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TWO NOTABLE CLASSES OF PROJECTIVE VECTOR FIELDS

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Abstract. Here, we find some necessary conditions for a projective vector field on a Randers metric to preserve the non-Riemannian quantities Ξ and H. They are known in the contexts as the C-projective and H-projective vector fields. We find all projective vector fields of the Funk type metrics on the Euclidean unit ball $\mathbb{B}^n(1)$.

Keywords: projective vector field; Randers metric; Funk type metrics; Euclidean unit ball.

1. Introduction

Beltrami [3] introduced the first examples of projective transformations. The projective Finsler geometry is much complicated than the projective Riemannian geometry. This complexity may impresses several projective properties which used to be proved in projective Riemannian geometry, for example Beltrami's theorem in Riemannian geometry states that the projective (i.e., locally projectively flat). Riemannian metrics are exactly those with constant sectional curvature, while this fails generally within Finsler geometry. This may even affect on the projective algebra (i.e. the lie algebra of the projective vector fields) $proj(M, F)$ and its subalgebras. To trace this, we may refer to the subalgebras of $proj(M, F)$ [7, 8]. The special projective algebra $sproj(M, F)$ consists of the projective vector fields preserving the Berwald curvature. In [11], it is proved that given any special projective vector field X on a Randers space $(M, F = \alpha + \beta)$ with the navigation data (h, W), either F is isotropic S-curvature or X is a conformal vector field for the Riemannian metric h. This result supports a Lichnerowicz-Obata type theorem for the special projective vector fields, see [4, 11]. It is guessed that some other Lichnerowicz-Obata type theorems may be established for our two notable projective subalgebras, namely, the C-projective and the H-projective algebras. We would like to examine another subalgebras of $proj(M, F)$ namely, the C-projective algebra and the H-invariant projective algebra $cproj(M, F)$ and $hproj(M, F)$, respectively. The former is shown here to be characterized by preserving the Ξ-curvature and the latter is defined by

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preserving the **H**-curvature. In [9], C-projective vector fields are studied. We prove the following infinitesimally stated results:

Theorem 1.1. *Let us suppose that $(M, F = \alpha+\beta)$ is a Randers space of dimension $n \geq 2$ and X is a C-projective vector field of (M, F). Then, at least one of the statements of (1.1) and (1.2) is held:*

$$(1.1)\qquad \begin{cases} 1.\ \ F\ is\ of\ isotropic\ S\text{-}curvature. \\ or \\ 2.\ \ b^2\alpha^2(\mathcal{L}_{\hat{X}}(\alpha^2-\beta^2)-(\beta^2\mathcal{L}_{\hat{X}}\alpha^2-\alpha^2\mathcal{L}_{\hat{X}}\beta^2)=\eta(\alpha^2-\beta^2), \end{cases}$$

where η is a polynomial of degree two on TM.

$$(1.2)\qquad \begin{cases} 3.\ \ \mathcal{L}_{\hat{X}}\alpha^2=\lambda\alpha^2. \\ or \\ 4.\ \ (8e_{00}\beta s_0-4e_{00}^2+e_{00;0}\beta)=\eta\alpha^2, \end{cases}$$

where $\lambda \in C^\infty(M)$ and $\eta(y)$ is a polynomials of degree two.

In the following, we investigate a bigger class of C-projective vector fields, namely, the vector fields which preserve H-curvature.

Theorem 1.2. *Let us suppose that $(M, F = \alpha+\beta)$ be a compact Randers space of dimension $n \geq 2$ and X is a H-projective vector field of (M, F). Then, at least one of the statements of (1.3) and (1.4) is held:*

$$(1.3)\qquad \begin{cases} 1.\ \ F\ is\ of\ isotropic\ S\text{-}curvature. \\ or \\ 2.\ \ (\alpha^2b^4+\beta^2)\mathcal{L}_{\hat{X}}(\alpha^2-\beta^2)+2b^2(\alpha^2\mathcal{L}_{\hat{X}}\beta^2-\beta^2\mathcal{L}_{\hat{X}}\alpha^2) \\ \qquad\qquad =\eta(x,y)(\alpha^2-\beta^2), \end{cases}$$

where η is a polynomial of degree two on TM and

$$(1.4)\qquad \begin{cases} 3.\ \ \mathcal{L}_{\hat{X}}\alpha^2=\lambda\alpha^2. \\ or \\ 4.\ \ -192e_{00}\beta s_0+96e_{00}^2-19e_{00;0}\beta=\lambda\alpha^2, \end{cases}$$

where $\lambda \in C^\infty(M)$ and $\eta(y)$ is a polynomials of degree two.

Here, we study a bigger class of projective transformation, namely, C-projective transformation of Randers space. C-projective algebra, i. e, the algebra of C-projective vector fields on an n-dimensional Finsler space is a sub-algebra of projective algebra, and its dimension is $n(n+2)$ at most. Let $F = \alpha+\beta$ be a Randers space. We find the conditions for a vector field to be a C-projective vector field.

2. Preliminaries

Let M be a smooth and connected manifold of dimension $n \geq 2$. T_xM denotes the tangent space of M at x. The tangent bundle of M is the union of tangent spaces $TM := \bigcup_{x\in M} T_xM$. We will denote the elements of TM by (x, y) where $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM_0 \to M$ is given by $\pi(x, y) := x$. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ with the following properties: (i) F is C^∞ on TM_0, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) the Hessian of F^2 with elements $g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^iy^j}$ is positive-definite matrix on TM_0. The pair (M, F) is then called a *Finsler space*. Throughout this paper, we denote a Riemannian metric by $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a 1-form by $\beta = b_i(x)y^i$. A globally defined spray $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where $G^i(x, y)$ are local functions on TM_0 satisfying $G^i(x, \lambda y) = \lambda^2G^i(x, y)$, $\lambda > 0$ and given by

$$G^i = \frac{1}{4}g^{ik}\{y^hF^2_{x^hy^k} - F^2_{x^k}\}. \tag{2.1}$$

Assume the following conventions:

$$G^i_j = \frac{\partial G^i}{\partial y^j}, \quad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}$$

The local functions G^i_j are coefficients of a connection in the pullback bundle $\pi^*TM \to M$ which is called the Berwald connection denoted by D. Recall that for instance, the derivatives of a vector field X and a 2-covariant tensor $T = T_{ij}dx^i\otimes dx^j$ are given by:

$$\begin{aligned} X_{i|j} &= \frac{\delta X_i}{\delta x^j} - X_rG^r_{ij} \\ T_{ij|k} &= \frac{\delta T_{ij}}{\delta x^k} - T_{rj}G^r_{ik} - T_{ir}G^r_{jk} \end{aligned} \tag{2.2}$$

where $\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - G^i_k\frac{\partial}{\partial y^i}$. Given a Finsler metric F on an n-dimensional manifold M, *the Busemann-Hausdorff volume form* $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n \mid F(y^i\frac{\partial}{\partial x^i}|_x) < 1\}}.$$

Define $g = \det(g_{ij}(x, y))$ and $\tau(x, y) := \ln\frac{\sqrt{g}}{\sigma_F(x)}$. Given a vector $y \in T_xM$, let $\gamma(t), -\epsilon < t < \epsilon$, denote the geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = y$. The function $S(x, y) := \frac{d}{dt}[\tau(\gamma(t), \dot\gamma(t))]_{|t=0}$ is called the S-curvature with respect to Busemann-Hausdorff volume form. A Finsler space is said to be *of isotropic S-curvature* if there is a function $c = c(x)$ defined on M such that $S = (n + 1)c(x)F$. It is called a Finsler space *of constant S-curvature* when c is constant. Let (M, α) be a Riemannian space and $\beta = b_i(x)y^i$ be a 1-form defined on M such that $\|\beta\|_x :=$

$\sup_{y\in T_xM}\beta(y)/\alpha(y)<1$. The Finsler metric $F=\alpha+\beta$ is called a Randers metric on the manifold M. Denote the geodesic spray coefficients of α and F by G^i_α and G^i, respectively. The Levi-Civita covariant derivative of α is denoted by ∇. Define $\nabla_j b_i$ by $(\nabla_j b_i)\theta^j := db_i - b_j\theta_i{}^j$, where $\theta^i := dx^i$ and $\theta_i{}^j := \tilde{\Gamma}^j_{ik}dx^k$ denote the Levi-Civita connection forms and ∇ denotes its associated covariant derivation of α. Let us put

$$\begin{aligned} r_{ij} &:= \frac{1}{2}(\nabla_j b_i + \nabla_i b_j),\ s_{ij} := \frac{1}{2}(\nabla_j b_i - \nabla_i b_j),\\ s^i{}_j &:= a^{ih}s_{hj},\ s_j := b_i s^i{}_j,\ e_{ij} := r_{ij} + b_i s_j + b_j s_i. \end{aligned}$$

Then G^i are given by

$$G^i = G^i_\alpha + \Big(\frac{e_{00}}{2F} - s_0\Big)y^i + \alpha s^i{}_0, \tag{2.3}$$

where $e_{00} := e_{ij}y^iy^j$, $s_0 := s_iy^i$, $s^i{}_0 := s^i{}_jy^j$ and G^i_α denote the geodesic coefficients of α.
Given any Randers space $(M, F=\alpha+\beta)$, the S-curvature takes the following form:

$$\mathbf{S} = (n+1)\{\frac{e_{00}}{2F} - s_0 - \rho_0\} \tag{2.4}$$

where $\rho = \ln(\sqrt{1-||\beta||^2_\alpha})$ and $\rho_0 = \rho_{x^i}y^i$. Akbar-Zadeh in [2] studied another non-Riemannian quantity H-curvature, which is invariant by special projective and C-projective algebras [15]. At every point $x\in M$, $\Xi_y = \Xi_i(y)dx^i$ and $H = H_{ij}dx^i\otimes dx^j$ are defined as follows:

$$\Xi_i = y^m S_{.i|m} - S_{|i} \tag{2.5}$$

$$H_{ij} = \frac{1}{2}S_{.i.j|m}y^m = \frac{1}{4}(\Xi_{i.j} + \Xi_{j.i}) \tag{2.6}$$

where "." and "|" denote the vertical and horizontal covariant derivatives, respectively, with respect to the Berwald connection. The quantity Ξ has been introduced by Zhongmin Shen using the S-curvature, cf. [14, 16]. The above quantities do not depend on the choice of connection for performing horizontal derivatives and can be derived for the Finsler metric itself.

The notion of Riemann curvature for Riemann metrics is extended to Finsler metrics. For a vector $y\in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by

$$R_y(u) := R^i_k(y)u^k\frac{\partial}{\partial x^i},$$

where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j\partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $R := \{R_y\}_{y\in TM_0}$ is called the Riemann curvature. We define the Ricci curvature as the trace of R_y, i.e., $Ric(x,y) := trace(R_y) = (n-1)R^m_m(y)$.

Every vector field X on a manifold M induces naturally an infinitesimal coordinate transformations on TM given by $(x^i, y^i) \to (\overline{x}^i, \overline{y}^i)$, given by

$$\overline{x}^i = x^i + V^i dt, \quad \overline{y}^i = y^i + y^k \frac{\partial V^i}{\partial x^k} dt.$$

Using this coordinates transformation, we may consider the notion of the complete lift $\hat{X}$ of V to a vector field on TM_0 given by

$$\hat{X} = X^i \frac{\partial}{\partial x^i} + y^k \frac{\partial X^i}{\partial x^k} \frac{\partial}{\partial y^i}$$

It is a notable remark on the Lie derivative computations that, $\mathcal{L}_{\hat{X}} y^i = 0$ and the differential operators $\mathcal{L}_{\hat{X}}$, $\frac{\partial}{\partial x^i}$, exterior differential operator d and $\frac{\partial}{\partial y^i}$ commute. The vector field $\hat{X}$ is called a projective vector field, if there is a function P (called the projective factor) on TM_0 such that $\mathcal{L}_{\hat{X}} G^i = P y^i$, cf. [1]. In this case, given any appropriate t, the local flow $\{\phi_t\}$ associated with X is a projective transformation. A projective vector field is said to be affine if $P = 0$. It is well-known that every Killing vector field is affine. Recall that, given any projective vector field X on a Riemannian space, the projective factor $P = P(x, y)$ is linear with respect to y and also it is the natural lift of a closed 1-form on M to a function on TM_0, while in the Finslerian setting these issues are non-Riemannian features. Consider the following conventional definitions of a projective vector field X; X is said to be (cf. [10])

1. special if $\mathcal{L}_{\hat{X}} E = 0$, or equivalently, $P(x, y) = P_i(x) y^i$.
2. C-projective if $P_{i|j} = P_{j|i}$.
3. H-invariant if $\mathcal{L}_{\hat{X}} H = 0$, equivalently, $P_{jk|l} = P_{jl|k}$.

The projective factor P for a projective vector field X on a Riemannian manifold is simultaneously a special and a C-projective vector field. The following theorem provides the equivalence conditions of C-projective vector fields [13]:

Theorem 2.1. *[13] Let $(M, F = \alpha + \beta)$ be a Randers space and V is a projective vector field $V \in \chi(M)$. The following statements are equivalent:*

1. V is C-projective,

2. $\mathcal{L}_V \Xi = 0$,

3. $\mathcal{L}_V \Sigma = 0$.

where $\Sigma = \Sigma_{ij} dx^i \otimes dx^j$ is defined as follows:

$$\Sigma_{ij} = \frac{1}{n+1}(S_{.i|j} - S_{.j|i})$$

Let $\hat{V}$ be a projective vector field. We have the following identities for arbitrary tensors $T_{i|j}$ and $T_{ij|k}$ [1]:

$$\begin{aligned} \mathcal{L}_{\hat{X}} T_{i|j} &= (\mathcal{L}_{\hat{X}} T_i)_{|j} - (\mathcal{L}_{\hat{X}} G^r_i) T_{r.j} - (\mathcal{L}_{\hat{X}} G^r_{ij}) T_r \\ \mathcal{L}_{\hat{X}} T_{ij|k} &= (\mathcal{L}_{\hat{X}} T_{ij})_{|k} - (\mathcal{L}_{\hat{X}} G^r_k) T_{ij.r} - (\mathcal{L}_{\hat{X}} G^r_i) T_{rj} - (\mathcal{L}_{\hat{X}} G^r_j) T_{ir} \end{aligned} \tag{2.7}$$

3. Projective vector field v.s. Ξ-curvature

Let $F = \alpha + \beta$ be a Randers metric. Due to (2.4), Ξ-curvature is directly obtained as follows:

$$\Xi_i = \frac{e_{i0|0}}{F} - \frac{e_{00|0}u_i}{2F^2} - \frac{e_{00|i}}{2F} - \Theta_i \tag{3.1}$$

where Θ_i is $\Theta_i := s_{i|0} - s_{0|i}$.

Using spray coefficients of Randers metric (2.3) and the general formula of horizontal derivatives (2.2) one can obtain the following:

$$\begin{aligned}
e_{00|0} &= e_{00;0} - \frac{4e_{00}^2}{F} + 8e_{00}s_0 - 4\alpha(q_{00} + \beta t_0 + s_0^2)\\
e_{00|i} &= e_{00;i} - \frac{5e_{00}e_{i0}}{F} + 2e_{i0}s_0 + \frac{2e_{00}^2u_i}{F^2} + 4e_{00}s_i\\
&\quad -2\alpha(q_{i0} + \beta t_i + s_0s_i) - 2\alpha_i(q_{00} + \beta t_0 + s_0^2)\\
e_{i0|0} &= e_{i0;0} - \frac{7e_{i0}e_{00}}{2F} + 3e_{i0}s_0 + \frac{e_{00}^2u_i}{F^2} + 2e_{00}s_i\\
&\quad -\alpha_i(q_{00} + \beta t_0 + s_0^2) - \alpha(q_{i0} + \beta t_i + s_0s_i)\\
e_{ij|0} &= e_{ij;0} - 2e_{ij}(\frac{e_{00}}{2F} - s_0) - e_{i0}(\frac{e_{j0}}{F} - \frac{e_{00}u_j}{2F^2} - s_j)\\
&\quad -e_{j0}(\frac{e_{i0}}{F} - \frac{e_{00}u_i}{2F^2} - s_i)\\
&\quad -\alpha_i(q_{j0} + b_jt_0 + s_0s_j) - \alpha_j(q_{i0} + b_it_0 + s_0s_i)\\
&\quad -\alpha(q_{ij} + q_{ji} + b_it_j + b_jt_i + 2s_is_j)
\end{aligned} \tag{3.2}$$

where ";" is the covariant derivatives with respect to α.

