FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 3 (2020), 727-740
https://doi.org/10.22190/FUMI2003727A

LEFT INVARIANT (α, β)-METRICS ON 4-DIMENSIONAL LIE GROUPS

Mona Atashafrouz, Behzad Najafi and Laurian-Ioan Pişcoran

Abstract. Let G be a 4-dimensional Lie group with an invariant para-hypercomplex structure and let $F = \beta + a\alpha + \beta^2/\alpha$ be a left invariant (α, β)-metric, where α is a Riemannian metric and β is a 1-form on G, and a is a real number. We prove that the flag curvature of F with parallel 1-form β is non-positive, except in Case 2, in which F admits both negative and positive flag curvature. Then, we determine all geodesic vectors of (G, F).

Keywords: para-hypercomplex structure; (α, β)-metric; Riemannian metric; flag curvature.

1. Introduction

Hypercomplex and para-hypercomplex structures are interesting and practical structures in differential geometry [13]. These structures have been used in theoretical physics and HKT-geometry, intensively [11]. According to V. V. Cortés and C. Mayer studies, the para-hypercomplex structures emerged as target manifold of hypermultiplets in Euclidean theories with rigid $N = 2$ supersymmetry [9]. M. L. Barberis classified the invariant hypercomplex structures on a simply-connected 4-dimensional real Lie group [3, 5]. In [6], N. Blažić and S. Vukmirović classified 4-dimensional Lie algebras admitting a para-hypercomplex structure.

Finsler geometry has many applications in mechanics, physics and biology [1]. Among Finsler metrics, (α, β)-metrics, which were first introduced by M. Matsumoto, are the important ones [16].
In [20] the third author introduced a new class of (α, β)-metrics given by $F = \beta + a\alpha + \beta^2/\alpha$ where $a \in (\frac{1}{4}, \infty)$ and studied the locally dually flatness for this type of metrics [21]. One of the key quantities in Riemannian geometry is the sectional curvature. In Finsler geometry, we have the notion of flag curvature as a natural extension of the notion of the sectional curvature [2].

Received September 11, 2019; accepted December 26, 2019
2010 *Mathematics Subject Classification.* Primary 53B40; Secondary 53C60

In the present study, we consider the left invariant 4-dimensional para-hypercomplex Lie groups and construct some Berwaldian left invariant (α,β)-metrics of type $F=\beta+a\alpha+\beta^2/\alpha$ on them. We get a formula for the flag curvature of F and prove that F is non- positive flag curvature except one case, consequently, F is not of constant Ricci curvature.

Let (G,α) be a Lie group G furnished by a left invariant Riemannian metric α. There is a natural kind of geodesics of (G,α) which are closely related to the algebraic ingredient of G. More precisely, we are interested in those geodesics which are in the form $\gamma(t)=exp(tX)$ for some tangent vector X in the Lie algebra of G, i.e., $\mathfrak{g}:=T_eG$. In other words, those geodesics which are orbits of one parameter subgroups of G. In this case, X is called a geodesic vector. This notion was extended to Finsler geometry by Latifi [14]. Here, we also obtain all geodesic vectors of the invariant (α,β)-metric $F=\beta+a\alpha+\beta^2/\alpha$.

2. Preliminaries

Let us recall some known facts about para-hypercomplex structures and Finsler spaces. Let M be a smooth manifold and $\{J_i\}_{i=1,2,3}$ be a family of fiberwise endomorphisms of TM such that

$$J_1^2=-Id_{TM}, \tag{2.1}$$

$$J_2^2=Id_{TM}, \qquad J_2\neq \pm Id_{TM}, \tag{2.2}$$

$$J_1J_2=-J_2J_1=J_3, \tag{2.3}$$

and

$$N_i=0 \qquad i=1,2,3, \tag{2.4}$$

where N_i is the Nijenhuis tensor corresponding to J_i defined as follows:

$$N_1(X,Y)=[J_1X,J_1Y]-J_1\big([X,J_1Y]+[J_1X,Y]\big)-[X,Y],$$

and

$$N_i(X,Y)=[J_iX,J_iY]-J_i\big([X,J_iY]+[J_iX,Y]\big)+[X,Y], \qquad i=2,3,$$

for all vector fields X, Y on M. A para-hypercomplex structure on a smooth manifold M is a triple $\{J_i\}_{i=1,2,3}$ such that J_1 is a complex structure and J_i, $i=2,3$, are two non-trivial integrable product structures on M satisfying (2.3).

