corollary 7.1. *An n-dimensional recurrent Lorentzian α-Sasakian manifold admitting an η-Ricci soliton (g,ξ,λ,μ) is an Einstein manifold.*

Example. We consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3 : z > 0\}$, where (x,y,z) are the standard coordinates of R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = e^{-z}\frac{\partial}{\partial y}, \quad e_2 = e^{-z}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \quad e_3 = \alpha\frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point p of M. Let g be the Lorentzian like (semi-Riemannian) metric defined by

$$g(e_1,e_1) = g(e_2,e_2) = 1, \quad g(e_3,e_3) = -1, \quad g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0.$$

Let η be the 1-form defined by $\eta(X) = g(X,e_3) = g(X,\xi)$ for all $X \in \chi(M)$, and let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = e_1, \quad \phi e_2 = e_2, \quad \phi e_3 = 0.$$

By applying linearity of ϕ and g, we have

$$\eta(\xi) = g(\xi,\xi) = -1, \quad \phi^2 X = X + \eta(X)\xi \quad \text{and} \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$. Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian almost paracontact metric structure on M.
Then we have

$$[e_1,e_2] = 0, \quad [e_1,e_3] = \alpha e_1, \quad [e_2,e_3] = \alpha e_2.$$

The Levi-Civita connection ∇ of the Lorentzian metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]),$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = \alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = \alpha e_1, \quad \nabla_{e_2} e_1 = 0, \tag{7.10}$$

$$\nabla_{e_2} e_2 = \alpha e_3, \quad \nabla_{e_2} e_3 = \alpha e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

Also, one can easily verify that

$$\nabla_X \xi = -\alpha\phi X \quad \text{and} \quad (\nabla_X \phi)Y = \alpha g(X,Y)\xi - \alpha\eta(Y)X.$$

Therefore, the manifold is a Lorentzian α-Sasakian manifold. From the above results, we can easily obtain the components of the curvature tensor as follows:

$$R(e_1,e_2)e_1 = -\alpha^2 e_2, \quad R(e_1,e_3)e_1 = -\alpha^2 e_3, \quad R(e_2,e_3)e_1 = 0, \tag{7.11}$$

$$R(e_1,e_2)e_2 = \alpha^2 e_1, \quad R(e_1,e_3)e_2 = 0, \quad R(e_2,e_3)e_2 = -\alpha^2 e_3,$$

$$R(e_1,e_2)e_3 = 0, \quad R(e_1,e_3)e_3 = -\alpha^2 e_1, \quad R(e_2,e_3)e_3 = -\alpha^2 e_2$$

from which it is clear that

$$R(X,Y)Z = \alpha^2[g(Y,Z)X - g(X,Z)Y]. \tag{7.12}$$

Hence the manifold (M, ϕ, ξ, g) is a Lorentzian α-Sasakian manifold of constant curvature α^2. With the help of the above results we get the components of Ricci tensor and scalar curvature as follows:

$$S(e_1, e_1) = S(e_2, e_2) = 2\alpha^2, \quad S(e_3, e_3) = -2\alpha^2, \tag{7.13}$$

Therefore, $r = \sum_{i=1}^{3} \epsilon_i S(e_i, e_i) = 6\alpha^2$, where $\epsilon_i = g(e_i, e_i)$. From the equation (3.3) and (7.13), we obtain $\lambda = \alpha(1 - 2\alpha)$ and $\mu = \alpha$. Thus the data (g, ξ, λ, μ) for $\lambda = \alpha(1 - 2\alpha)$ and $\mu = \alpha$ defines an $\eta-$Ricci soliton on (M, ϕ, ξ, η, g).

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REFERENCES

1. A. Barman, *On Lorentzian $\alpha-$Sasakian manifolds admitting a type of semi-symmetric metric connection*, Novi Sad J. Math., **44**(2014), 77-88.
2. A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata, **7**(1978), 259-280.
3. A. Haseeb and R. Prasad, *On concircular curvature tensor in a Lorentzian $\alpha-$Sasakian manifold with respect to the quarter-symmetric non-metric connection*, Acta et Commen. Univ. Tart. de Mathematica, **22**(2018), 279-292.
4. A. Singh and S. Kishor, *Some types of η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Facta Universitatis (NIS), **33**(2018), 217-230.
5. A. Yildiz and C. Murathan, *On Lorentzian $\alpha-$Sasakian manifolds*, Kyungpook Math. J., **45**(2005), 95-103.
6. A. Yildiz, M. Turan and C. Murathan, *A class of Lorentzian $\alpha-$Sasakian manifolds*, Kyungpook Math. J., **49**(2009), 789-799.
7. A. Yildiz, U. C. De and E. Ata, *On a type of Lorentzain para-Sasakian Manifolds*, Math. Reports, 16(66), **1**(2014), 61-67.
8. A. G. Walkar, *On Ruse's spaces of recurrent curvature*, Proc. London Math. Soc., **52**(1950), 36-64.
9. A. M. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat, **30**(2016), 489-496.
10. D. E. Blair, *Riemannian Geometry of contact and symlectic manifolds*, Birkhauser, Boston, (2002).
11. D. G. Prakasha and B. S. Hadimani, *η-Ricci solitons on para-Sasakian manifolds*, J. Geom., **108**(2017), 383-392.
12. G. Perelman, *Ricci flow with surgery on three manifolds*, http://arXiv.org/abs/math/0303109, 2003, 1-22.
13. G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, http://arXiv.org/abs/math/0211159, 2002, 1-39.
14. J. A. Oubina, *New classes of contact metric structures*, Publ. Math. Debrecen, **32**(1985), 187-193.

15. J. T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J., **61**(2009), 205-212.

16. M. Ali and Z. Ahsan, *Gravitational field of Schwarzschild soliton*, Arab J. Math. Sci., **21**(2015), 15-21.

17. M. Turan, U. C. De and A. Yildiz, *Ricci solitons and gradient Ricci solitons in three-dimensional trans-Sasakian manifolds*, Filomat, **26**(2012), 363-370.

18. R. Sharma, *Certain results on K-contact and (K, μ)-contact manifolds*, J. Geom., **89**(2008), 138-147.

19. S. Ghosh, *$\eta-$Ricci solitons on quasi-Sasakian manifolds*, Analele Universitatii de Vest, Timisoara Seria Matematica Informatica, LVI(2018), 73-85.

20. R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and general relativity*, Contemp. Math., **71**, American Math. Soc., (1988), 237-262.

21. U. C. De and P. Majhi, *$\phi-$semisymmetric generalized Sasakian space-forms*, Arab J. Math. Sci., **21**(2015), 170-178.

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