By substituting (3.2) in (3.1), we obtain:

$$\begin{aligned}
\Xi_i &= \frac{1}{2F^3}\big[(-e_{00;i} + 2e_{i0;0} - 4e_{i0}s_0)F^2 - u_i(-4(\beta t_0 + s_0^2 + q_{00})\alpha + 8s_0e_{00} + e_{00;0})F\\
&+ 4e_{00}^2u_i + 2e_{i0}e_{00}F + 4F^2\alpha(b_it_0 + s_is_0 + q_{0i}) - 2A_{i0}F^3\big]
\end{aligned}$$

where $u_i = \frac{y_i}{F}$.

Proof of Theorem 1.1 Let us suppose X be a C-projective vector field. Then by Theorem 2.1, we have $\mathcal{L}_{\hat{X}}\Xi_i = 0$. Suppose $\mathcal{L}_{\hat{X}}\alpha^2 = t_{00}$. It is easy to see that every C-projective vector field is a projective vector field and every projective vector field of $F = \alpha + \beta$, is a projective vector field of α (see [12]), then there is a projective factor $\eta = \eta(x, y)$ which is linear with respect to y. By using the Lie identity of (2.7) we obtain:

$$\begin{aligned}
\mathcal{L}_{\hat{X}}e_{i0;0} &= (\mathcal{L}_{\hat{X}}e_{i0})_{;0} - 2e_{00}\eta_i - 3e_{i0}\eta\\
\mathcal{L}_{\hat{X}}e_{00;i} &= (\mathcal{L}_{\hat{X}}e_{00})_{;0} - 4e_{00}\eta_i - 2e_{i0}\eta\\
\mathcal{L}_{\hat{X}}e_{00;0} &= (\mathcal{L}_{\hat{X}}e_{00})_{;0} - 8e_{00}\eta
\end{aligned} \tag{3.3}$$

Then by substituting (3.3) and using the Maple program, we obtain the following:

$$
\begin{aligned}
\mathcal{L}_{\hat{X}}\Xi_i &= \frac{3\Xi_i\mathcal{L}_{\hat{X}}F}{F} - \\
&- \frac{1}{2F^3}(-8u_ie_{00}\mathcal{L}_{\hat{X}}e_{00} - 4F^2s_0\mathcal{L}_{\hat{X}}e_{i0} - 4F^2e_{i0}\mathcal{L}_{\hat{X}}s_0 + 2Fe_{i0}\mathcal{L}_{\hat{X}}e_{00} \\
&+ 2Fe_{00}\mathcal{L}_{\hat{X}}e_{i0} + 6F^2\mathcal{L}_{\hat{X}}FA_{i0} + \\
&+ (-4F\alpha\mathcal{L}_{\hat{X}}u_i - 4u_i\alpha\mathcal{L}_{\hat{X}}F - 4Fu_i\frac{t_{00}}{2\alpha})(\beta t_0 + s_0^2 + q_{00}) \\
&+ 4F^2\alpha(\mathcal{L}_{\hat{X}}b_it_0 + \mathcal{L}_{\hat{X}}b_it_0 + \mathcal{L}_{\hat{X}}s_0s_i + s_0\mathcal{L}_{\hat{X}}s_i + \mathcal{L}_{\hat{X}}q_{i0}) + 2e_{i0}e_{00}\mathcal{L}_{\hat{X}}F \\
&- 8F\mathcal{L}_{\hat{X}}Fs_0e_{i0} + s_0\mathcal{L}_{\hat{X}}Fu_ie_{00} + 8Fs_0e_{00}\mathcal{L}_{\hat{X}}u_i + \\
&+ 8Fu_ie_{00}\mathcal{L}_{\hat{X}}s_0 + 8Fu_is_0\mathcal{L}_{\hat{X}}e_{00} \\
&- 4Fu_i\alpha(\mathcal{L}_{\hat{X}}\beta t_0 + 2s_0\mathcal{L}_{\hat{X}}s_0 + \mathcal{L}_{\hat{X}}t_0\beta + \mathcal{L}_{\hat{X}}q_{00}) + \\
&+ 2F^3\mathcal{L}_{\hat{X}}A_{i0} + 2F\mathcal{L}_{\hat{X}}Fe_{00;i} \\
&- 4F\mathcal{L}_{\hat{X}}Fe_{i0;0} + Fu_i((\mathcal{L}_{\hat{X}}e_{00})_{;0}) - 8\eta e_{00} + e_{00;0}F\mathcal{L}_{\hat{X}}u_i + e_{00;0}u_i\mathcal{L}_{\hat{X}}F \\
&+ (4(2F\mathcal{L}_{\hat{X}}F\alpha + F^2\frac{t_{00}}{2\alpha}))(b_it_0 + s_0s_i + q_{i0}) + \\
&+ F^2(\mathcal{L}_{\hat{X}}e_{00})_{;i} - 2\eta_ie_{00} - 3\eta e_{i0}) \\
&- 2F^2(\mathcal{L}_{\hat{X}}e_{i0})_{;0} - 4\eta_ie_{00} - 2\eta e_{i0}) - 4e_{00}^2\mathcal{L}_{\hat{X}}u_i) \\
&= -2F^4(Rat_i + \alpha Irrat_i)
\end{aligned}
$$

where $Rat_i = A_0 + A_2\alpha^2 + A_4\alpha^4 + A_6\alpha^6$ and $Irrat_i = A_7\alpha^6 + A_5\alpha^4 + A_3\alpha^2 + A_1$ and the terms $A_0, \cdots, A_6$ are respectively given in Appendix 1.

$$
\begin{aligned}
A_0 &= 8e_{00}t_{00}\beta^2s_0y_i - 4e_{00}^2t_{00}y_i\beta + e_{00;0}t_{00}\beta^2y_i \\
A_1 &= -2t_{00}(8e_{00}\beta s_0y_i - 4e_{00}^2y_i + e_{00;0}\beta y_i)
\end{aligned}
$$

The equation (3.4) is equivalent to $Rat_i = 0$ and $Irrat_i = 0$, $(i = 1, ..., n)$. The system of equations $Rat_i = 0$ and $Irrat_i = 0$ is itself equivalent to the system of equations $Rat_i - \beta Irrat_i = 0$ and $Irrat_i = 0$. By using Maple, we obtain the followings:

$$
\begin{aligned}
&Rat_i - \beta Irrat_i = \\
&= (\alpha^2 - \beta^2)\big\{\big[-32\mathcal{L}_{\hat{X}}\beta b_it_0 - 40\mathcal{L}_{\hat{X}}\beta s_0s_i - 24\mathcal{L}_{\hat{X}}b_i\beta t_0 + 8\mathcal{L}_{\hat{X}}b_is_0^2 \\
&-16\mathcal{L}_{\hat{X}}s_0\beta s_i + 16\mathcal{L}_{\hat{X}}s_0b_is_0 - 16\mathcal{L}_{\hat{X}}s_i\beta s_0 + 8\mathcal{L}_{\hat{X}}t_0\beta b_i - 24a_{i0}\mathcal{L}_{\hat{X}}\beta \\
&+8e_{i0}\mathcal{L}_{\hat{X}}s_0 - 8e_{i0}\eta - 12\mathcal{L}_{\hat{X}}a_{i0}\beta + 8\mathcal{L}_{\hat{X}}e_{i0}s_0 - 24\mathcal{L}_{\hat{X}}\beta q_{0i} - 16\mathcal{L}_{\hat{X}}\beta q_{i0} \\
&+8\mathcal{L}_{\hat{X}}b_iq_{00} + 8\mathcal{L}_{\hat{X}}q_{00}b_i - 16\mathcal{L}_{\hat{X}}q_{i0}\beta - 2\mathcal{L}_{\hat{X}}e_{00;i} + 4\mathcal{L}_{\hat{X}}e_{i0;0}\big]\alpha^4 \\
&+\big[-24a_{i0}(\mathcal{L}_{\hat{X}}\beta)\beta^2 - 64e_{00}\mathcal{L}_{\hat{X}}\beta b_is_0 - 16e_{00}\mathcal{L}_{\hat{X}}b_i\beta s_0 - 16e_{00}\mathcal{L}_{\hat{X}}s_0\beta b_i \\
&+16e_{00}\beta b_i\eta + 40e_{i0}(\mathcal{L}_{\hat{X}}\beta)\beta s_0 + 8e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^2 - 8e_{i0}\beta^2\eta - 4\mathcal{L}_{\hat{X}}a_{i0}\beta^3
\end{aligned}
$$

$$
\begin{aligned}
& -16\mathcal{L}_{\hat{X}}e_{00}\beta b_i s_0 + 8\mathcal{L}_{\hat{X}}e_{i0}\beta^2 s_0 + 40(\mathcal{L}_{\hat{X}}\beta)\beta t_0 y_i + 32\mathcal{L}_{\hat{X}}\beta s_0^2 y_i \\
& +16\mathcal{L}_{\hat{X}}s_0\beta s_0 y_i + 8\mathcal{L}_{\hat{X}}t_0\beta^2 y_i + 8\mathcal{L}_{\hat{X}}y_i\beta^2 t_0 + 8\mathcal{L}_{\hat{X}}y_i\beta s_0^2 - 8t_{00}\beta b_i t_0 \\
- \quad & 28t_{00}\beta s_0 s_i + 20t_{00}b_i s_0^2 - 24a_{i0}t_{00}\beta + 8e_{00}^2\mathcal{L}_{\hat{X}}b_i - 16e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta \\
& +16e_{00}\mathcal{L}_{\hat{X}}e_{00}b_i - 4e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta - 16e_{00}\mathcal{L}_{\hat{X}}s_0 y_i - 16e_{00}\mathcal{L}_{\hat{X}}y_i s_0 \\
& +16e_{00}\eta y_i - 8e_{00;0}\mathcal{L}_{\hat{X}}\beta b_i - 2e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta - 10e_{00;i}(\mathcal{L}_{\hat{X}}\beta)\beta - 4e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta \\
& +20e_{i0}t_{00}s_0 + 20e_{i0;0}(\mathcal{L}_{\hat{X}}\beta)\beta - 16\mathcal{L}_{\hat{X}}e_{00}s_0 y_i - 2\mathcal{L}_{\hat{X}}e_{00;0}\beta b_i - 2\mathcal{L}_{\hat{X}}e_{00;i}\beta^2 \\
& +4\mathcal{L}_{\hat{X}}e_{i0;0}\beta^2 + 32\mathcal{L}_{\hat{X}}\beta q_{00}y_i + 8\mathcal{L}_{\hat{X}}q_{00}\beta y_i + 8\mathcal{L}_{\hat{X}}y_i\beta q_{00} - 12t_{00}\beta q_{0i} \\
& -16t_{00}\beta q_{i0} + 20t_{00}b_i q_{00} - 2e_{00;0}\mathcal{L}_{\hat{X}}y_i - 5e_{00;i}t_{00} + 10e_{i0;0}t_{00} - 2\mathcal{L}_{\hat{X}}e_{00;0}y_i\big]\alpha^2 \\
- \quad & 24e_{00}^2\mathcal{L}_{\hat{X}}(\beta)\beta b_i + 24e_{00}^2\mathcal{L}_{\hat{X}}\beta y_i + 12e_{00}^2 t_{00}b_i - 24e_{00}t_{00}s_0 y_i - 3e_{00;0}t_{00}y_i\big\} \\
+ \quad & 12e_{00}^2\{[(-2\beta b_i + 2y_i)\mathcal{L}_{\hat{X}}\beta + b_i t_{00}]\alpha^2 - y_i t_{00}\beta\}
\end{aligned}
$$

By the above equation, for any point $x \in M$, the irreducible polynomial $\alpha^2 - \beta^2$ divides e_{00} or $[(-2\beta b_i + 2y_i)\mathcal{L}_{\hat{X}}\beta + b_i t_{00}]\alpha^2 - y_i t_{00}\beta$. In the first case, for a function $c \in C^\infty(M)$, $e_{00} = 2c(x)(\alpha^2 - \beta^2)$ which means that F is of isotropic S-curvature. In the second case, we have

$$
[(-2\beta b_i + 2y_i)\mathcal{L}_{\hat{X}}\beta + b_i t_{00}]\alpha^2 - y_i t_{00}\beta = \eta_i(x,y)(\alpha^2 - \beta^2) \tag{3.4}
$$

where η_i are polynomials of degree two. By contracting above equation with b^i, we obtain

$$
b^2(\mathcal{L}_{\hat{X}}(\alpha^2 - \beta^2) - (\beta^2\mathcal{L}_{\hat{X}}\alpha^2 - \alpha^2\mathcal{L}_{\hat{X}}\beta^2) = \eta(x,y)(\alpha^2 - \beta^2)
$$

where $\eta = \eta_i(x,y)b^i$ is a polynomial of degree two on TM.

By $Irrat_i = 0$, we have $\mathcal{L}_{\hat{X}}\alpha^2 = \sigma(x)\alpha^2$ which means that $\hat{X}$ is a conformal vector field of α, or α^2 divides $(8e_{00}\beta s_0 y_i - 4e_{00}^2 y_i + e_{00;0}\beta y_i)$. By contracting it with b^i, we obtain

$$
(8e_{00}\beta s_0 - 4e_{00}^2 + e_{00;0}\beta) = \lambda(x,y)\alpha^2
$$

where $\lambda(x,y)$ is a polynomial of degree two. $\square$

Example 3.1. *Let $M = \mathbb{R}^2$ and $F(x,y,u,v) = \sqrt{u^2+v^2} + au + bv$, where $a, b \in \mathbb{R}$ and $a^2 + b^2 < 1$. Since F is Berwaldian, $S = 0$ and $\Xi = 0$, i.e., every projective vector field is a C-projective vector field. Let $V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Then by a dircet calculation, we have $\mathcal{L}_{\hat{X}}\alpha^2 = 2\alpha^2$ and $\mathcal{L}_{\hat{X}}\beta = \beta$. If we substitute them in (3.4) we obtain:*

$$
[(-2b_i\beta + 2y_i)\beta + 2b_i\alpha^2]\alpha^2 - 2y_i\alpha^2\beta = \eta_i(x,y)(\alpha^2 - \beta^2) \tag{3.5}
$$

By contracting (4.4) with y^i we have $\eta(x,y) = 2\beta\alpha^2$. In this case all of the conditions Theorem 1.1 is held.

Here we make an example which satiesfies condition 1 and 3 of Theorem 1.1.