Definition 2.1. A para-hypercomplex structure $\{J_i\}_{i=1,2,3}$ on a Lie group G is said to be left invariant if for any $a\in G$ the following diagram is commutative:

$$\begin{array}{ccc} TG & \xrightarrow{TL_a} & TG \\ {\scriptstyle J_i}\downarrow & & \downarrow{\scriptstyle J_i} \\ TG & \xrightarrow[TL_a]{} & TG \end{array}$$

That is

$$J_i = TL_a \circ J_i \circ TL_{a^{-1}}, \qquad i = 1, 2, 3$$

where $L_a : G \to G$ given by $L_a(x) = ax$ is the left translation along a and TL_a is its derivation.

A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on $TM_0 := TM \setminus \{0\}$; (ii) F is positively 1-homogeneous on the fibers of the tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0} = g_{ij}(x, y)u^i v^j,$$
$$u = u^i \frac{\partial}{\partial x^i}, v = v^j \frac{\partial}{\partial x^j} \in T_xM,$$

where $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is called the fundamental tensor of F.

An important class of Finsler metrics is the class of (α, β)-metrics which was first introduced by M. Matsumoto in 1992 [16]. An (α, β)-metric on a manifold M is a Finsler metric with the form $F = \alpha\phi(\frac{\beta}{\alpha})$, where $\alpha(x, y) = \sqrt{g_{ij}(x)y^i y^j}$, $\beta(x, y) = b_i(x)y^i$ is a Riemannian metric and a 1-form on the manifold M, respectively and $\phi : (-b_0, b_0) \to \mathbb{R}^+$ is a C^∞ function satisfying

$$\phi(s) - s\phi^{'}(s) > 0, \ \phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \tag{2.5}$$

for all $|s| \leq b < b_0$ in which $b := \|\beta\|$ denotes the norm of β with respect to α (see [17], [26] and [28]).

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4}g^{il}\Big\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big\}y^j y^k \tag{2.6}$$

is called the associated spray to (M, F). The projection of an integral curve of $\mathbf{G}$ is called a *geodesic* in M. For Riemannian metrics, $G^i(x, y)$ are quadratic with respect to y. For a general Finsler metric F, we define the Berwald curvature of F by

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \tag{2.7}$$

A Finsler metric is called Berwald metric if its Berwald curvature vanishes [18].

A Finsler metric F on a Lie group G is called left invariant if for all $a \in G$ and $Y \in T_aG$

$$F(a, Y) = F(e, (L_{a^{-1}})_{*a}Y). \tag{2.8}$$

One of the main quantities in Finsler geometry is the flag curvature which is defined as follows:

$$K(P,Y)=\frac{g_Y(R(U,Y)Y,U)}{g_Y(Y,Y).g_Y(U,U)-g_Y^2(Y,U)}, \tag{2.9}$$

where $P = span\{U,Y\}$ is a 2-plane in T_xM, $R(U,Y)Y = \nabla_U\nabla_Y Y - \nabla_Y\nabla_U Y - \nabla_{[U,Y]}Y$and ∇ is the Chern connection induced by F (for more details, see [4, 25]).

In [6], N. Blažić and S. Vukmirović classified 4-dimensional Lie algebras admitting left invariant para-hypercomplex structures. H. R. Salimi Moghaddam obtained some curvature properties of left invariant Riemannian metrics on such Lie groups [23]. In each case, let G_i be the connected 4-dimensional Lie group corresponding to the considered Lie algebra $\mathfrak{g}_i$ and $\langle,\rangle$ is an inner product on $\mathfrak{g}_i$ such that $\{X,Y,Z,W\}$ is an orthonormal basis for $\mathfrak{g}_i$. Additionally, we use g for the left invariant Riemannian metric on G_i induced by $\langle,\rangle$ and use ∇ for its Levi-Civita connection. Let us denote the Riemannian curvature tensor of g by R. Furthermore, suppose that $U = aX + bY + cZ + dW$and $V = \tilde{a}X + \tilde{b}Y + \tilde{c}Z + \tilde{d}W$ are any two independent vectors in $\mathfrak{g}_i$.
Now, we list all five classes of 4-dimensional Lie algebras admitting an invariant para-hypercomplex structure and non-zero parallel vector fields. These classes of Lie algebras were first introduced in [6].

Case 1. [6] Let $\mathfrak{g}_1$ be the Lie algebra spanned by the basis $\{X,Y,Z,W\}$ with the following Lie algebra structure:

$$[X,Y]=Y, \qquad [X,W]=W. \tag{2.10}$$

Hence, using Koszula's formula, we have

Table 2.1: Taken from [23]

	X	Y	Z	W
∇_X	0	0	0	0
∇_Y	$-Y$	X	0	0
∇_Z	0	0	0	0
∇_W	$-W$	0	0	X

Therefore, for U and V we have

$$R(V,U)U=(a\tilde{b}-b\tilde{a})(bX-aY)+(a\tilde{d}-d\tilde{a})(dX-aW)+(b\tilde{d}-d\tilde{b})(dY-bW).$$

Case 2. [6] The Lie algebra of Case 2 has the following Lie bracket:

$$[X,Y]=Z. \tag{2.11}$$

Therefore

Table 2.2: Taken from [23]

	X	Y	Z	W
∇_X	0	$\frac{1}{2}Z$	$-\frac{1}{2}Y$	0
∇_Y	$-\frac{1}{2}Z$	0	$\frac{1}{2}X$	0
∇_Z	$-\frac{1}{2}Y$	$\frac{1}{2}X$	0	0
∇_W	0	0	0	0

Hence for U and V we have

$$R(V,U)U = \frac{3}{4}(\tilde{a}b - b\tilde{a})(bX - aY) + \frac{1}{4}(a\tilde{c} - c\tilde{a})(aZ - cX) + \frac{1}{4}(b\tilde{c} - c\tilde{b})(bZ - cY).$$

Case 3. [6] The Lie algebra structure of g_3 is in the following form:

$$[X,Y] = X. \tag{2.12}$$

Hence,

Table 2.3: Taken from [23]

	X	Y	Z	W
∇_X	$-Y$	X	0	0
∇_Y	0	0	0	0
∇_Z	0	0	0	0
∇_W	0	0	0	0

as a result, for U and V we have

$$R(V,U)U = (a\tilde{b} - b\tilde{a})(bX - aY).$$

Case 4. [6] In the Lie algebra structure of Case 4, there are two real parameters λ and η. This Lie algebra has the following structure:

$$[X,Z] = X, \qquad [X,W] = Y, \qquad [Y,Z] = Y, \qquad [Y,W] = \lambda X + \eta\beta Y, \qquad \lambda,\eta \in \mathbb{R},$$

thus

Table 2.4: Taken from [23]

	X	Y	Z	W
∇_X	$-Z$	$\frac{-(1+\lambda)}{2}W$	X	$\frac{1+\lambda}{2}Y$
∇_Y	$\frac{-(1+\lambda)}{2}W$	$-(Z+\eta W)$	Y	$\frac{(1+\lambda)}{2}X+\eta Y$
∇_Z	0	0	0	0
∇_W	$\frac{\lambda-1}{2}Y$	$\frac{1-\lambda}{2}X$	0	0

Therefore, we have

$$
\begin{aligned}
R(V,U)U &= -\Big\{(a\tilde{b}-b\tilde{a})\Big(b\frac{(1+\lambda)^2-4}{4}X+a\frac{4-(1+\lambda)^2}{4}Y\Big)\\
&+ (a\tilde{c}-c\tilde{a})\Big(aZ+b\frac{1+\lambda}{2}W-cX-d\frac{1+\lambda}{2}Y\Big)+(a\tilde{d}-d\tilde{a})\\
&\times \Big(a\frac{-\lambda^2+2\lambda+3}{4}W+b\frac{1+\lambda}{2}Z+b\eta W-c\frac{1+\lambda}{2}Y\\
&+ d\frac{(1+\lambda)(\lambda-3)}{4}X-d\eta Y\Big)\\
&+ (b\tilde{c}-c\tilde{b})\Big(a\frac{1+\lambda}{2}W+bZ+b\eta W-cY-d\frac{1+\lambda}{2}X-d\beta Y\Big)\\
&+ (b\tilde{d}-d\tilde{b})\Big(a\frac{1+\lambda}{2}Z+a\eta W+b\eta Z+b\frac{3\lambda^2+4\eta^2+2\lambda-1}{4}W\\
&- c\frac{1+\lambda}{2}X-c\eta Y-d\frac{3\lambda^2+4\eta^2+2\lambda-1}{4}Y-\eta dX\Big)\Big\}. \qquad (2.13)
\end{aligned}
$$

Case 5. [6] The last Lie algebra is $\mathfrak{g}_5$ with the following Lie algebra structure:

$$[X,Y]=W, \qquad [X,W]=-Y, \qquad [Y,W]=-X. \qquad (2.14)$$

Thus

Table 2.5: Taken from [23]

	X	Y	Z	W
∇_X	0	$\frac{3}{2}W$	X	$-\frac{3}{2}Y$
∇_Y	$\frac{1}{2}W$	$-W$	0	$-\frac{1}{2}X$
∇_Z	0	0	0	0
∇_W	$-\frac{1}{2}Y$	$\frac{1}{2}X$	0	0

Thus for U and V we have

$$R(V,U)U=-\frac{1}{4}\{(a\tilde{b}-b\tilde{a})(bX-aY)+(a\tilde{d}-d\tilde{a})(dX-aW)+7(b\tilde{d}-d\tilde{b})(-dY+bW)\}.$$

3. Flag curvature of $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$

Let us give a formula for the fundamental tensor of invariant (α,β)-metrics of type $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$, where $a\in(\frac{1}{4},\infty)$ and α is a left invariant Riemannian metric on a 4-dimensional Lie group G. We consider a left invariant vector field B on G and we let β be the 1-form associated to B with respect to α, that is, for any $x\in G$ and $y\in T_xG$, $\beta_x(y)=\alpha_x(B(x),y)$. Moreover, in the reminder of this section, we require B to be parallel with respect to α, i.e., $\nabla_B B=0$, where ∇ is the Levi-Civita connection of α. It is known that in this case, the Chern connection of F coincides to the Levi-Civita connection of α, hence F is a Berwald metric [1].