Example 3.2. *Let $M = \mathbb{B}^2(1)$ open ball on $\mathbb{R}^2$ and let F be Funk metric as:*

$$
\theta = \frac{\sqrt{|y|^2 - |x|^2|y|^2 + \langle x, y\rangle^2}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2} \tag{3.6}
$$

By [6] one can see that Funk metric is of constant S-curvature $S = \frac{1}{2}\theta$. *It is easy to see that the vector field* $V = (x^i < a, x > -a^i)\frac{\partial}{\partial x^i}$ *is a Killing vector field of* $\alpha := \frac{\sqrt{|y|^2 - |x|^2|y|^2 + <x,y>^2}}{1-|x|^2}$. *By direct calculation we have*

$$\mathcal{L}_{\hat{V}}(\beta) = - < a, y > \tag{3.7}$$

where $\beta := \frac{<x,y>}{1-|x|^2}$. *So conditions 1 and 3 of Theorem 1.1 hold. Here, we examine condition 2 and 4. Since* $s_j = 0$ *and F is of constant S-curvature, we have* $r_{00} = \alpha^2 - \beta^2$. *By taking the covariant derivative with respect to* α, *we have*

$$r_{00;0} = -2\beta(\alpha^2 - \beta^2). \tag{3.8}$$

If we substitute (3.8) in condition 4, we have

$$\frac{|y|^2}{1-|x|^2}\{2|y|^2(1-|x|^2) + < x, y >^2\} = (|y|^2(1-|x|^2) + < x, y >^2)\lambda(x, y)$$

but $\lambda(x, y)$ *should be a polynomial of degree 2, which is a contradiction. Now we consider condition 2. Since* $\mathcal{L}_{\hat{V}}\alpha^2 = 0$, *condition 2 reduces to*

$$\alpha^2(1-b^2)\mathcal{L}_{\hat{V}}\beta^2 = \eta(x, y)(\alpha^2 - \beta^2)$$

Then by substituting $b^2 = |x|^2$ *and* $\mathcal{L}_{\hat{V}}\beta^2$ *we have*

$$-2 < a, y >< x, y > \{|y|^2 + \frac{< x, y >^2}{1-|y|^2}\} = |y|^2\eta_V(x, y) \tag{3.9}$$

where η_V *is a polynomial of degree two. If we write the extended form of (3.9) we have*

$$\begin{aligned}(a_1y_1 + a_2y_2) \qquad &(x_1y_1 + x_2y_2)\{y_1^2 + y_2^2 + \frac{x_1^2y_1^2 + x_2^2y_2^2 + 2x_1x_2y_1y_2}{1-|x|^2}\}\\ &= (y_1^2 + y_2^2)(a_{11}(x)y_1^2 + 2a_{12}(x)y_1y_2 + a_{22}y_2^2)\end{aligned}$$

By comparing left and right side of above equation we have $a_1x_1x_2^2 + a_2x_1^2x_2 = 0$, *i.e.* $a = 0$ *which is contradition. Therefore, there is not any* $\eta(x, y)$ *to applies condition 2.*

Here we make an example which satisfies condition 2 and 3 of Theorem 1.1.

Example 3.3. *Let* α *be the Bergman metric on* $D = \{x \in \mathbb{R}^{2N} : |x| < 1\}$ *and* $f = \frac{1}{2}ln(1-|x|^2)$ *be the potential of* α, *and J be the complex structure. B. Chen and L. Zhao [5] proved Randers metric*

$$F_\epsilon(x, y) = \sqrt{\alpha(y, y)} + df(\epsilon y - Jy), \qquad \epsilon \neq 0$$

is of scalar flag curvature and neither projectively flat nor of isotropic S-curvature. That is the condition 1 of Theorem 1.1 is not held. Let $\hat{V}$ be a Killing vector field of F, i. e. condition 2 and 3 are established automatically. By [5] we have

$$b_k = -J^i_k f_i, \quad s_k = -f_k, \quad a_{kj} = -\tfrac{1}{2}(f_{si}J^s_k J^i_j + f_{jk}), \quad r_{00} + 2s_0\beta = 0$$

$$e_{00}(\epsilon) = 2\epsilon\beta(\beta + \epsilon f_0), \quad e_{00;0}(\epsilon) = 4\epsilon\beta f_0(\beta + \epsilon f_0 + f_0) \tag{3.10}$$

By substituting (3.10) in condition 4 Theorem 1.1, we find out that the term

$$4\epsilon\beta^2(-3f_0(\beta + \epsilon f_0) + f_0^2 - 4\epsilon(\beta + \epsilon f_0)^2) \tag{3.11}$$

should be a multiple of α^2, which is impossible. Thus condition 4 is not true.

4. Projective vector fields vs. H-projective

Let $F = \alpha + \beta$ be a Randers metric. Due to (2.6), H-curvature can be directly obtained as follows:

$$4H_{ij} = \frac{2e_{ij|0}}{F} - \frac{2e_{i0|0}u_j}{F^2} - \frac{2e_{j0|0}u_i}{F^2} - \frac{e_{00|0}F_{ij}}{F^2} + \frac{2e_{00|0}u_iu_j}{F^3} \tag{4.1}$$

By substituting (3.2) in (4.1) we obtain:

Proof of Theorem 1.2 Let us suppose X be a H-invariant vector field, i. e. $\mathcal{L}_{\hat{X}}H_{ij} = 0$ Suppose $\mathcal{L}_{\hat{X}}\alpha^2 = t_{00}$, then by using Maple program and using equations (4.1), (3.2) and (3.3) we obtain the following:

$$H_{ij} = -2F^4(Rat_{ij} + \alpha Irrat_{ij}) \tag{4.2}$$

where $Rat_{ij} = A_0 + A_2\alpha^2 + \cdots + A_{10}\alpha^{10}$ and $Irrat_{ij} = A_9\alpha^8 + \cdots + A_1$ and the terms $A_0, \cdots, A_{10}$ are respectively given in Appendix 2.

$$A_1 = 3t_{00}\beta y_i y_j(-40e_{00}\beta s_0 + 16e_{00}^2 - 5e_{00;0}\beta) \tag{4.3}$$

The equation (4.2) is equivalent to $Rat_{ij} = 0$ and $Irrat_{ij} = 0$. The system of equations $Rat_{ij} = 0$ and $Irrat_{ij} = 0$ is itself equivalent to the system of equations $Rat_{ij} - \beta Irrat_{ij} = 0$ and $Irrat_{ij} = 0$. By using Maple we obtain the followings:

$$\begin{aligned}
&Rat_{ij} - \beta Irrat_{ij} = \\
&= (\alpha^2 - \beta^2)\big\{\big[-2(\mathcal{L}_{\hat{X}}b_it_j + \mathcal{L}_{\hat{X}}b_jt_i + 2\mathcal{L}_{\hat{X}}s_is_j + 2\mathcal{L}_{\hat{X}}s_js_i + \mathcal{L}_{\hat{X}}t_ib_j \\
&+\mathcal{L}_{\hat{X}}t_jb_i + \mathcal{L}_{\hat{X}}q_{ij} + \mathcal{L}_{\hat{X}}q_{ji})\big]\alpha^8 + \big[12\mathcal{L}_{\hat{X}}\beta\beta b_it_j + 12\mathcal{L}_{\hat{X}}\beta\beta b_jt_i + 40\mathcal{L}_{\hat{X}}\beta\beta s_is_j \\
&-72\mathcal{L}_{\hat{X}}\beta b_ib_jt_0 - 44\mathcal{L}_{\hat{X}}\beta b_is_0s_j - 44\mathcal{L}_{\hat{X}}\beta b_js_0s_i - 10e_{i0}\mathcal{L}_{\hat{X}}\beta s_j - 20e_{ij}\mathcal{L}_{\hat{X}}\beta s_0 \\
&-10e_{j0}\mathcal{L}_{\hat{X}}\beta s_i + 20\mathcal{L}_{\hat{X}}\beta\beta q_{ij} + 20\mathcal{L}_{\hat{X}}\beta\beta q_{ji} - 24\mathcal{L}_{\hat{X}}\beta b_iq_{0j} - 12\mathcal{L}_{\hat{X}}\beta b_iq_{j0} \\
&-28\mathcal{L}_{\hat{X}}\beta b_jq_{0i} - 12\mathcal{L}_{\hat{X}}\beta b_jq_{i0} + 5t_{00}b_it_j + 5t_{00}b_jt_i + 10t_{00}s_is_j - 12\mathcal{L}_{\hat{X}}s_i\beta^2s_j \\
&-12\mathcal{L}_{\hat{X}}s_j\beta^2s_i - 6\mathcal{L}_{\hat{X}}t_i\beta^2b_j - 2\mathcal{L}_{\hat{X}}t_j\beta^2b_i + 4\mathcal{L}_{\hat{X}}t_j\beta^2b_j
\end{aligned}$$

$$
\begin{aligned}
&+4a_{ij}\mathcal{L}_{\hat{X}}\beta t_0+8a_{ij}\mathcal{L}_{\hat{X}}s_0s_0\\
&+4a_{ij}\mathcal{L}_{\hat{X}}t_0\beta-4e_{00}\mathcal{L}_{\hat{X}}b_is_j-4e_{00}\mathcal{L}_{\hat{X}}b_js_i\\
&-4e_{00}\mathcal{L}_{\hat{X}}s_ib_j-4e_{00}\mathcal{L}_{\hat{X}}s_jb_i+4e_{00}b_i\eta_j\\
&+4e_{00}b_j\eta_i-6e_{i0}\mathcal{L}_{\hat{X}}b_js_0-6e_{i0}\mathcal{L}_{\hat{X}}s_0b_j+6e_{i0}\mathcal{L}_{\hat{X}}s_j\beta+2\mathcal{L}_{\hat{X}}t_j\beta y_i+2\mathcal{L}_{\hat{X}}t_j\beta y_j\\
&+2\mathcal{L}_{\hat{X}}y_i\beta t_j-2\mathcal{L}_{\hat{X}}y_ib_jt_0+4\mathcal{L}_{\hat{X}}y_is_0s_j+4\mathcal{L}_{\hat{X}}y_is_jt_0\\
&-6e_{i0}\beta\eta_j+6e_{i0}b_j\eta+12e_{ij}\mathcal{L}_{\hat{X}}s_0\beta\\
&-12e_{ij}\beta\eta-6e_{j0}\mathcal{L}_{\hat{X}}b_is_0-6e_{j0}\mathcal{L}_{\hat{X}}s_0b_i+6e_{j0}\mathcal{L}_{\hat{X}}s_i\beta\\
&-6e_{j0}\beta\eta_i+6e_{j0}b_i\eta+4\mathcal{L}_{\hat{X}}a_{ij}\beta t_0\\
&-4\mathcal{L}_{\hat{X}}e_{00}b_is_j-4\mathcal{L}_{\hat{X}}e_{00}b_js_i+6\mathcal{L}_{\hat{X}}e_{i0}\beta s_j+4\mathcal{L}_{\hat{X}}a_{ij}\beta t_0+12e_{ij}\mathcal{L}_{\hat{X}}s_0\beta\\
&-6\mathcal{L}_{\hat{X}}e_{i0}b_js_0+12\mathcal{L}_{\hat{X}}e_{ij}\beta s_0+6\mathcal{L}_{\hat{X}}e_{j0}\beta s_i\\
&-6\mathcal{L}_{\hat{X}}e_{j0}b_is_0+2\mathcal{L}_{\hat{X}}\beta t_iy_j-2\mathcal{L}_{\hat{X}}b_i\beta^2t_j\\
&-8\mathcal{L}_{\hat{X}}b_ib_js_0^2-2\mathcal{L}_{\hat{X}}b_j\beta^2t_i-8\mathcal{L}_{\hat{X}}b_jb_is_0^2+2\mathcal{L}_{\hat{X}}\beta t_jy_i+8\mathcal{L}_{\hat{X}}b_i\beta q_{0j}+4\mathcal{L}_{\hat{X}}b_i\beta q_{j0}\\
&-8\mathcal{L}_{\hat{X}}b_ib_jq_{00}-2\mathcal{L}_{\hat{X}}b_it_0y_j+8\mathcal{L}_{\hat{X}}b_j\beta q_{0i}\\
&+4\mathcal{L}_{\hat{X}}b_j\beta q_{i0}-8\mathcal{L}_{\hat{X}}b_jb_iq_{00}-2\mathcal{L}_{\hat{X}}b_jt_0y_i\\
&-8\mathcal{L}_{\hat{X}}q_{00}b_ib_j+8\mathcal{L}_{\hat{X}}q_{0i}\beta b_j+8\mathcal{L}_{\hat{X}}q_{0j}\beta b_i+4\mathcal{L}_{\hat{X}}q_{i0}\beta b_j+4\mathcal{L}_{\hat{X}}q_{j0}\beta b_i+4\mathcal{L}_{\hat{X}}s_is_0y_j\\
&+4\mathcal{L}_{\hat{X}}s_it_0y_j+4\mathcal{L}_{\hat{X}}s_js_0y_i+4\mathcal{L}_{\hat{X}}s_jt_0y_i-2\mathcal{L}_{\hat{X}}t_0b_iy_j-2\mathcal{L}_{\hat{X}}t_0b_jy_i+4\mathcal{L}_{\hat{X}}t_0s_iy_j\\
&+4\mathcal{L}_{\hat{X}}t_0s_jy_i+2\mathcal{L}_{\hat{X}}y_j\beta t_i-2\mathcal{L}_{\hat{X}}y_jb_it_0+4\mathcal{L}_{\hat{X}}y_js_0s_i+4\mathcal{L}_{\hat{X}}y_js_it_0-2(\mathcal{L}_{\hat{X}}e_{j0})_{;0}b_i\\
&+4\mathcal{L}_{\hat{X}}q_{0i}y_j+4\mathcal{L}_{\hat{X}}q_{0j}y_i+4\mathcal{L}_{\hat{X}}y_iq_{0j}+4\mathcal{L}_{\hat{X}}y_jq_{0i}\\
&+4\mathcal{L}_{\hat{X}}a_{ij}s_0^2-6\mathcal{L}_{\hat{X}}q_{ij}\beta^2-10e_{ij;0}\mathcal{L}_{\hat{X}}\beta\\
&+5t_{00}q_{ij}+5t_{00}q_{ji}-6\mathcal{L}_{\hat{X}}q_{ji}\beta^2+4a_{ij}\mathcal{L}_{\hat{X}}q_{00}\\
&-2e_{00}\mathcal{L}_{\hat{X}}e_{ij}-2e_{i0;0}\mathcal{L}_{\hat{X}}b_j-2e_{ij}\mathcal{L}_{\hat{X}}e_{00}\\
&-2e_{j0;0}\mathcal{L}_{\hat{X}}b_i+4\mathcal{L}_{\hat{X}}a_{ij}q_{00}-2\mathcal{L}_{\hat{X}}e_{i0;0}b_j\\
&+6\mathcal{L}_{\hat{X}}e_{ij;0}\beta+4\mathcal{L}_{\hat{X}}s_0\beta b_js_i-16\mathcal{L}_{\hat{X}}s_0b_ib_js_0\\
&+12\mathcal{L}_{\hat{X}}s_i\beta b_js_0+8\mathcal{L}_{\hat{X}}s_i\beta b_jt_0+12\mathcal{L}_{\hat{X}}s_j\beta b_is_0\\
&+8\mathcal{L}_{\hat{X}}\beta b_is_jt_0+8\mathcal{L}_{\hat{X}}\beta b_js_it_0-8\mathcal{L}_{\hat{X}}b_i\beta b_jt_0\\
&+12\mathcal{L}_{\hat{X}}b_i\beta s_0s_j+8\mathcal{L}_{\hat{X}}b_i\beta s_jt_0-8\mathcal{L}_{\hat{X}}b_j\beta b_it_0\\
&+12\mathcal{L}_{\hat{X}}b_j\beta s_0s_i+8\mathcal{L}_{\hat{X}}b_j\beta s_it_0+4\mathcal{L}_{\hat{X}}s_0\beta b_is_j\\
&+8\mathcal{L}_{\hat{X}}s_j\beta b_it_0-8\mathcal{L}_{\hat{X}}t_0\beta b_ib_j+8\mathcal{L}_{\hat{X}}t_0\beta b_is_j\\
&+8\mathcal{L}_{\hat{X}}t_0\beta b_js_i-4\mathcal{L}_{\hat{X}}e_{i0}e_{j0}-4\mathcal{L}_{\hat{X}}e_{j0}e_{i0}\big]\alpha^6\\
&+\big[12e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta b_j-26e_{ij}t_{00}\beta s_0+36e_{j0}\mathcal{L}_{\hat{X}}\beta s_0y_i\\
&-13e_{j0}t_{00}\beta s_i+21e_{j0}t_{00}b_is_0-20a_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2t_0\\
&-24a_{ij}\mathcal{L}_{\hat{X}}\beta\beta s_0^2-10e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2s_j-20e_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2s_0\\
&-10e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^2s_i-10\mathcal{L}_{\hat{X}}\beta\beta^2t_iy_j\\
&-10\mathcal{L}_{\hat{X}}\beta\beta^2t_jy_i+44\mathcal{L}_{\hat{X}}\beta b_is_0^2y_j+44\mathcal{L}_{\hat{X}}\beta b_js_0^2y_i\\
&+12e_{j0;0}\mathcal{L}_{\hat{X}}\beta\beta b_i-28\mathcal{L}_{\hat{X}}\beta\beta q_{0i}y_j-24\mathcal{L}_{\hat{X}}\beta\beta q_{0j}y_i\\
&+8\mathcal{L}_{\hat{X}}\beta\beta q_{i0}y_j+8\mathcal{L}_{\hat{X}}\beta\beta q_{j0}y_i+44\mathcal{L}_{\hat{X}}\beta b_iq_{00}y_j
\end{aligned}
$$