For any non- zero tangent vector $Y\in T_xM$, denote the fundamental tensors of F and α by g_Y and g, respectively. By definition, we have

$$g_Y(U,V):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(Y+sU+tV)\Big]|_{s,t=0},\quad U,V\in T_xM. \tag{3.1}$$

$$g(U,V):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[\alpha^2(Y+sU+tV)\Big]|_{s,t=0},\quad U,V\in T_xM. \tag{3.2}$$

It is easy to see that

$$\begin{aligned}
g_Y(U,V) &= \frac{4g(X,Y)^4g(U,Y)g(V,Y)}{g(X,Y)^3}+\frac{3g(X,Y)^3g(U,Y)g(V,Y)}{g(Y,Y)^{\frac{5}{2}}}\\
&- \frac{g(X,Y)^3}{g(Y,Y)^2}\Big(g(X,Y)g(U,V)+4g(X,U)g(V,Y)+4g(X,V)g(U,Y)\Big)\\
&- \frac{g(X,Y)}{g(Y,Y)^{\frac{3}{2}}}\Big(ag(U,Y)g(V,Y)+g(X,Y)^2g(U,V)+3g(X,Y)g(X,U)g(V,Y)\\
&+ 3g(X,Y)g(X,V)g(U,Y)\Big)+\frac{6}{g(Y,Y)}\Big(g(X,Y)^2g(X,U)g(X,V)\Big)\\
&+ \frac{1}{\sqrt{g(Y,Y)}}\Big(ag(X,Y)g(U,V)+6g(X,Y)g(X,U)g(X,V)+ag(X,U)g(V,Y)\\
(3.3)&+ {}^ag(X,V)g(U,Y)\Big)+a^2g(U,V)+g(X,U)g(X,V)+2ag(X,U)g(X,V).
\end{aligned}$$

Remark 3.1. We know that $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$ is an (α,β)-metric with $\phi(s)=s^2+s+a$, i.e., $F=\alpha\phi(\frac{\beta}{\alpha})$. By applying the formula obtained by Z. Shen [7], we can also get the formula of $g_Y(U,V)$. Indeed, we have

$$\begin{aligned} g_Y(U,V) &= \phi^2(s)g(U,V)+\phi(s)\phi'(s)\Big(-sg(U,V)+g(X,U)\frac{g(V,Y)}{\sqrt{g(Y,Y)}} \\ &+ g(X,V)\frac{g(U,Y)}{\sqrt{g(Y,Y)}}-s\frac{g(U,V)g(V,Y)}{g(Y,Y)}\Big) \\ &+ \Big(\phi(s)\phi''(s)+(\phi'(s))^2\Big)\Big(g(X,U)g(X,V)-sg(X,U)\frac{g(V,Y)}{\sqrt{g(Y,Y)}} \\ &- sg(X,V)\frac{g(U,Y)}{\sqrt{g(Y,Y)}}+s^2\frac{g(U,V)g(V,Y)}{g(Y,Y)}\Big), \end{aligned} \tag{3.4}$$

where $s=\frac{g(X,Y)}{\sqrt{g(Y,Y)}}$. It is easy to see that 3.3 and 3.4 coincide.

Let G be a 4-dimensional Lie group admitting an invariant para-hypercomplex structure. As mentioned above, all such Lie groups are classified in [6] [23]. Now we consider the cases 1-5 discussed in [23] and give the explicit formula for their flag curvature in each case. Let $Y:=U$ in (3.4), in all cases.

Case 1. Here, the only left invariant and parallel vector field with respect to α is given by $B=qZ$ with $\frac{1}{4}<|q|<\infty$. Note that here $s=\frac{g(qZ,U)}{\sqrt{g(U,U)}}=cq$, where we have used $g(U,U)=1$. In this case, it follows from (3.4)

$$\begin{aligned} g_U(R(V,U)U,V) &= -\big((\phi(s))^2-s\phi(s)\phi'(s)\big)\Big((a\tilde{b}-b\tilde{a})^2+(a\tilde{d}-d\tilde{a})^2+(b\tilde{d}-d\tilde{b})^2\Big) \\ g_U(U,U) &= (\phi(s))^2 \\ g_U(V,V) &= \big(\phi(s)\phi''(s)+(\phi'(s))^2\big)(\tilde{c}q)^2+(\phi(s))^2-cq\phi(s)\phi'(s) \\ g_U(U,V) &= \phi(s)\phi'(s)(\tilde{c}q). \end{aligned}$$

Let $P=span\{U,V\}$. In [5], Latifi gives a formula for the flag curvature of a left invariant (α,β)-metric. Using this formula, we get the following

$$K(P,U)=\frac{-\big((\phi(s))^2-s\phi(s)\phi'(s)\big)\big\{(a\tilde{b}-b\tilde{a})^2+(a\tilde{d}-d\tilde{a})^2+(b\tilde{d}-d\tilde{b})^2\big\}}{(\phi(s))^2\big\{(\phi(s)\phi''(s)+(\phi'(s))^2\big)(\tilde{c}q)^2+(\phi(s))^2-cq\phi(s)\phi'(s)\big\}-\big(\phi(s)\phi'(s)(\tilde{c}q)\big)^2}.$$

Hence, $K(P,U)\leqslant 0$. It means that (G,F) has non-positive flag curvature.

Remark 3.2. In [12], L. Huang proved that a left invariant Finsler metric F on a Lie group G admits a direction in which the flag curvature is non-negative, provided $dim[\mathfrak{g},\mathfrak{g}]\leq dim\mathfrak{g}-2$. Thus, Case 1 shows that we can not replace non-negative with positive in Huang's theorem.

Case 2. We see that the only left invariant and parallel vector field with respect to α is given by $X = qW$ with $\frac{1}{4} < |q| < \infty$. Thus $s = \frac{g(qW,U)}{\sqrt{g(U,U)}} = qd$. A similar argument as in the Case 1 yields

$$\begin{aligned}
g_(RV,U)U,V) &= \big((\phi(s))^2 - \phi(s)\phi^{'}(s)\big)\big\{-\frac{3}{4}(a\tilde{b}-b\tilde{a})^2 + \frac{1}{4}(a\tilde{c}-c\tilde{a})^2 + \frac{1}{4}(b\tilde{c}-c\tilde{b})^2\big\}\\
g_U(U,U) &= (\phi(s))^2\\
g_U(V,V) &= \big(\phi(s)\phi^{''}(s) + (\phi^{'}(s))^2\big)(\tilde{d}q)^2 + (\phi(s))^2 - dq\phi(s)\phi^{'}(s)\\
g_U(U,V) &= \phi(s)\phi^{'}(s)(\tilde{d}q).
\end{aligned}$$

We obtain the flag curvature as follows:

$$K(P,U) = \frac{\big((\phi(s))^2 - \phi(s)\phi^{'}(s)\big)\big\{-\frac{3}{4}(a\tilde{b}-b\tilde{a})^2 + \frac{1}{4}(a\tilde{c}-c\tilde{a})^2 + \frac{1}{4}(b\tilde{c}-c\tilde{b})^2\big\}}{\phi^2(s)\big\{(\phi(s))^2 - dq\phi(s)\phi'(s) + (\phi(s)\phi''(s) + (\phi'(s))^2\big)(\tilde{d}q)^2\big\} - \big(\phi(s)\phi'(s)(\tilde{d}q)\big)^2}.$$

Unlike Case 1, in this case (G,F) admits both positive and negative flag curvature.

Case 3. According to [23], (G_3,g) admits a parallel left invariant vector field $X = q_1Z + q_2W$ such that $\frac{1}{4} < |q_1^2 + q_2^2| < \infty$. As in the previous cases, we get $s = cq_1 + dq_2$.

$$\begin{aligned}
g_U(RV,U)U,V) &= -\big((\phi(s))^2 - s\phi(s)\phi^{'}(s)\big)\big(a\tilde{b}-b\tilde{a}\big)^2\\
g_U(U,U) &= (\phi(s))^2\\
g_U(V,V) &= \big(\phi(s)\phi^{''}(s) + (\phi^{'}(s))^2\big)(\tilde{c}q_1 + \tilde{d}q_2) + (\phi(s))^2 - \big(cq_1 + dq_2\big)\phi(s)\phi^{'}(s)\\
g_U(U,V) &= \phi(s)\phi^{'}(s)\big(\tilde{c}q_1 + \tilde{d}q_2\big)
\end{aligned}$$

Therefore, the flag curvature of F is as follows:

$$K(P,U) = \frac{-\big\{\phi^2(s) - s\phi(s)\phi^{'}(s)\big\}(a\tilde{b}-b\tilde{a})^2}{\Psi}, \tag{3.5}$$

where
$\Psi := \phi^2\{(\phi\phi^{''} + \phi'^2)(\tilde{c}q_1 + \tilde{d}q_2) + \phi^2 - (cq_1 + dq_2)\phi\phi^{'}\} - (\phi\phi^{'}(\tilde{c}q_1 + \tilde{d}q_2))^2 \geqslant 0.$