$$
\begin{aligned}
&+44\mathcal{L}_{\hat{X}}\beta b_j q_{00} y_i - 12t_{00}\beta b_i q_{0j} - 6t_{00}\beta b_i q_{j0}\\
&-18t_{00}\beta b_j q_{0i} - 6t_{00}\beta b_j q_{i0} - 7t_{00}\beta t_i y_j - 7t_{00}\beta t_j y_i\\
&+28t_{00} b_i b_j q_{00} - 13t_{00} b_i t_0 y_j - 13t_{00} b_j t_0 y_i\\
&-22t_{00} s_0 s_i y_j - 22t_{00} s_0 s_j y_i - t_{00}\beta^2 b_i t_j - t_{00}\beta^2 b_j t_i\\
&+10t_{00}\beta^2 s_i s_j + 28t_{00} b_i b_j s_0^2 + 48a_{ij} e_{00}\mathcal{L}_{\hat{X}}\beta s_0\\
&-24a_{ij}\mathcal{L}_{\hat{X}}\beta\beta q_{00} - 14a_{ij} t_{00}\beta t_0 - 56e_{00} e_{i0}\mathcal{L}_{\hat{X}}\beta b_j\\
&+12e_{00} e_{ij}\mathcal{L}_{\hat{X}}\beta\beta - 56e_{00} e_{j0}\mathcal{L}_{\hat{X}}\beta b_i + 24e_{00}\mathcal{L}_{\hat{X}}\beta s_i y_j\\
&+24e_{00}\mathcal{L}_{\hat{X}}\beta s_j y_i + 14e_{00} t_{00} b_i s_j + 14e_{00} t_{00} b_j s_i\\
&-14e_{00;0}\mathcal{L}_{\hat{X}}\beta b_i b_j + 24e_{i0} e_{j0}\mathcal{L}_{\hat{X}}\beta\beta + 36e_{i0}\mathcal{L}_{\hat{X}}\beta s_0 y_j\\
&-13e_{i0} t_{00}\beta s_j + 21e_{i0} t_{00} b_j s_0 + 28\mathcal{L}_{\hat{X}}\beta\beta b_i t_0 y_j\\
&+28\mathcal{L}_{\hat{X}}\beta\beta b_j t_0 y_i + 12e_{j0;0}\mathcal{L}_{\hat{X}}\beta y_i + 7e_{j0;0} t_{00} b_i\\
&-18t_{00} q_{0i} y_j - 14t_{00} q_{0j} y_i - 14a_{ij} t_{00} s_0^2 + 4a_{ij}\mathcal{L}_{\hat{X}} t_0\beta^3\\
&-10e_{ij;0}\mathcal{L}_{\hat{X}}\beta\beta^2 + 5t_{00}\beta^2 q_{ij} + 5t_{00}\beta^2 q_{ji}\\
&+4a_{ij} e_{00;0}\mathcal{L}_{\hat{X}}\beta - 14a_{ij} t_{00} q_{00} + 7e_{00} e_{ij} t_{00}\\
&+14e_{i0} e_{j0} t_{00} + 12e_{i0;0}\mathcal{L}_{\hat{X}}\beta y_j + 7e_{i0;0} t_{00} b_j - 13e_{ij;0} t_{00}\beta\\
&+2\mathcal{L}_{\hat{X}} t_j\beta^3 y_i + 2\mathcal{L}_{\hat{X}} t_j\beta^3 y_j + 4\mathcal{L}_{\hat{X}} a_{ij}\beta^3 t_0 + 4\mathcal{L}_{\hat{X}} a_{ij}\beta^2 s_0^2\\
&+4a_{ij}\mathcal{L}_{\hat{X}} q_{00}\beta^2 - 12e_{00}^2\mathcal{L}_{\hat{X}} b_i b_j - 12e_{00}^2\mathcal{L}_{\hat{X}} b_j b_i\\
&-2e_{00}\mathcal{L}_{\hat{X}} e_{ij}\beta^2 - 4e_{i0}\mathcal{L}_{\hat{X}} e_{j0}\beta^2 - 2e_{i0;0}\mathcal{L}_{\hat{X}} b_j\beta^2\\
&+8e_{00} e_{j0}\mathcal{L}_{\hat{X}} y_i + 8e_{00}\mathcal{L}_{\hat{X}} e_{i0} y_j + 8e_{00}\mathcal{L}_{\hat{X}} e_{j0} y_i\\
&-2e_{00;0}\mathcal{L}_{\hat{X}} a_{ij}\beta + 2e_{00;0}\mathcal{L}_{\hat{X}} b_i y_j + 2e_{00;0}\mathcal{L}_{\hat{X}} b_j y_i\\
&+2e_{00;0}\mathcal{L}_{\hat{X}} y_i b_j + 2e_{00;0}\mathcal{L}_{\hat{X}} y_j b_i + 8e_{i0}\mathcal{L}_{\hat{X}} e_{00} y_j\\
&-2e_{ij}\mathcal{L}_{\hat{X}} e_{00}\beta^2 - 4e_{j0}\mathcal{L}_{\hat{X}} e_{i0}\beta^2 - 2e_{j0;0}\mathcal{L}_{\hat{X}} b_i\beta^2\\
&+4\mathcal{L}_{\hat{X}} a_{ij}\beta^2 q_{00} - 2\mathcal{L}_{\hat{X}} e_{i0;0}\beta^2 b_j - 2\mathcal{L}_{\hat{X}} e_{j0;0}\beta^2 b_i\\
&+4\mathcal{L}_{\hat{X}} q_{0i}\beta^2 y_j + 4\mathcal{L}_{\hat{X}} q_{0j}\beta^2 y_i - 4\mathcal{L}_{\hat{X}} q_{i0}\beta^2 y_j - 4\mathcal{L}_{\hat{X}} q_{j0}\beta^2 y_i\\
&+2\mathcal{L}_{\hat{X}} y_i\beta^3 t_j + 2\mathcal{L}_{\hat{X}} y_j\beta^3 t_i - 4e_{i0;0}\mathcal{L}_{\hat{X}} y_j\beta\\
&-4e_{j0;0}\mathcal{L}_{\hat{X}} y_i\beta + 8e_{j0}\mathcal{L}_{\hat{X}} e_{00} y_i + 2\mathcal{L}_{\hat{X}} e_{00;0} b_i y_j\\
&+2(\mathcal{L}_{\hat{X}} e_{00})_{;0} b_j y_i - 4\mathcal{L}_{\hat{X}} e_{i0;0}\beta y_j - 4\mathcal{L}_{\hat{X}} e_{j0;0}\beta y_i\\
&-8\mathcal{L}_{\hat{X}} q_{00} y_i y_j - 8\mathcal{L}_{\hat{X}} y_i q_{00} y_j - 8\mathcal{L}_{\hat{X}} y_j q_{00} y_i + 4\mathcal{L}_{\hat{X}} y_i\beta^2 q_{0j}\\
&-4\mathcal{L}_{\hat{X}} y_i\beta^2 q_{j0} - 8\mathcal{L}_{\hat{X}} y_i s_0^2 y_j + 4\mathcal{L}_{\hat{X}} y_j\beta^2 q_{0i}\\
&-4\mathcal{L}_{\hat{X}} y_j\beta^2 q_{i0} - 8\mathcal{L}_{\hat{X}} y_j s_0^2 y_i + 8a_{ij} e_{00}\mathcal{L}_{\hat{X}} e_{00}\\
&-2a_{ij}\mathcal{L}_{\hat{X}} e_{00;0}\beta + 8e_{00} e_{i0}\mathcal{L}_{\hat{X}} y_j + 16e_{00}\mathcal{L}_{\hat{X}} b_i\beta b_j s_0\\
&+16e_{00}\mathcal{L}_{\hat{X}} b_j\beta b_i s_0 + 16e_{00}\mathcal{L}_{\hat{X}} s_0\beta b_i b_j - 16e_{00}\beta b_i b_j\eta\\
&+12t_{00}\beta b_i s_j t_0 + 12t_{00}\beta b_j s_i t_0 + 16\mathcal{L}_{\hat{X}} e_{00}\beta b_i b_j s_0\\
&+8\mathcal{L}_{\hat{X}}\beta\beta s_i t_0 y_j + 8\mathcal{L}_{\hat{X}}\beta\beta s_j t_0 y_i - 8\mathcal{L}_{\hat{X}} s_0\beta b_i s_0 y_j\\
&-8\mathcal{L}_{\hat{X}} s_0\beta b_j s_0 y_i + 36e_{i0}\mathcal{L}_{\hat{X}}\beta\beta b_j s_0 + 36e_{j0}\mathcal{L}_{\hat{X}}\beta\beta b_i s_0\\
&-24\mathcal{L}_{\hat{X}}\beta\beta s_0 s_i y_j - 24\mathcal{L}_{\hat{X}}\beta\beta s_0 s_j y_i - 20t_{00}\beta b_i b_j t_0 - 30t_{00}\beta b_i s_0 s_j
\end{aligned}
$$

$$
\begin{aligned}
&-30t_{00}\beta b_j s_0 s_i + 24e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_i s_j \\
&+24e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_j s_i - 112e_{00}\mathcal{L}_{\hat{X}}\beta b_i b_j s_0 + 4e_{00}^2\mathcal{L}_{\hat{X}}a_{ij} \\
&+8a_{ij}\mathcal{L}_{\hat{X}}s_0\beta^2 s_0 - 4e_{00}\mathcal{L}_{\hat{X}}b_i\beta^2 s_j - 4e_{00}\mathcal{L}_{\hat{X}}b_j\beta^2 s_i \\
&-4e_{00}\mathcal{L}_{\hat{X}}s_i\beta^2 b_j - 4e_{00}\mathcal{L}_{\hat{X}}s_j\beta^2 b_i + 4e_{00}\beta^2 b_i\eta_j \\
&+4e_{00}\beta^2 b_j\eta_i - 10\mathcal{L}_{\hat{X}}b_j\beta^2 t_0 y_i - 4\mathcal{L}_{\hat{X}}b_j\beta s_0^2 y_i \\
&-4\mathcal{L}_{\hat{X}}s_0\beta^2 s_i y_j - 4\mathcal{L}_{\hat{X}}s_0\beta^2 s_j y_i - 6e_{i0}\mathcal{L}_{\hat{X}}b_j\beta^2 s_0 \\
&-6e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^2 b_j + 6e_{i0}\beta^2 b_j\eta - 6e_{j0}\mathcal{L}_{\hat{X}}b_i\beta^2 s_0 \\
&-6e_{j0}\mathcal{L}_{\hat{X}}s_0\beta^2 b_i + 6e_{j0}\beta^2 b_i\eta - 4\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_i s_j \\
&-4\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_j s_i - 6\mathcal{L}_{\hat{X}}e_{i0}\beta^2 b_j s_0 - 6\mathcal{L}_{\hat{X}}e_{j0}\beta^2 b_i s_0 \\
&+4\mathcal{L}_{\hat{X}}s_i\beta^2 t_0 y_j + 4\mathcal{L}_{\hat{X}}s_j\beta^2 t_0 y_i - 10\mathcal{L}_{\hat{X}}t_0\beta^2 b_i y_j \\
&-10\mathcal{L}_{\hat{X}}t_0\beta^2 b_j y_i + 4\mathcal{L}_{\hat{X}}t_0\beta^2 s_i y_j + 4\mathcal{L}_{\hat{X}}t_0\beta^2 s_j y_i \\
&-10\mathcal{L}_{\hat{X}}y_i\beta^2 b_j t_0 + 4\mathcal{L}_{\hat{X}}y_i\beta^2 s_j t_0 - 4\mathcal{L}_{\hat{X}}y_i\beta b_j s_0^2 \\
&-10\mathcal{L}_{\hat{X}}y_j\beta^2 b_i t_0 + 4\mathcal{L}_{\hat{X}}y_j\beta^2 s_i t_0 - 10\mathcal{L}_{\hat{X}}b_i\beta^2 t_0 y_j \\
&-4\mathcal{L}_{\hat{X}}b_i\beta s_0^2 y_j + 8e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta b_j + 8e_{00}\mathcal{L}_{\hat{X}}e_{j0}\beta b_i \\
&+16e_{00}\mathcal{L}_{\hat{X}}b_i s_0 y_j + 16e_{00}\mathcal{L}_{\hat{X}}b_j s_0 y_i + 16e_{00}\mathcal{L}_{\hat{X}}s_0 b_i y_j \\
&+16e_{00}\mathcal{L}_{\hat{X}}s_0 b_j y_i - 8e_{00}\mathcal{L}_{\hat{X}}s_i\beta y_j - 8e_{00}\mathcal{L}_{\hat{X}}s_j\beta y_i \\
&-8e_{00}\mathcal{L}_{\hat{X}}y_i\beta s_j + 16e_{00}\mathcal{L}_{\hat{X}}y_i b_j s_0 - 8e_{00}\mathcal{L}_{\hat{X}}y_j\beta s_i \\
&+16e_{00}\mathcal{L}_{\hat{X}}y_j b_i s_0 + 8e_{00}\beta\eta_i y_j + 8e_{00}\beta\eta_j y_i - 16e_{00}b_i\eta y_j \\
&-16e_{00}b_j\eta y_i + 2e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta b_j - 4\mathcal{L}_{\hat{X}}y_j\beta b_i s_0^2 \\
&-8a_{ij}e_{00}\mathcal{L}_{\hat{X}}s_0\beta + 16a_{ij}e_{00}\beta\eta - 16a_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta s_0 \\
&+8e_{00}e_{i0}\mathcal{L}_{\hat{X}}b_j\beta + 2(\mathcal{L}_{\hat{X}}e_{00})_{;0}\beta b_i b_j \\
&-12\mathcal{L}_{\hat{X}}e_{i0}\beta s_0 y_j - 12\mathcal{L}_{\hat{X}}e_{j0}\beta s_0 y_i - 8\mathcal{L}_{\hat{X}}\beta t_0 y_i y_j \\
&-4\mathcal{L}_{\hat{X}}b_i\beta q_{00} y_j - 4\mathcal{L}_{\hat{X}}b_j\beta q_{00} y_i \\
&-4\mathcal{L}_{\hat{X}}q_{00}\beta b_i y_j - 4\mathcal{L}_{\hat{X}}q_{00}\beta b_j y_i - 16\mathcal{L}_{\hat{X}}s_0 s_0 y_i y_j \\
&-8\mathcal{L}_{\hat{X}}t_0\beta y_i y_j + 2e_{00;0}\mathcal{L}_{\hat{X}}b_j\beta b_i \\
&+8e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta b_j - 12e_{i0}\mathcal{L}_{\hat{X}}s_0\beta y_j - 12e_{i0}\mathcal{L}_{\hat{X}}y_j\beta s_0 \\
&+12e_{i0}\beta\eta y_j + 8e_{j0}\mathcal{L}_{\hat{X}}e_{00}\beta b_i \\
&-12e_{j0}\mathcal{L}_{\hat{X}}s_0\beta y_i - 12e_{j0}\mathcal{L}_{\hat{X}}y_i\beta s_0 + 12e_{j0}\beta\eta y_i \\
&+8e_{00}e_{j0}\mathcal{L}_{\hat{X}}b_i\beta + 16\mathcal{L}_{\hat{X}}e_{00}b_j s_0 y_i \\
&-16e_{00}\mathcal{L}_{\hat{X}}a_{ij}\beta s_0 - 24e_{00}\mathcal{L}_{\hat{X}}e_{00}b_i b_j - 4\mathcal{L}_{\hat{X}}y_i\beta b_j q_{00} \\
&-8\mathcal{L}_{\hat{X}}y_i\beta t_0 y_j - 4\mathcal{L}_{\hat{X}}y_j\beta b_i q_{00} \\
&+6t_{00}s_i t_0 y_j + 6t_{00}s_j t_0 y_i - 8\mathcal{L}_{\hat{X}}e_{00}\beta s_i y_j \\
&-8\mathcal{L}_{\hat{X}}e_{00}\beta s_j y_i + 16\mathcal{L}_{\hat{X}}e_{00}b_i s_0 y_j \\
&-2e_{j0}\beta^3\eta_i + 2\mathcal{L}_{\hat{X}}e_{i0}\beta^3 s_j + 4\mathcal{L}_{\hat{X}}e_{ij}\beta^3 s_0 \\
&+2\mathcal{L}_{\hat{X}}e_{j0}\beta^3 s_i + 2e_{i0}\mathcal{L}_{\hat{X}}s_j\beta^3 - 2e_{i0}\beta^3\eta_j \\
&+4e_{ij}\mathcal{L}_{\hat{X}}s_0\beta^3 - 4e_{ij}\beta^3\eta + 2e_{j0}\mathcal{L}_{\hat{X}}s_i\beta^3 + 2\mathcal{L}_{\hat{X}}e_{ij;0}\beta^3\big]\alpha^4
\end{aligned}
$$