Case 4. In [23], it has been shown that vector fields which are parallel to (G_4,g), are of the form $X = qW$ such that $\frac{1}{4} < |q| < \infty$. Thus $s = dq$ and we have:

$$\begin{aligned}
g_U(RV,U)U,V) &= -((\phi(s))^2 - s\phi(s)\phi^{'}(s))((a\tilde{c}-c\tilde{a})^2 + (b\tilde{c}-c\tilde{b})^2)\\
g_U(U,U) &= (\phi(s))^2\\
g_U(V,V) &= \big(\phi(s)\phi^{''}(s) + (\phi^{'}(s))^2\big)\big(\tilde{d}q)^2 + (\phi(s))^2 - dq\phi(s)\phi^{'}(s)\big)\\
g_U(U,V) &= \phi(s)\phi^{'}(s)(\tilde{d}q).
\end{aligned}$$

We have the flag curvature of F as follows:

$$K(P,U)=\frac{\big(-(\phi(s))^2-dq\phi(s)\phi^{'}(s)\big)\big\{(a\tilde{c}-c\tilde{a})^2+(b\tilde{c}-c\tilde{b})^2\big\}}{(\phi(s))^2\{\big(\phi(s)\phi''(s)+\phi'(s)^2\big)(\tilde{d}q)^2+\phi^2(s)-dq\phi(s)\phi'(s)\}-(\phi(s)\phi'(s)(\tilde{d}q))^2},$$

which are always non-positive.

Case 5. In [23], it has been shown that the parallel left invariant vector fields are of the form $X=qZ$ such that $\frac{1}{4}<|q|<\infty$. Thus $s=cq$ and we get:

$$\begin{aligned} g_U(RV,U)U,V) &= -\frac{1}{4}\big((\phi(s))^2-cq\phi(s)\phi^{'}(s)\big)\big\{(a\tilde{b}-b\tilde{a})^2+(a\tilde{d}-d\tilde{a})^2+7(b\tilde{d}-d\tilde{b})^2\big\} \\ g_U(U,U) &= (\phi(s))^2 \\ g_U(V,V) &= \big(\phi(s)\phi^{''}(s)+(\phi^{'}(s))^2\big)(\tilde{c}q)^2+(\phi(s))^2-cq\phi(s)\phi^{'}(s) \\ g_U(U,V) &= \phi(s)\phi^{'}(s)(\tilde{c}q), \end{aligned}$$

Moreover, the flag curvature is given by the following:

$$K(P,U)=\frac{-\frac{1}{4}\big((\phi(s))^2-cq\phi(s)\phi^{'}(s)\big)\big\{(a\tilde{b}-b\tilde{a})^2+(a\tilde{d}-d\tilde{a})^2+7(b\tilde{d}-d\tilde{b})^2\big\}}{(\phi(s))^2\big\{\big(\phi(s)\phi''(s)+\phi'(s)^2\big)(\tilde{c}q)^2+(\phi(s))^2-cq\phi(s)\phi'(s)\big\}-\big(\phi(s)\phi'(s)(\tilde{c}q)\big)^2}.$$

which are always non-positive.

Sumarizing the above results, we get the following.

Theorem 3.1. *In all above cases, except for the Case 2, the flag curvature of F is non-positive. Moreover, in Case 2, (G,F) admits both positive and negative flag curvature.*

Remark 3.3. In [10], S. Deng proved that if a G-invariant Randers metric $F=\alpha+\beta$ on a homogeneous manifold $\frac{G}{H}$, which is Douglas type, has negative flag curvature, then the sectional curvature of α is negative. Case 5 shows that this fact is no longer true for (α,β)-metric of type $F=\beta+a\alpha+\frac{\beta^2}{\alpha}$.

4. Geodesic vectors

In this section, we discuss the geodesic vectors of a left invariant Finsler metric $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$ on a 4-dimensional Lie group G admitting an invariant para-hypercomplex structure. We still asume that β is parallel with respect to α. Let us recall the definition of geodesic vectors.

Definition 4.1. Let F be a left invariant Finsler metric on a Lie group G. A non-zero tangent vector $B \in T_eG$ is said to be a geodesic vector of F, if the 1-parameter subgroup $t \longrightarrow exp(tB)$, $t \in \mathbb{R}_+$ is a geodesic of F.

To find all geodesic vectors of a left invariant Finsler metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ on a 4-dimensional Lie group G admitting an invariant para-hypercomplex structure, we need the following propositions.

Proposition 4.1. *(see [14]) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let F be a left-invariant Finsler metric on G. Then a non-zero vector $B \in \mathfrak{g}$ is a geodesic vector of F if and only if for every $Z \in \mathfrak{g}$*

$$g_B([B, Z], B) = 0, \tag{4.1}$$

Proposition 4.2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let F be a left-invariant (α, β)- Berwald Finsler metric on G. Then a non-zero vector $B \in \mathfrak{g}$ is a geodesic vector of F if and only if it is a geodesic vector of α.*

Now, we find all geodesic vectors in each case of all five classes given in [23], while they equipped with Left invariant Finsler metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$. Using Proposition 4.1 and 4.2, we obtain all geodesic vectors of $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ as follows.

Theorem 4.1. *The geodesic vectors of left invariant finsler metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ are given by the following*

	geodesic vectors
Case 1	$\{aX + cZ \mid a, c \in \mathbb{R}\}$
Case 2	$\{aX + bY + cZ + dW \mid bc = ac = 0\}$
Case 3	$\{bY + cZ + dW \mid b, c, d \in \mathbb{R}\}$
Case 4	$\{aX + cZ + dW \mid ac = ad\lambda = 0\}$
Case 5	$\{aX + bY + cZ + dW \mid ad = ab = 0\}$

Now, we obtain a relation between the geodesic vectors of a general (α, β)-metric F and a Riemannian metric g.

Theorem 4.2. *Let G be a Lie group and $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β)-metric arising from a Riemannian metric g and a left invariant vector filed B, i.e., $\alpha(x, y) = \sqrt{g_x(y, y)}$ and $\beta(x, y) = \alpha_x(B, y)$ Suppose that $Y \in \mathfrak{g}$ is a unit vector for which $g(B, [Y, Z]) = 0$ for all $Z \in \mathfrak{g}$. Then Y is a geodesic vector of (M, F) if and only if Y is a geodesic vector of (M, g).*

Proof. Using (3.3) and taking into account $g(B, [Y, Z]) = 0$ for all $Z \in \mathfrak{g}$, we have

$$g_Y(Y, [Y, Z]) = \Big(\phi^2(s) - \phi(s)\phi^{'}(s)\frac{g(B,Y)}{\sqrt{g(Y,Y)}}\Big)g(Y, [Y, Z]), \tag{4.2}$$

Let Y be a geodesic vector of g. Replacing (4.1) into (4.2) and using $g(B,[Y,Z]) = 0$, we have Y is a geodesic vector of (M,F).
Conversely, let Y be a unit geodesic vector of (M,F). We have

$$\Big(\phi^2(s) - \phi(s)\phi^{'}(s)g(X,Y)\Big)g(Y,[Y,Z]) = 0, \tag{4.3}$$

This completes the proof. □

Theorem 4.3. *Let (G,F) be a connected Lie group and $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ be a left- invariant Finsler metric of Berwald type on G defined by the Riemannian metric α and the vector field B. Then (G,F) is complete.*

Proof. Since F is of the Berwald type then (G,F) and (G,α) have the same connection also $\nabla B = 0$ where ∇ is Riemannian connection of α. On the other hand (G,α) is a Lie group and hence a complete space. As (G,F) and (G,α) have the same geodesics. We show that these geodesics have constant Finsler speed. Let $\sigma(t)$, $-\infty < t < \infty$ be a geodesic for F , we have

$$F(\sigma(t),\dot\sigma(t)) = g_{\sigma(t)}(B,\dot\sigma(t)) + a\sqrt{g_{\sigma(t)}(\dot\sigma(t)),\dot\sigma(t))} + \frac{g^2_{\sigma(t)}(B,\dot\sigma(t))}{\sqrt{g_{\sigma(t)}(\dot\sigma(t)),\dot\sigma(t))}}$$

Since $g_{\sigma(t)}(\dot\sigma(t),\dot\sigma(t))$ is constant, it is enough to show that $g_{\sigma(t)}(B,\dot\sigma(t))$ is also constant. we have

$$\frac{d}{dt}(g_{\sigma(t)}(B,\dot\sigma(t)) = g_{\sigma(t)}(\nabla_{\dot\sigma(t)}B,\dot\sigma(t)) + g_{\sigma(t)}(B,\nabla_{\dot\sigma(t)}\dot\sigma(t)) = 0 \tag{4.4}$$

Then this yields that these geodesics have constant Finsler speed. □
The following Proposition can be found in [14].

Proposition 4.3. *Let (M,F) be a forward geodesically complete Finsler manifold. If X is a vector field such that $F(X)$ is bounded, then X is a forward complete vector field.*

Using Proposition 4.3, we get the following.