$$
\begin{aligned}
&+\big[-e_{i0}t_{00}\beta^3 s_j - 2e_{ij}t_{00}\beta^3 s_0 - 9e_{00;0}t_{00}b_j y_i \\
&+10e_{i0;0}t_{00}\beta y_j + 10e_{j0;0}t_{00}\beta y_i - t_{00}\beta^3 t_i y_j \\
&-t_{00}\beta^3 t_j y_i - 2a_{ij}t_{00}\beta^2 q_{00} + 96e_{00}^2\mathcal{L}_{\hat{X}}\beta b_i y_j \\
&+96e_{00}^2\mathcal{L}_{\hat{X}}\beta b_j y_i + 54e_{00}^2 t_{00}b_i b_j + e_{00}e_{ij}t_{00}\beta^2 \\
&+2e_{i0}e_{j0}t_{00}\beta^2 + e_{i0;0}t_{00}\beta^2 b_j + e_{j0;0}t_{00}\beta^2 b_i \\
&-e_{j0}t_{00}\beta^3 s_i - 2a_{ij}t_{00}\beta^3 t_0 - 2a_{ij}t_{00}\beta^2 s_0^2 \\
&-96\beta e_{00}^2\mathcal{L}_{\hat{X}}\beta b_i b_j - e_{ij}\mathcal{L}_{\hat{X}}\mathcal{L}_{\hat{X}}0 t_{00}\beta^3 \\
&-18a_{ij}e_{00}^2 t_{00} - 16e_{00}^2\mathcal{L}_{\hat{X}}y_i y_j - 16e_{00}^2\mathcal{L}_{\hat{X}}y_j y_i \\
&+36t_{00}\beta t_0 y_i y_j - 6t_{00}\beta b_j q_{00}y_i + t_{00}\beta^2 b_i t_0 y_j \\
&+t_{00}\beta^2 b_j t_0 y_i - 6t_{00}\beta b_i s_0^2 y_j - 6t_{00}\beta b_j s_0^2 y_i \\
&+40a_{ij}e_{00}t_{00}\beta s_0 - 4e_{00}e_{i0}t_{00}\beta b_j - 4e_{00}e_{j0}t_{00}\beta b_i + 2e_{00}t_{00}\beta^2 b_i s_j \\
&+2e_{00}t_{00}\beta^2 b_j s_i \\
&+3e_{i0}t_{00}\beta^2 b_j s_0 + 3e_{j0}t_{00}\beta^2 b_i s_0 - 160e_{00}\mathcal{L}_{\hat{X}}\beta s_0 y_i y_j \\
&+20e_{00}t_{00}\beta s_i y_j + 20e_{00}t_{00}\beta s_j y_i \\
&-72e_{00}t_{00}b_i s_0 y_j - 72e_{00}t_{00}b_j s_0 y_i - e_{00;0}t_{00}\beta b_i b_j \\
&+30e_{i0}t_{00}\beta s_0 y_j + 30e_{j0}t_{00}\beta s_0 y_i \\
&-6t_{00}\beta b_i q_{00}y_j + 4t_{00}\beta^2 s_0 s_i y_j + 4t_{00}\beta^2 s_0 s_j y_i \\
&+32e_{00}\mathcal{L}_{\hat{X}}y_j\beta s_0 y_i - 32e_{00}\beta\eta y_i y_j \\
&+2t_{00}\beta^2 s_i t_0 y_j + 2t_{00}\beta^2 s_j t_0 y_i + 32\mathcal{L}_{\hat{X}}e_{00}\beta s_0 y_i y_j \\
&+24e_{00}\mathcal{L}_{\hat{X}}s_0\beta y_i y_j - 18e_{00;0}\mathcal{L}_{\hat{X}}\beta y_i y_j \\
&+32e_{00}\mathcal{L}_{\hat{X}}y_i\beta s_0 y_j - 4t_{00}\beta^2 q_{0i}y_j - 2t_{00}\beta^2 q_{0j}y_i \\
&+10t_{00}\beta^2 q_{i0}y_j + 10t_{00}\beta^2 q_{j0}y_i + 3a_{ij}e_{00;0}t_{00}\beta \\
&-36e_{00}e_{i0}t_{00}y_j - 36e_{00}e_{j0}t_{00}y_i \\
&-9e_{00;0}t_{00}b_i y_j - 32e_{00}\mathcal{L}_{\hat{X}}e_{00}y_i y_j + 4e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta y_j \\
&+4e_{00;0}\mathcal{L}_{\hat{X}}y_j\beta y_i + 4\mathcal{L}_{\hat{X}}e_{00;0}\beta y_i y_j \\
&+36t_{00}q_{00}y_i y_j + 36t_{00}s_0^2 y_i y_j - 8e_{00}t_{00}\beta b_i b_j s_0\big]\alpha^2 \\
&-96e_{00}^2\mathcal{L}_{\hat{X}}\beta\beta^3 b_i b_j + (48(b_i b_j t_{00} \\
&+2\mathcal{L}_{\hat{X}}\beta(b_i y_j + b_j y_i)))e_{00}^2\beta^2 + (((-48b_i y_j - 48b_j y_i)t_{00} - 96y_i y_j\mathcal{L}_{\hat{X}}\beta)e_{00}^2 \\
&-80e_{00}t_{00}s_0 y_i y_j - 8e_{00;0}t_{00}y_i y_j)\beta + 88e_{00}^2 t_{00}y_i y_j\big\} \\
&+48e_{00}^2\big\{\big[(-2\beta b_i b_j + 2b_i y_j + 2b_j y_i)\mathcal{L}_{\hat{X}}\beta + b_i b_j t_{00}\big]\alpha^2 - 2y_i y_j\beta\mathcal{L}_{\hat{X}}\beta \\
&-\big[(b_i y_j + b_j y_i)\beta - y_i y_j\big]t_{00}\big\}\alpha^2
\end{aligned}
$$

By above equation, for any point $x \in M$, the irreducible polynomial $(\alpha^2 - \beta^2)$ divides e_{00}, or irreducible polynomial $(\alpha^2 - \beta^2)$ divides last equation. In the first case, for a function $c \in C^\infty(M)$, $e_{00} = 2c(x)(\alpha^2 - \beta^2)$ which means that F is of isotropic S-curvature. In the second case, the irreducible polynomial $(\alpha^2 - \beta^2)$

divides:

$$[(-2\beta b_i b_j + 2b_i y_j + 2b_j y_i)\mathcal{L}_{\hat{X}}\beta + b_i b_j t_{00}]\alpha^2$$
$$-2y_i y_j \beta \mathcal{L}_{\hat{X}}\beta - [(b_i y_j + b_j y_i)\beta - y_i y_j]t_{00}$$

By contracting previous equation with $b^i b^j$ we obtain

$$(\alpha^2 b^4 + \beta^2)\mathcal{L}_{\hat{X}}(\alpha^2 - \beta^2) + 2b^2(\alpha^2 \mathcal{L}_{\hat{X}}\beta^2 - \beta^2 \mathcal{L}_{\hat{X}}\alpha^2) = \eta(x,y)(\alpha^2 - \beta^2)$$

where $\eta(x,y)$ is a quadratic form.

By $Irrat = 0$, then α^2 divides A_1, where

$$A_1 = \frac{1}{2}\beta t_{00} y_i y_j(-192 e_{00}\beta s_0 + 96 e_{00}^2 - 19 e_{00;0}\beta)$$

In this case, $\mathcal{L}_{\hat{X}}\alpha^2 = \sigma(x)\alpha^2$ which means that $\hat{X}$ is a Killing vector field of α, or α^2 divides $-192 e_{00}\beta s_0 + 96 e_{00}^2 - 19 e_{00;0}\beta$. We have

$$-192 e_{00}\beta s_0 + 96 e_{00}^2 - 19 e_{00;0}\beta = \lambda(x,y)\alpha^2$$

where $\lambda(x,y)$ is a quadratic form.

□

Example 4.1. *Let $M = \mathbb{R}^2$ and $F(x,y,u,v) = \sqrt{u^2+v^2} + au + bv$, as it defined in Example 4.1. Then by the same argument, one can see the condition 1, 2 and 3 of Theorem 1.2 are held. If we substitute $\mathcal{L}_{\hat{X}}\alpha^2 = 2\alpha^2$ and $\mathcal{L}_{\hat{X}}\beta = \beta$ in 4 we have:*

$$[(\alpha^2 b^4 + \beta^2)(2\alpha^2 - \beta^2) = \eta^2(x,y)(\alpha^2 - \beta^2) \tag{4.4}$$

which means that $\eta(x,y) = 2(\alpha^2 b^4 + \beta^2)$. In this case, all of the conditions Theorem 2.1 hold.

Here we give an example which satisfies condition 1 and 3 of Theorem 2.1.

Example 4.2. *Let (θ, M) be the Funk metric as Example 3.2, i. e. condition 1 and 3 of Theorem 1.2 are held. If we substitute (3.8) in condition 4, we have*

$$\frac{|y|^2}{1-|x|^2}\{48|y|^2(1-|x|^2) + 19 < x,y >^2\} = (|y|^2(1-|x|^2) + < x,y >^2)\lambda(x,y)$$

but $\lambda(x,y)$ should be a polynomial of degree 2, which is a contradiction. Now we consider condition 2. Since $\mathcal{L}_{\hat{V}}\alpha^2 = 0$, condition 2 is reduced to

$$(2b^2\alpha^2 - b^4\alpha^2 - beta^2)\mathcal{L}_{\hat{V}}\beta^2 = \eta(x,y)(\alpha^2 - \beta^2)$$

Then by substituting $b^2 = |x|^2$ and $\mathcal{L}_{\hat{V}}\beta^2$ we have

$$\begin{aligned}&\{(\frac{|x|^2|y|^2}{1-|x|^2} + \frac{|x|^2 < x,y >^2}{(1-|x|^2)^2})(2-|x|^2) - \frac{< x,y >^2}{(1-|x|^2)^2}\}\\ &\times(-2 < x,y >< a,y >) = |y|^2\eta(x,y)\end{aligned} \tag{4.5}$$

where η is a polynomial of degree two. If we compute the extended form of (4.5) we have

$$a_1x_1(x_1^2 - 3x_2^2 + a_2x_2(x_2^2 - 3x_1^2) = 0$$

which means that $a = 0$, which is a contradiction. Therefore, there is not any $\eta(x, y)$ to applies condition 2.

Here we make an example which satisfies condition 2 and 3 of Theorem 1.2.

Example 4.3. *We consider Theorem 1.2 for Bergman metric. As we see in Example 3.3, conditions 2 and 3 are established automatically for a Killing vector field of F and the condition 1 of Theorem 1.1 do not hold. Then by substituting (3.10) in condition 4, we find out that the term*

$$\epsilon\beta^2(77f_0(\beta + \epsilon f_0) - 19f_0 + 96\epsilon(\beta + \epsilon f_0)^2) \tag{4.6}$$

a should be a multiple of α^2, which is impossible. Thus condition 4, does not exist.

5. C-projective vector fields of Funk type Finsler metrics

Let M be a bounded convex in $\mathbb{R}^n$ and θ be the Funk type Finsler metric on M, i.e.,

$$\theta_{x^k} = \theta\theta_{y^k} \tag{5.1}$$

The spray of a Funk type Finsler metric is given by $G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$ where $G^i = \frac{1}{2}\theta y^i$. Thus, every Funk type Finsler metric is locally projectively flat. It is easy to see that every Funk type Finsler metric is also of constant flag curvature $K = -\frac{1}{4}$. We are going to characterize all projective vector fields of Funk type Finsler metrics.

Theorem 5.1. *Let $V = V^i\frac{\partial}{\partial x^i}$ be a vector field on M. Then the complete lift of V, i.e., $\hat{V}$ is a projective vector field of (M, θ) if and only if there is a 1-form $\eta := \eta_t(x)y^t$ on M which satiesties*

$$2\frac{\partial^2 V^i}{\partial x^t\partial x^s} = \eta_t(x)\delta^i{}_s + \eta_s(x)\delta^i{}_t \tag{5.2}$$

Proof. Suppose that $\hat{V} = V^i\frac{\partial}{\partial x^i} + y^k\frac{\partial V^i}{\partial x^k}\frac{\partial}{\partial y^i}$ is a projective vector field of (M, θ). Then by [1], $\hat{V}$ is a projective vector field if and only if there exsits a function $P(x, y)$ satisfying

$$[\hat{V}, G] = Py^i. \tag{5.3}$$

By a direct calculation, we have

$$[\hat{V}, G] = (Qy^i - A^i)\frac{\partial}{\partial y^i}, \tag{5.4}$$

where $Q = (V^j\theta + y^k\frac{\partial V^j}{\partial x^k})\theta_j$ and $A^i = y^jy^k\frac{\partial^2 V^i}{\partial x^k\partial x^j}$. Comparing (5.3) with (5.4), we get

$$Qy^i - A^i = Py^i \tag{5.5}$$

By differentiating (5.5) with respect to y^s we have

$$2y^k\frac{\partial^2 V^i}{\partial x^k\partial x^s} = (P_s - Q_s)y^i + (P - Q){\delta^i}_s \tag{5.6}$$

Again by differentiating (5.6) with respect to y^t we have

$$2\frac{\partial^2 V^i}{\partial x^t\partial x^s} = (P_{st} - Q_{st})y^i + (P_s - Q_s){\delta^i}_t + (P_t - Q_t){\delta^i}_s \tag{5.7}$$

By differentiating (5.7) with respect to y^r we have

$$(P_{rst} - Q_{rst})y^i + (P_{st} - Q_{st}){\delta^i}_r + (P_{sr} - Q_{sr}){\delta^i}_t + (P_{tr} - Q_{tr}){\delta^i}_s = 0 \tag{5.8}$$

Let $i = r$ in (5.8), then we have $(n+1)(P_{st} - Q_{st}) = 0$, which means that

$$P = Q + \eta_j(x)y^j, \tag{5.9}$$

for some 1-form η on M. If we put (5.9) into (5.5), we have

$$y^jy^k\frac{\partial^2 V^i}{\partial x^k\partial x^j} = \eta_j(x)y^jy^i. \tag{5.10}$$

By differentiating (5.10) with respect to y^s we have

$$2y^k\frac{\partial^2 V^i}{\partial x^k\partial x^s} = \eta_s(x)y^i + \eta_j(x)y^j{\delta^i}_s \tag{5.11}$$

Again by differentiating (5.11) with respect to y^t we have

$$2\frac{\partial^2 V^i}{\partial x^t\partial x^s} = \eta_s(x){\delta^i}_t + \eta_t(x){\delta^i}_s \tag{5.12}$$

Thus if $\hat{V}$ is a projective vector field of the Funk type metric θ, then there is a 1-form $\eta_t(x)$ which satisfies (5.12).

The converse is trivial. □

Theorem 5.2. *Let θ be a Funk type metric on a bounded convex domain M in $\mathbb{R}^n$. Then $V = V^i\frac{\partial}{\partial x^i}$ is a projective vector field of θ if and only if V^i's are given by*

$$V^i = x^i < a, x > + Q^i_jx^j + c^i, \tag{5.13}$$

where $a, c = (c^1, \cdots, c^n) \in \mathbb{R}^n$ are two fixed vectors, (Q^i_j) is a fixed $n\times n$ real matrix and $< . >$ is the Euclidean inner product on $\mathbb{R}^n$. 1

$$[\hat{V}, G] = PY$$

where

$$\begin{aligned} P &= -2 < a, y > -\theta_{x^k}(x^k < a, x > + Q^k_jx^j + c^k) \\ &\quad -\theta_{y^k}(y^k < a, x > + x^k < a, y > + Q^k_jy^j). \end{aligned}$$

Proof. By [12] the maximum degree of projective vector fields is $n(n+2)$, thus the projective vector field of Funk type metrics are exactly as (5.13) $\square$

Appendix 1

$$
\begin{aligned}
A_0 &= 8e_{00}t_{00}\beta^2 s_0 y_i - 4e_{00}^2 t_{00} y_i\beta + e_{00;0}t_{00}\beta^2 y_i\\
A_1 &= -2t_{00}(8e_{00}\beta s_0 y_i - 4e_{00}^2 y_i + e_{00;0}\beta y_i)\\
A_2 &= -4t_{00}\beta^3 s_0 s_i + 4t_{00}\beta^2 b_i s_0^2 - 12A_{i0}t_{00}\beta^3\\
&\quad -64e_{00}(\mathcal{L}_{\hat{X}}\beta)\beta s_0 y_i - 16e_{00}\mathcal{L}_{\hat{X}}s_0\beta^2 y_i\\
&\quad -16e_{00}\mathcal{L}_{\hat{X}}y_i\beta^2 s_0 - 32e_{00}t_{00}\beta b_i s_0 + 16e_{00}\beta^2\eta y_i\\
&\quad +20e_{i0}t_{00}\beta^2 s_0 - 16\mathcal{L}_{\hat{X}}e_{00}\beta^2 s_0 y_i\\
&\quad -4t_{00}\beta^3 q_{i0} + 4t_{00}\beta^2 b_i q_{00} + 16t_{00}\beta^2 t_0 y_i\\
&\quad +16t_{00}\beta s0^2 y_i + 24e_{00}^2\mathcal{L}_{\hat{X}}\beta y_i + 8e_{00}^2\mathcal{L}_{\hat{X}}y_i\beta\\
&\quad +12e_{00}^2 t_{00}b_i - 8e_{00}e_{i0}t_{00}\beta + 16e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta y_i\\
&\quad -24e_{00}t_{00}s_0 y_i - 8e_{00;0}(\mathcal{L}_{\hat{X}}\beta)\beta y_i - 2e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta^2\\
&\quad -4e_{00;0}t_{00}\beta b_i - 5e_{00;i}t_{00}\beta^2 + 10e_{i0;0}t_{00}\beta^2\\
&\quad -2\mathcal{L}_{\hat{X}}e_{00;0}\beta^2 y_i + 16t_{00}\beta q_{00}y_i - 3e_{00;0}t_{00}y_i\\
A_3 &= 8\mathcal{L}_{\hat{X}}y_i\beta^3 t_0 + 8\mathcal{L}_{\hat{X}}y_i\beta^2 s_0^2 + 8\mathcal{L}_{\hat{X}}y_i\beta^2 q_{00}\\
&\quad +16t_{00}s_0^2 y_i + 16t_{00}q_{00}y_i - 2e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta^2\\
&\quad -8e_{00}e_{i0}t_{00} + 20e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta^2 - 24A_{i0}\mathcal{L}_{\hat{X}}\beta\beta^3\\
&\quad -36A_{i0}t_{00}\beta^2 - 10e_{00;i}\mathcal{L}_{\hat{X}}\beta\beta^2 - 8e_{00;0}\mathcal{L}_{\hat{X}}\beta y_i\\
&\quad +20e_{i0;0}t_{00}\beta - 10e_{00;i}t_{00}\beta - 4\mathcal{L}_{\hat{X}}e_{00;0}y_i\beta\\
&\quad -2e_{i0}\beta^3\eta + 24e_{00}^2\mathcal{L}_{\hat{X}}\beta b_i - 12t_{00}\beta^2 q_{0i}\\
&\quad -4e_{00;0}t_{00}b_i + 16e_{00}\mathcal{L}_{\hat{X}}e_{00}y_i - 2\mathcal{L}_{\hat{X}}e_{00;0}\beta^2 b_i\\
&\quad -20t_{00}\beta^2 q_{i0} + 8e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^3 + 8\mathcal{L}_{\hat{X}}e_{i0}\beta^3 s_0\\
&\quad -4e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta^2 - 4e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta^2 + 8\mathcal{L}_{\hat{X}}t_0\beta^3 y_i\\
&\quad +8\mathcal{L}_{\hat{X}}q_{00}\beta^2 y_i - 4e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta + 8e_{00}^2\mathcal{L}_{\hat{X}}b_i\beta\\
&\quad -12e_{00}\beta^3\eta_i - 4\mathcal{L}_{\hat{X}}A_{i0}\beta^4 - 2\mathcal{L}_{\hat{X}}e_{00;i}\beta^3\\
&\quad +4\mathcal{L}_{\hat{X}}e_{i0;0}\beta^3 + 8e_{00}^2\mathcal{L}_{\hat{X}}y_i - 64e_{00}\mathcal{L}_{\hat{X}}\beta b_i s_0\beta\\
&\quad +16\mathcal{L}_{\hat{X}}s_0\beta^2 s_0 y_i - 64e_{00}\mathcal{L}_{\hat{X}}\beta s_0 y_i - 32e_{00}t_{00}b_i s_0 + 40e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2 s_0\\
&\quad +40e_{i0}t_{00}\beta s_0 - 8t_{00}\beta^2 b_i t_0\\
&\quad +24t_{00}\beta b_i s_0^2 + 24t_{00}\beta b_i q_{00} + 16t_{00}\beta t_0 y_i + 32\mathcal{L}_{\hat{X}}\beta s_0^2 y_i\beta\\
&\quad +32\mathcal{L}_{\hat{X}}\beta q_{00}y_i\beta + 16e_{00}\beta^2 b_i\eta\\
&\quad +32e_{00}\beta\eta y_i - 16e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta\beta - 8e_{00;0}\mathcal{L}_{\hat{X}}\beta b_i\beta\\
&\quad +16e_{00}\mathcal{L}_{\hat{X}}e_{00}b_i\beta - 32t_{00}\beta^2 s_0 si\\
&\quad -16e_{00}\mathcal{L}_{\hat{X}}b_i\beta^2 s_0 - 32e_{00}\mathcal{L}_{\hat{X}}y_i\beta s_0 - 16e_{00}\mathcal{L}_{\hat{X}}s_0\beta^2 b_i\\
&\quad -32e_{00}\mathcal{L}_{\hat{X}}s_0\beta y_i s_0 - 16e_{00}\mathcal{L}_{\hat{X}}y
\end{aligned}
$$

$$
\begin{aligned}
A_4 \;=\;& -8e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta - 8e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta + 32\mathcal{L}_{\hat{X}}\beta s_0^2 y_i \\
& +8\mathcal{L}_{\hat{X}}b_i\beta^2 q_{00} + 16\mathcal{L}_{\hat{X}}y_i\beta^2 t_0 + 20t_{00}b_i s_0^2 \\
& +32\mathcal{L}_{\hat{X}}\beta q_{00}y_i + 20t_{00}b_i q_{00} - 16e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta \\
& -72A_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2 - 8e_{00;0}\mathcal{L}_{\hat{X}}\beta b_i \\
& +(40e_{i0;0}\mathcal{L}_{\hat{X}}\beta)\beta - 36A_{i0}t_{00}\beta - 20e_{00;i}(\mathcal{L}_{\hat{X}}\beta)\beta \\
& -4\mathcal{L}_{\hat{X}}e_{00;0}b_i\beta - 28t_{00}q_{i0}\beta - 8\mathcal{L}_{\hat{X}}s_0\beta^3 s_i \\
& -36e_{00}\beta^2\eta_i - 8\mathcal{L}_{\hat{X}}b_i\beta^3 t_0 - 6e_{i0}\beta^2\eta \\
& +8\mathcal{L}_{\hat{X}}b_i\beta^2 s_0^2 - 24(\mathcal{L}_{\hat{X}}\beta)\beta^2 q_{0i} - 24t_{00}\beta q_{0i} \\
& +16e_{00}\mathcal{L}_{\hat{X}}e_{00}b_i + 16e_{00}\eta y_i - (16\mathcal{L}_{\hat{X}}\beta)\beta^2 q_{i0} \\
& -16e_{00}\mathcal{L}_{\hat{X}}y_i s_0 + 24e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^2 \\
& +24\mathcal{L}_{\hat{X}}e_{i0}\beta^2 s_0 - 16e_{00}\mathcal{L}_{\hat{X}}s_0 y_i - 16\mathcal{L}_{\hat{X}}e_{00}s_0 y_i \\
& +8\mathcal{L}_{\hat{X}}t_0\beta^3 b_i + 8\mathcal{L}_{\hat{X}}q_{00}\beta^2 b_i + 16\mathcal{L}_{\hat{X}}t_0\beta^2 y_i \\
& +20e_{i0}t_{00}s_0 + 16\mathcal{L}_{\hat{X}}y_i s_0^2\beta + 16\mathcal{L}_{\hat{X}}y_i q_{00}\beta \\
& -4e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta + 16\mathcal{L}_{\hat{X}}q_{00}y_i\beta - 8\mathcal{L}_{\hat{X}}s_i\beta^3 s_0 \\
& -8\mathcal{L}_{\hat{X}}q_{i0}\beta^3 - 16\mathcal{L}_{\hat{X}}A_{i0}\beta^3 - 2e_{00;0}\mathcal{L}_{\hat{X}}y_i + 8e_{00}^2\mathcal{L}_{\hat{X}}b_i \\
& -5e_{00;i}t_{00} - 2\mathcal{L}_{\hat{X}}e_{00;0}y_i - 6\mathcal{L}_{\hat{X}}e_{00;i}\beta^2 \\
& +12\mathcal{L}_{\hat{X}}e_{i0;0}\beta^2 + 10e_{i0;0}t_{00} + 32\mathcal{L}_{\hat{X}}s_0\beta s_0 y_i \\
& -64e_{00}\mathcal{L}_{\hat{X}}\beta b_i s_0 + 80e_{i0}(\mathcal{L}_{\hat{X}}\beta)\beta s_0 - 32t_{00}\beta b_i t_0 \\
& +32e_{00}\beta b_i\eta + 32\mathcal{L}_{\hat{X}}\beta b_i s_0^2\beta + 32\mathcal{L}_{\hat{X}}\beta b_i q_{00}\beta \\
& -40(\mathcal{L}_{\hat{X}}\beta)\beta^2 s_0 s_i - 52t_{00}\beta s_0 s_i - 32e_{00}\mathcal{L}_{\hat{X}}b_i\beta s_0 \\
& -32e_{00}\mathcal{L}_{\hat{X}}s_0\beta b_i - 32\mathcal{L}_{\hat{X}}e_{00}\beta b_i s_0 \\
& +16\mathcal{L}_{\hat{X}}s_0\beta^2 b_i s_0 + 48(\mathcal{L}_{\hat{X}}\beta)\beta t_0 y_i \\
A_5 \;=\;& -36e_{00}\eta_i\beta - 6e_{i0}\eta\beta + 8\mathcal{L}_{\hat{X}}y_i\beta t_0 - \\
& 72A_{i0}(\mathcal{L}_{\hat{X}}\beta)\beta - 32\mathcal{L}_{\hat{X}}\beta q_{i0}\beta + 24e_{i0}\mathcal{L}_{\hat{X}}s_0\beta \\
& +24\mathcal{L}_{\hat{X}}e_{i0}s_0\beta + 16\mathcal{L}_{\hat{X}}b_i s_0^2\beta - 24\mathcal{L}_{\hat{X}}s_i\beta^2 s_0 \\
& -24\mathcal{L}_{\hat{X}}s_0\beta^2 s_i + 32\mathcal{L}_{\hat{X}}\beta b_i s_0^2 - 16e_{00}\mathcal{L}_{\hat{X}}s_0 b_i \\
& +32\mathcal{L}_{\hat{X}}\beta b_i q_{00} - 48(\mathcal{L}_{\hat{X}}\beta)\beta q_{0i} + 16e_{00}b_i\eta \\
& -24t_{00}b_i t_0 - 24t_{00}s_0 s_i - 16e_{00}\mathcal{L}_{\hat{X}}b_i s_0 \\
& -16\mathcal{L}_{\hat{X}}e_{00}b_i s_0 + 16\mathcal{L}_{\hat{X}}t_0\beta^2 b_i + 8\mathcal{L}_{\hat{X}}\beta t_0 y_i \\
& +16(\mathcal{L}_{\hat{X}}s_0)s_0 y_i + 8\mathcal{L}_{\hat{X}}t_0\beta y_i + 40e_{i0}\mathcal{L}_{\hat{X}}\beta s_0 \\
& -32\mathcal{L}_{\hat{X}}b_i\beta^2 t_0 + 16\mathcal{L}_{\hat{X}}b_i q_{00}\beta + 16\mathcal{L}_{\hat{X}}q_{00}b_i\beta \\
& -12t_{00}q_{0i} - 4e_{00}\mathcal{L}_{\hat{X}}e_{i0} - 4e_{i0}\mathcal{L}_{\hat{X}}e_{00} \\
& -24\mathcal{L}_{\hat{X}}q_{i0}\beta^2 + 8\mathcal{L}_{\hat{X}}y_i s_0^2 + 8\mathcal{L}_{\hat{X}}y_i q_{00} - 2e_{00;0}\mathcal{L}_{\hat{X}}b_i \\
& +8\mathcal{L}_{\hat{X}}q_{00}y_i - 10e_{00;i}\mathcal{L}_{\hat{X}}\beta \\
& -2\mathcal{L}_{\hat{X}}e_{00;0}b_i - 12t_{00}q_{i0} - 6\mathcal{L}_{\hat{X}}e_{00;i}\beta + 12\mathcal{L}_{\hat{X}}e_{i0;0}\beta \\
& -24\mathcal{L}_{\hat{X}}A_{i0}\beta^2 + 20e_{i0;0}\mathcal{L}_{\hat{X}}\beta - 12A_{i0}t_{00}
\end{aligned}
$$

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\begin{aligned}
& -80\mathcal{L}_{\hat{X}}\beta)\beta s_0 s_i - 32\mathcal{L}_{\hat{X}}\beta b_i t_0\beta + 32\mathcal{L}_{\hat{X}} s_0\beta b_i s_0 \\
A_6 \;=\; & -32\mathcal{L}_{\hat{X}}\beta b_i t_0 - 40\mathcal{L}_{\hat{X}}\beta s_0 s_i - 40\mathcal{L}_{\hat{X}} b_i\beta t_0 \\
& +8\mathcal{L}_{\hat{X}} b_i s_0^2 - 24\mathcal{L}_{\hat{X}} s_0\beta s_i + 16\mathcal{L}_{\hat{X}} s_0 b_i s_0 \\
& -24\mathcal{L}_{\hat{X}} s_i\beta s_0 + 8\mathcal{L}_{\hat{X}} t_0\beta b_i - 24A_{i0}\mathcal{L}_{\hat{X}}\beta \\
& -12e_{00}\eta_i + 8e_{i0}\mathcal{L}_{\hat{X}} s_0 - 2e_{i0}\eta - 16\mathcal{L}_{\hat{X}} A_{i0}\beta \\
& +8\mathcal{L}_{\hat{X}} e_{i0} s_0 - 24\mathcal{L}_{\hat{X}}\beta q_{0i} - 16\mathcal{L}_{\hat{X}}\beta q_{i0} + 8\mathcal{L}_{\hat{X}} b_i q_{00} + 8\mathcal{L}_{\hat{X}} q_{00} b_i \\
& -24\mathcal{L}_{\hat{X}} q_{i0}\beta - 2\mathcal{L}_{\hat{X}} e_{00;i} + 4\mathcal{L}_{\hat{X}} e_{i0;0} \\
A_7 \;=\; & -16\mathcal{L}_{\hat{X}} b_i t_0 - 8\mathcal{L}_{\hat{X}} s_0 s_i - 8\mathcal{L}_{\hat{X}} s_i s_0 - 4\mathcal{L}_{\hat{X}} A_{i0} - 8\mathcal{L}_{\hat{X}} q_{i0}
\end{aligned}
$$