Theorem 4.4. *Let (G,F) be a connected Lie group and $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ be a left- invariant Finsler metric of Berwald type. Then the vector field B is complete.*

REFERENCES

1. P. L. Antonelli, R. S. Ingarden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Academic Publishers, 1993.
2. G. S. Asanov, *Finsler geometry, Relativity and Gauge Theories*, D. Reidel Publishing Company, 1985.
3. M. L. Barberis, *Hypercomplex structures on four-dimensional Lie groups*, Proc. Amer. Math. Soc. **125** (4) (1997), 1043–1054.
4. D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer, New York, 2000.
5. M. L. Barberis, *Hyper-Kähler metrics conformal to left invariant metrics on four-dimensional Lie groups*, Math. Phys. Anal. Geom. **6** (2003), 1–8.
6. N. Blažić, S. Vukmirović, *Four-dimensional Lie algebras with a para-hypercomplex structure*, Rocky J. of Math. **4** (2010), 1391–1439.
7. S. S. Chern, Z. Shen, *Riemann-Finsler geometry*, World Scientific Publishing, 2004.
8. X. Cheng, Z. Shen, *A class of Finsler metrics with isotropic S-curvature*, Israel Journal of Mathematics. **169** (2009), 317–340.
9. V. V. Cortés,C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of Euclidean supersymmetry II. Hypermultiplets and the c-map*, Tech. report, Institute of Physics Publishing for SISSA, 2005.
10. S. Deng, Z. Hu, *On flag curvature of homogeneous Randers spaces*, Canad. J. Math. **65** (1) (2013), 66–81.
11. G. W. Gibbons, G. Papadopoulos and K. S. Stelle, *HKT and OKT geometries on soliton black hole moduli spaces*, Nuclear Phys. B **508** (1997), 623–658.
12. L. Huang, *Flag curvatures of homogeneous Finsler spaces*, European Journal of Mathematics. **3**(2017), 1000–1029.
13. D. Joyce, *Compact hypercomplex and quaternionic manifolds*, J.Differential Geom. **3** (1992), 743–761.
14. D. Latifi, *Homogeneous geodesics in homogeneous Finsler spaces*, Journal of Geometry and Physics. **57** (2007), 1421–1433.
15. D. Latifi, A. Razavi, *On homogeneous Finsler spaces*, Rep. Math. Phys. **57**(2006), 357–366.
16. M. Matsumoto, *On C-reducible Finsler spaces*, Tensor N.S. **24** (1972), 29–37.
17. B. Najafi, A. Tayebi, *Some curvature properties of (α, β)-metrics*, Bulletin Math de la Societe des Sciences Math de Roumanie. Tome 60 (108) No. 3, (2017), 277–291.
18. B. Najafi, A. Tayebi, *Weakly stretch Finsler metrics*, Publ Math Debrecen. **91**(2017), 441–454.
19. M. Parhizkar, D. Latifi, *On the flag curvature of invariant (α, β)-metrics*, International Journal of Geometric Methods in Modern Physics. **4**(1650039) (2016), (11 pages).
20. L.I. Pişcoran, V. N. Mishra, *Projective flatness of a new class of (α, β)-metrics*, Georgian Math Journal. DOI: 10.1515/gmj-2017-0034 (in press).

21. L.I. Pişcoran, V. N. Mishra, *The Variational Problem in Lagrange Spaces Endowed with a Special Type of* (α, β)*-metrics*, Filomat. **32**(2) (2018), 643–652.

22. L. I. Pişcoran, V. N. Mishra, *S-curvature for a new class of* (α, β)*-metrics*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A, Matematica. **111**(2017) 1187–1200.

23. H. R. Salimi Moghaddam, *On the geometry of some para-hypercomplex Lie groups*, Archivum Mathematicum (Brno). Tomus. **45** (2009), 159–170.

24. H. R. Salimi Moghaddam, *Randers metrics of Berwald type on 4-dimensional hypercomplex Lie groups*, J. Phys. A: Math. Theor. **42**(2009), 095212(7 pp).

25. Z. Shen, *Lectures on Finsler Geometry*, World Sci. Publishing, Singapore, 2001.

26. A. Tayebi, M. Barzegari, *Generalized Berwald spaces with* (α, β)*-metrics*, Indagationes Mathematicae. **27** (2016), 670–683.

27. A. Tayebi, M. Rafie Rad, *S-curvature of isotropic Berwald metrics*, Science in China, Series A: Math. **51**(2008), 2198–2204.

28. A. Tayebi, H. Sadeghi, *On generalized Douglas-Weyl* (α, β)*-metrics*, Acta Mathematica Sinica, English Series, **31**(2015), 1611-1620.

Mona Atashafrouz
Department of Mathematics and Computer Sciences
Amirkabir University (Tehran Polytechnic)
Tehran. Iran
m.atashafrooz@aut.ac.ir

Behzad Najafi
Department of Mathematics and Computer Sciences
Amirkabir University (Tehran Polytechnic)
Tehran. Iran
behzad.najafi@aut.ac.ir

Laurian-Ioan Pişcoran
North University Center of Baia Mare Technical University of Cluj Napoca
Department of Mathematics and Informatics
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania
plaurian@yahoo.com