Appendix 2

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\begin{aligned}
A_0 \;=\; & \frac{1}{2} t_{00}\beta^2 y_i y_j(-32e_{00}\beta s_0 + 16e_{00}^2 - 3e_{00;0}\beta) \\
A_1 \;=\; & \frac{1}{2}\beta t_{00} y_i y_j(-192e_{00}\beta s_0 + 96e_{00}^2 - 19e_{00;0}\beta) \\
A_2 \;=\; & 8a_{ij}e_{00}t_{00}\beta^3 s_0 + 28e_{00}^2\mathcal{L}_{\hat{X}}\beta\beta y_i y_j - 4e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta^2 y_i y_j \\
& +2t_{00}\beta^4 s_0 s_i y_j + 2t_{00}\beta^4 s_0 s_j y_i \\
& -2t_{00}\beta^3 b_i s_0^2 y_j - 2t_{00}\beta^3 b_j s_0^2 y_i + 6e_{j0}t_{00}\beta^3 s_0 y_i \\
& +4e_{00}t_{00}\beta^3 s_i y_j + 4e_{00}t_{00}\beta^3 s_j y_i + 6e_{i0}t_{00}\beta^3 s_0 y_j \\
& +8e_{00}\mathcal{L}_{\hat{X}} y_j\beta^3 s_0 y_i + 12e_{00}^2 t_{00} b_i y_j\beta + 12e_{00}^2 t_{00} b_j y_i\beta \\
& -8e_{00}\beta^3\eta y_i y_j - 2t_{00}\beta^3 b_i q_{00} y_j \\
& -2t_{00}\beta^3 b_j q_{00} y_i - 8e_{00}e_{i0}t_{00}\beta^2 y_j - 8e_{00}e_{j0}t_{00}\beta^2 y_i \\
& -2e_{00;0}t_{00}\beta^2 b_i y_j - 2e_{00;0}t_{00}\beta^2 b_j y_i \\
& +8\mathcal{L}_{\hat{X}} e_{00}\beta^3 s_0 y_i y_j - 8e_{00}\mathcal{L}_{\hat{X}} e_{00}\beta^2 y_i y_j \\
& +4e_{00}\mathcal{L}_{\hat{X}} s_0\beta^3 y_i y_j + 8e_{00}\mathcal{L}_{\hat{X}} y_i\beta^3 s_0 y_j - 4a_{ij}e_{00}^2 t_{00}\beta^2 \\
& +(1/2)a_{ij}e_{00;0}t_{00}\beta^3 + 2e_{i0;0}t_{00}\beta^3 y_j + 2e_{j0;0}t_{00}\beta^3 y_i \\
& +2t_{00}\beta^4 q_{i0} y_j + 2t_{00}\beta^4 q_{j0} y_i - 4e_{00}^2\mathcal{L}_{\hat{X}} y_i\beta^2 y_j \\
& -4e_{00}^2\mathcal{L}_{\hat{X}} y_j\beta^2 y_i + e_{00;0}\mathcal{L}_{\hat{X}} y_i\beta^3 y_j + e_{00;0}\mathcal{L}_{\hat{X}} y_j\beta^3 y_i \\
& +(\mathcal{L}_{\hat{X}} e_{00})_{;0}\beta^3 y_i y_j + 8t_{00}\beta^3 t0 y_i y_j \\
& +8t_{00}\beta^2 q_{00} y_i y_j + 8t_{00}\beta^2 s_0^2 y_i y_j + 88e_{00}^2 t_{00} y_i y_j \\
& -16e_{00}t_{00}\beta^2 b_i s_0 y_j - 16e_{00}t_{00}\beta^2 b_j s_0 y_i \\
& -200e_{00}t_{00}\beta s_0 y_i y_j - 48e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2 s_0 y_i y_j - (43/2e_{00;0})t_{00}\beta y_i y_j \\
A_3 \;=\; & -e_{ij;0}t_{00}\beta^4 - 8t_{00}\beta^2 b_i q_{00} y_j - 8t_{00}\beta^2 b_j q_{00} y_i + 6t_{00}\beta^3 s_0 s_i y_j \\
& +6t_{00}\beta^3 s_0 s_j y_i - 8t_{00}\beta^2 b_i s_0^2 y_j \\
& -8t_{00}\beta^2 b_j s_0^2 y_i + 48a_{ij}e_{00}t_{00}\beta^2 s_0 - 4e_{00}e_{i0}t_{00}\beta^2 b_j \\
& -4e_{00}e_{j0}t_{00}\beta^2 b_i + 44t_{00}\beta^2 t_0 y_i y_j \\
& +44t_{00}\beta s_0^2 y_i y_j - 44e_{00}e_{i0}t_{00}\beta y_j - 44e_{00}e_{j0}t_{00}\beta y_i \\
& -22e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta y_i y_j - 11e_{00;0}t_{00}\beta b_i y_j
\end{aligned}
$$

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\begin{aligned}
& -11e_{00;0}t_{00}\beta b_jy_i+44t_{00}\beta q_{00}y_iy_j+24e_{00}t_{00}\beta^2s_iy_j\\
& +24e_{00}t_{00}\beta^2s_jy_i+36e_{i0}t_{00}\beta^2s_0y_j\\
& +36e_{j0}t_{00}\beta^2s_0y_i+t_{00}\beta^3b_it_0y_j+t_{00}\beta^3b_jt_0y_i\\
& -e_{00;0}t_{00}\beta^2b_ib_j+2e_{00}t_{00}\beta^3b_is_j+2e_{00}t_{00}\beta^3b_js_i\\
& +3e_{i0}t_{00}\beta^3b_js_0+3e_{j0}t_{00}\beta^3b_is_0+40e_{00}\mathcal{L}_{\hat{X}}y_j\beta^2s_0y_i\\
& -40e_{00}\beta^2etay_iy_j+2t_{00}\beta^3s_it_0y_j\\
& +2t_{00}\beta^3s_jt_0y_i+6e_{00}^2t_{00}b_ib_j\beta+40\mathcal{L}_{\hat{X}}e_{00}\beta^2s_0y_iy_j\\
& -40e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta y_iy_j+28e_{00}\mathcal{L}_{\hat{X}}s_0\beta^2y_iy_j\\
& +40e_{00}\mathcal{L}_{\hat{X}}y_i\beta^2s_0y_j-2t_{00}\beta^3q_{0j}y_i-e_{j0}t_{00}\beta^4s_i\\
& +2e_{i0}e_{j0}t_{00}\beta^3+e_{00}e_{ij}t_{00}\beta^3-e_{i0}t_{00}\beta^4s_0\\
& -2e_{ij}t_{00}\beta^4s_0-t_{00}\beta^4t_iy_j-t_{00}\beta^4t_jy_i+e_{i0;0}t_{00}\beta^3b_j\\
& +e_{j0;0}t_{00}\beta^3b_i-4t_{00}\beta^3q_{0i}y_j-2a_{ij}t_{00}\beta^3q_{00}\\
& -2a_{ij}t_{00}\beta^4t_0-2a_{ij}t_{00}\beta^3s_0^2+12t_{00}\beta^3q_{i0}y_j\\
& +12t_{00}\beta^3q_{j0}y_i-22a_{ij}e_{00}^2t_{00}\beta+124e_{00}^2\mathcal{L}_{\hat{X}}\beta y_iy_j\\
& +60e_{00}^2t_{00}b_iy_j+60e_{00}^2t_{00}b_jy_i-20e_{00}^2\mathcal{L}_{\hat{X}}y_i\beta y_j\\
& -20e_{00}^2\mathcal{L}_{\hat{X}}y_j\beta y_i+5e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta^2y_j\\
& +5e_{00;0}\mathcal{L}_{\hat{X}}y_j\beta^2y_i+5(\mathcal{L}_{\hat{X}}e_{00})_{;0}\beta^2y_iy_j\\
& +(7/2a_{ij})e_{00;0}t_{00}\beta^2-120e_{00}t_{00}s_0y_iy_j\\
& -8e_{00}t_{00}\beta^2b_ib_js_0-208e_{00}\mathcal{L}_{\hat{X}}\beta\beta s_0y_iy_j-88e_{00}t_{00}\beta b_is_0y_j\\
& -88e_{00}t_{00}\beta b_js_0y_i+12e_{i0;0}t_{00}\beta^2y_j\\
& +12e_{j0;0}t_{00}\beta^2y_i-(27/2e_{00;0})t_{00}y_iy_j
\end{aligned}
$$

$$
\begin{aligned}
A_4 \;=\; & -9e_{ij;0}t_{00}\beta^3-18a_{ij}e_{00}^2t_{00}-2\mathcal{L}_{\hat{X}}q_{i0}\beta^4y_j\\
& -2\mathcal{L}_{\hat{X}}q_{j0}\beta^4y_i-2e_{i0;0}\mathcal{L}_{\hat{X}}y_j\beta^3-2e_{j0;0}\mathcal{L}_{\hat{X}}y_i\beta^3\\
& -2(\mathcal{L}_{\hat{X}}e_{i0})_{;0}\beta^3y_j-2\mathcal{L}_{\hat{X}}e_{j0;0}\beta^3y_i-16e_{00}^2\mathcal{L}_{\hat{X}}y_iy_j\\
& -16e_{00}^2\mathcal{L}_{\hat{X}}y_jy_i-2\mathcal{L}_{\hat{X}}y_i\beta^4q_{j0}-2\mathcal{L}_{\hat{X}}y_j\beta^4q_{i0}\\
& -a_{ij}(\mathcal{L}_{\hat{X}}e_{00})_{;0}\beta^3+4e_{00}^2\mathcal{L}_{\hat{X}}a_{ij}\beta^2-e_{00;0}\mathcal{L}_{\hat{X}}a_{ij}\beta^3\\
& +17t_{00}\beta^2b_it_0y_j+17t_{00}\beta^2b_jt_0y_i+18t_{00}\beta b_is_0^2y_j\\
& +18t_{00}\beta b_js_0^2y_i+72a_{ij}e_{00}t_{00}\beta s_0-56e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta\beta y_j\\
& -36e_{00}e_{i0}t_{00}\beta b_j-56e_{00}e_{j0}\mathcal{L}_{\hat{X}}\beta\beta y_i\\
& -36e_{00}e_{j0}t_{00}\beta b_i+18e_{00}t_{00}\beta^2b_is_j+18e_{00}t_{00}\beta^2b_js_i\\
& +36e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2s_0y_j+27e_{i0}t_{00}\beta^2b_js_0\\
& +36e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^2s_0y_i+27e_{j0}t_{00}\beta^2b_is_0-12\mathcal{L}_{\hat{X}}\beta\beta^2b_iq_{00}y_j\\
& -12\mathcal{L}_{\hat{X}}\beta\beta^2b_jq_{00}y_i+48\mathcal{L}_{\hat{X}}\beta\beta^2t_0y_iy_j\\
& +56\mathcal{L}_{\hat{X}}\beta\beta s_0^2y_iy_j-160e_{00}\mathcal{L}_{\hat{X}}\beta s_0y_iy_j+36e_{00}t_{00}\beta s_iy_j\\
& +36e_{00}t_{00}\beta s_jy_i-72e_{00}t_{00}b_is_0y_j\\
& -72e_{00}t_{00}b_js_0y_i-14e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta b_iy_j-14e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta b_jy_i
\end{aligned}
$$

$$
\begin{aligned}
&-9e_{00;0}t_{00}\beta b_i b_j + 54e_{i0}t_{00}\beta s_0 y_j\\
&+54e_{j0}t_{00}\beta s_0 y_i + 56\mathcal{L}_{\hat{X}}\beta\beta q_{00}y_i y_j + 18t_{00}\beta b_i q_{00}y_j\\
&+18t_{00}\beta b_j q_{00}y_i + 4\mathcal{L}_{\hat{X}}s_0\beta^3 b_j s_0 y_i\\
&-16t_{00}\beta^2 s_0 s_i y_j - 16t_{00}\beta^2 s_0 s_j y_i - 8t_{00}\beta^3 b_i b_j t_0 + 4t_{00}\beta^3 b_i s_j t_0\\
&+4t_{00}\beta^3 b_j s_i t_0 - 4t_{00}\beta^3 b_i s_0 s_j\\
&-4t_{00}\beta^3 b_j s_0 s_i + 4\mathcal{L}_{\hat{X}}s_0\beta^3 b_i s_0 y_j + 56e_{00}\mathcal{L}_{\hat{X}}y_j\beta s_0 y_i\\
&-56e_{00}\beta\eta y_i y_j + 16\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_i s_0 y_j\\
&+16\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_j s_0 y_i - 16\mathcal{L}_{\hat{X}}s_0\beta^2 s_0 y_i y_j + 10t_{00}\beta^2 s_i t_0 y_j\\
&+10t_{00}\beta^2 s_j t_0 y_i + 16e_{00}\mathcal{L}_{\hat{X}}b_i\beta^2 s_0 y_j\\
&+16e_{00}\mathcal{L}_{\hat{X}}b_j\beta^2 s_0 y_i + 16e_{00}\mathcal{L}_{\hat{X}}s_0\beta^2 b_i y_j + 16e_{00}\mathcal{L}_{\hat{X}}s_0\beta^2 b_j y_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}y_i\beta^2 b_j s_0 + 16e_{00}\mathcal{L}_{\hat{X}}y_j\beta^2 b_i s_0\\
&-16e_{00}\beta^2 b_i\eta y_j - 16e_{00}\beta^2 b_j\eta y_i + 10\mathcal{L}_{\hat{X}}\beta\beta^3 s_0 s_i y_j\\
&+10\mathcal{L}_{\hat{X}}\beta\beta^3 s_0 s_j y_i - 12\mathcal{L}_{\hat{X}}\beta\beta^2 b_i s_0^2 y_j\\
&-12\mathcal{L}_{\hat{X}}\beta\beta^2 b_j s_0^2 y_i + 48a_{ij}e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2 s_0 + 24e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2 s_i y_j\\
&+24e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2 s_j y_i + 56\mathcal{L}_{\hat{X}}e_{00}\beta s_0 y_i y_j\\
&-24e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta b_i y_j - 24e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta b_j y_i + 44e_{00}\mathcal{L}_{\hat{X}}s_0\beta y_i y_j\\
&+56e_{00}\mathcal{L}_{\hat{X}}y_i\beta s_0 y_j - 2t_{00}\beta^3 b_j q_{0i}\\
&-26t_{00}\beta^2 q_{0i}y_j - 18t_{00}\beta^2 q_{0j}y_i + 18t_{00}\beta^2 q_{i0}y_j\\
&+18t_{00}\beta^2 q_{j0}y_i - 36e_{00}e_{i0}t_{00}y_j - 36e_{00}e_{j0}t_{00}y_i\\
&-2\mathcal{L}_{\hat{X}}s_0\beta^4 s_i y_j - 2\mathcal{L}_{\hat{X}}s_0\beta^4 s_j y_i - 2\mathcal{L}_{\hat{X}}y_i\beta^4 s_0 s_j\\
&+2\mathcal{L}_{\hat{X}}y_i\beta^3 b_j s_0^2 + 2\mathcal{L}_{\hat{X}}b_i\beta^3 s_0^2 y_j + 2\mathcal{L}_{\hat{X}}b_j\beta^3 s_0^2 y_i\\
&-4e_{00}\mathcal{L}_{\hat{X}}y_i\beta^3 s_j - 4e_{00}\mathcal{L}_{\hat{X}}y_j\beta^3 s_i + 4e_{00}\beta^3\eta_i y_j\\
&+4e_{00}\beta^3\eta_j y_i - 2\mathcal{L}_{\hat{X}}y_j\beta^4 s_0 s_i + 2\mathcal{L}_{\hat{X}}y_j\beta^3 b_i s_0^2\\
&-4a_{ij}e_{00}\mathcal{L}_{\hat{X}}s_0\beta^3 + 8a_{ij}e_{00}\beta^3\eta - 8a_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta^3 s_0\\
&-2\mathcal{L}_{\hat{X}}s_i\beta^4 s_0 y_j - 2\mathcal{L}_{\hat{X}}s_j\beta^4 s_0 y_i + 2\mathcal{L}_{\hat{X}}q_{00}\beta^3 b_i y_j\\
&+2\mathcal{L}_{\hat{X}}q_{00}\beta^3 b_j y_i - 8\mathcal{L}_{\hat{X}}t_0\beta^3 y_i y_j - 6e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^3 y_j\\
&-6e_{i0}\mathcal{L}_{\hat{X}}y_j\beta^3 s_0 + 6e_{i0}\beta^3\eta y_j - 6e_{j0}\mathcal{L}_{\hat{X}}s_0\beta^3 y_i\\
&-6e_{j0}\mathcal{L}_{\hat{X}}y_i\beta^3 s_0 + 6e_{j0}\beta^3\eta y_i - 8e_{00}\mathcal{L}_{\hat{X}}a_{ij}\beta^3 s_0\\
&-4e_{00}\mathcal{L}_{\hat{X}}s_i\beta^3 y_j - 4e_{00}\mathcal{L}_{\hat{X}}s_j\beta^3 y_i + 8e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta^2 y_j\\
&+8e_{00}\mathcal{L}_{\hat{X}}e_{j0}\beta^2 y_i + 2e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta^2 y_j + 2e_{00;0}\mathcal{L}_{\hat{X}}b_j\beta^2 y_i\\
&+2e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta^2 b_j + 2e_{00;0}\mathcal{L}_{\hat{X}}y_j\beta^2 b_i\\
&+2\mathcal{L}_{\hat{X}}y_i\beta^3 b_j q_{00} - 8\mathcal{L}_{\hat{X}}y_i\beta^3 t_0 y_j - 8\mathcal{L}_{\hat{X}}y_i\beta^2 s_0^2 y_j 2\mathcal{L}_{\hat{X}}y_j\beta^3 b_i q_{00}\\
&-8\mathcal{L}_{\hat{X}}y_j\beta^3 t_0 y_i - 8\mathcal{L}_{\hat{X}}y_j\beta^2 s_0^2 y_i\\
&+8a_{ij}e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta^2 - 4\mathcal{L}_{\hat{X}}e_{00}\beta^3 s_i y_j - 4\mathcal{L}_{\hat{X}}e_{00}\beta^3 s_j y_i\\
&-6\mathcal{L}_{\hat{X}}e_{i0}\beta^3 s_0 y_j - 6\mathcal{L}_{\hat{X}}e_{j0}\beta^3 s_0 y_i\\
&+2\mathcal{L}_{\hat{X}}b_i\beta^3 q_{00}y_j + 2\mathcal{L}_{\hat{X}}b_j\beta^3 q_{00}y_i - 32e_{00}\mathcal{L}_{\hat{X}}e_{00}y_i y_j
\end{aligned}
$$

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\begin{aligned}
& +7e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta y_j + 7e_{00;0}\mathcal{L}_{\hat{X}}y_j\beta y_i \\
& +7\mathcal{L}_{\hat{X}}e_{00;0}\beta y_i y_j + 8e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta^2 y_j + 8e_{j0}\mathcal{L}_{\hat{X}}e_{00}\beta^2 y_i \\
& +2\mathcal{L}_{\hat{X}}e_{00;0}\beta^2 b_i y_j + 2\mathcal{L}_{\hat{X}}e_{00;0}\beta^2 b_j y_i \\
& -8\mathcal{L}_{\hat{X}}q_{00}\beta^2 y_i y_j - 8\mathcal{L}_{\hat{X}}y_i\beta^2 q_{00}y_j - 8\mathcal{L}_{\hat{X}}y_j\beta^2 q_{00}y_i \\
& -12e_{00}^2\mathcal{L}_{\hat{X}}b_i\beta y_j - 12e_{00}^2\mathcal{L}_{\hat{X}}b_j\beta y_i \\
& -12e_{00}^2\mathcal{L}_{\hat{X}}y_i\beta b_j - 12e_{00}^2\mathcal{L}_{\hat{X}}y_j\beta b_i + 8e_{00}e_{i0}\mathcal{L}_{\hat{X}}y_j\beta^2 \\
& +8e_{00}e_{j0}\mathcal{L}_{\hat{X}}y_i\beta^2 + (11/2a_{ij})e_{00;0}t_{00}\beta \\
& +36t_{00}\beta t_0 y_i y_j + 36t_{00}q_{00}y_i y_j + 36t_{00}s_0^2 y_i y_j - 112e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_i s_0 y_j \\
& -112e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_j s_0 y_i \\
& -72e_{00}t_{00}\beta b_i b_j s_0 - 9e_{i0}t_{00}\beta^3 s_j - 18e_{ij}t_{00}\beta^3 s_0 \\
& +10\mathcal{L}_{\hat{X}}\beta\beta^3 q_{i0}y_j + 10\mathcal{L}_{\hat{X}}\beta\beta^3 q_{j0}y_i - 18a_{ij}t_{00}\beta^3 t_0 \\
& -18a_{ij}t_{00}\beta^2 s_0^2 - 18e_{00;0}\mathcal{L}_{\hat{X}}\beta y_i y_j - 9e_{00;0}t_{00}b_i y_j \\
& -9e_{00;0}t_{00}b_j y_i + 18e_{i0;0}t_{00}\beta y_j + 18e_{j0;0}t_{00}\beta y_i \\
& -9t_{00}\beta^3 tiy_j - 9t_{00}\beta^3 tjy_i - 28a_{ij}e_{00}^2\mathcal{L}_{\hat{X}}\beta\beta \\
& +4a_{ij}e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta^2 - 18a_{ij}t_{00}\beta^2 q_{00} + 96e_{00}^2\mathcal{L}_{\hat{X}}\beta b_i y_j \\
& +96e_{00}^2\mathcal{L}_{\hat{X}}\beta b_j y_i + 54e_{00}^2 t_{00}b_i b_j + 9e_{00}e_{ij}t_{00}\beta^2 \\
& +18e_{i0}e_{j0}t_{00}\beta^2 + 12e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta^2 y_j + 9e_{i0;0}t_{00}\beta^2 b_j \\
& +12e_{j0;0}\mathcal{L}_{\hat{X}}\beta\beta^2 y_i + 9e_{j0;0}t_{00}\beta^2 b_i - 9e_{j0}t_{00}\beta^3 s_i
\end{aligned}
$$

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\begin{aligned}
A_5 \quad = \quad & 4a_{ij}\mathcal{L}_{\hat{X}}t_0\beta^4 - 10e_{ij;0}\mathcal{L}_{\hat{X}}\beta\beta^3 + 5t_{00}\beta^3 q_{ij} \\
& 5t_{00}\beta^3 q_{ji} - 28a_{ij}e_{00}^2\mathcal{L}_{\hat{X}}\beta - 21e_{ij;0}t_{00}\beta^2 \\
& +8e_{i0;0}t_{00}y_j + 8e_{j0;0}t_{00}y_i - 2e_{i0}\beta^4\eta_j + 4e_{ij}\mathcal{L}_{\hat{X}}s_0\beta^4 \\
& -4e_{ij}\beta^4 eta + 2e_{j0}\mathcal{L}_{\hat{X}}s_i\beta^4 - 2e_{j0}\beta^4\eta_i \\
& +4\mathcal{L}_{\hat{X}}a_{ij}\beta^4 t_0 + 4\mathcal{L}_{\hat{X}}a_{ij}\beta^3 s_0^2 + 2\mathcal{L}_{\hat{X}}e_{i0}\beta^4 s_j \\
& +4\mathcal{L}_{\hat{X}}e_{ij}\beta^4 s_0 + 2\mathcal{L}_{\hat{X}}e_{j0}\beta^4 s_i - 2e_{00}\mathcal{L}_{\hat{X}}e_{ij}\beta^3 \\
& -4e_{i0}\mathcal{L}_{\hat{X}}e_{j0}\beta^3 + 2\mathcal{L}_{\hat{X}}t_j\beta^4 y_i + 2\mathcal{L}_{\hat{X}}t_j\beta^4 y_j \\
& +2e_{i0}\mathcal{L}_{\hat{X}}s_j\beta^4 - 2e_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta^3 - 4e_{j0}\mathcal{L}_{\hat{X}}e_{i0}\beta^3 \\
& -2e_{j0;0}\mathcal{L}_{\hat{X}}b_i\beta^3 + 4\mathcal{L}_{\hat{X}}a_{ij}\beta^3 q_{00} - 2\mathcal{L}_{\hat{X}}e_{i0;0}\beta^3 b_j \\
& -2\mathcal{L}_{\hat{X}}e_{j0;0}\beta^3 b_i + 4\mathcal{L}_{\hat{X}}q_{0i}\beta^3 y_j + 4\mathcal{L}_{\hat{X}}q_{0j}\beta^3 y_i \\
& +2\mathcal{L}_{\hat{X}}y_i\beta^4 t_j + 2\mathcal{L}_{\hat{X}}y_j\beta^4 t_i + 4a_{ij}\mathcal{L}_{\hat{X}}q_{00}\beta^3 \\
& -6e_{i0;0}\mathcal{L}_{\hat{X}}y_j\beta^2 - 6e_{j0;0}\mathcal{L}_{\hat{X}}y_i\beta^2 - 6(\mathcal{L}_{\hat{X}}e_{i0})_{;0}\beta^2 y_j \\
& -6(\mathcal{L}_{\hat{X}}e_{j0})_{;0}\beta^2 y_i + 3e_{00;0}\mathcal{L}_{\hat{X}}y_i y_j + 3e_{00;0}\mathcal{L}_{\hat{X}}y_j y_i \\
& +3\mathcal{L}_{\hat{X}}e_{00;0}y_i y_j + 4\mathcal{L}_{\hat{X}}y_i\beta^3 q_{0j} + 4\mathcal{L}_{\hat{X}}y_j\beta^3 q_{0i} \\
& -3a_{ij}\mathcal{L}_{\hat{X}}e_{00;0}\beta^2 + 8e_{00}^2\mathcal{L}_{\hat{X}}a_{ij}\beta - 12e_{00}^2\mathcal{L}_{\hat{X}}b_i y_j \\
& -12e_{00}^2\mathcal{L}_{\hat{X}}b_j y_i - 12e_{00}^2\mathcal{L}_{\hat{X}}y_i b_j - 12e_{00}^2\mathcal{L}_{\hat{X}}y_j b_i \\
& -3e_{00;0}\mathcal{L}_{\hat{X}}a_{ij}\beta^2 - 2e_{i0;0}\mathcal{L}_{\hat{X}}b_j\beta^3 - 14e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta b_i b_j \\
& +72e_{i0}\mathcal{L}_{\hat{X}}\beta\beta s_0 y_j + 45e_{i0}t_{00}\beta b_j s_0
\end{aligned}
$$

$$
\begin{aligned}
&+72e_{j0}\mathcal{L}_{\hat{X}}\beta\beta s_0 y_i + 45e_{j0}t_{00}\beta b_i s_0 + 32\mathcal{L}_{\hat{X}}\beta\beta b_i q_{00} y_j\\
&+32\mathcal{L}_{\hat{X}}\beta\beta b_j q_{00} y_i + 40\mathcal{L}_{\hat{X}}\beta\beta t_0 y_i y_j\\
&+28t_{00}\beta b_i b_j q_{00} + 3t_{00}\beta b_i t_0 y_j + 3t_{00}\beta b_j t_0 y_i\\
&-42t_{00}\beta s_0 s_i y_j - 42t_{00}\beta s_0 s_j y_i + 16e_{00}\mathcal{L}_{\hat{X}} b_i\beta^2 b_j s_0\\
&+16e_{00}\mathcal{L}_{\hat{X}} b_j\beta^2 b_i s_0 + 16e_{00}\mathcal{L}_{\hat{X}} s_0\beta^2 b_i b_j - 16e_{00}\beta^2 b_i b_j\eta\\
&+16t_{00}\beta^2 b_i s_j t_0 + 16t_{00}\beta^2 b_j s_i t_0\\
&-24e_{00}\mathcal{L}_{\hat{X}} e_{00}\beta b_i b_j + 32e_{00}\mathcal{L}_{\hat{X}} b_i\beta s_0 y_j + 16\mathcal{L}_{\hat{X}} e_{00}\beta^2 b_i b_j s_0\\
&-14\mathcal{L}_{\hat{X}}\beta\beta^2 s_0 s_i y_j - 14\mathcal{L}_{\hat{X}}\beta\beta^2 s_0 s_j y_i\\
&+8\mathcal{L}_{\hat{X}}\beta\beta^2 s_i t_0 y_j + 8\mathcal{L}_{\hat{X}}\beta\beta^2 s_j t_0 y_i - 4\mathcal{L}_{\hat{X}} s_0\beta^2 b_i s_0 y_j\\
&-4\mathcal{L}_{\hat{X}} s_0\beta^2 b_j s_0 y_i + 32\mathcal{L}_{\hat{X}} e_{00}\beta b_i s_0 y_j\\
&+32\mathcal{L}_{\hat{X}} e_{00}\beta b_j s_0 y_i - 28t_{00}\beta^2 b_i b_j t_0 - 32\mathcal{L}_{\hat{X}} s_0\beta s_0 y_i y_j\\
&+14t_{00}\beta s_i t_0 y_j + 14t_{00}\beta s_j t_0 y_i\\
&+32e_{00}\mathcal{L}_{\hat{X}} b_j\beta s_0 y_i + 32e_{00}\mathcal{L}_{\hat{X}} s_0\beta b_i y_j + 32e_{00}\mathcal{L}_{\hat{X}} s_0\beta b_j y_i\\
&+32e_{00}\mathcal{L}_{\hat{X}} y_i\beta b_j s_0 + 32e_{00}\mathcal{L}_{\hat{X}} y_j\beta b_i s_0\\
&-32e_{00}\beta b_i\eta y_j - 32e_{00}\beta b_j\eta y_i + 36e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2 b_j s_0\\
&+36e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^2 b_i s_0 + 28\mathcal{L}_{\hat{X}}\beta\beta^2 b_i t_0 y_j\\
&+28\mathcal{L}_{\hat{X}}\beta\beta^2 b_j t_0 y_i + 32\mathcal{L}_{\hat{X}}\beta\beta b_i s_0^2 y_j + 32\mathcal{L}_{\hat{X}}\beta\beta b_j s_0^2 y_i\\
&-34t_{00}\beta^2 b_i s_0 s_j - 34t_{00}\beta^2 b_j s_0 s_i\\
&+28t_{00}\beta b_i b_j s_0^2 + 96a_{ij}e_{00}\mathcal{L}_{\hat{X}}\beta\beta s_0 - 56e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta\beta b_j\\
&-56e_{00}e_{j0}\mathcal{L}_{\hat{X}}\beta\beta b_i + 48e_{00}\mathcal{L}_{\hat{X}}\beta\beta s_i y_j\\
&+48e_{00}\mathcal{L}_{\hat{X}}\beta\beta s_j y_i - 112e_{00}\mathcal{L}_{\hat{X}}\beta b_i s_0 y_j - 112e_{00}\mathcal{L}_{\hat{X}}\beta b_j s_0 y_i\\
&+30e_{00}t_{00}\beta b_i s_j + 30e_{00}t_{00}\beta b_j s_i\\
&-64e_{00}t_{00}b_i b_j s_0 + 24e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2 b_i s_j + 24e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2 b_j s_i\\
&-4e_{00}\mathcal{L}_{\hat{X}} b_j\beta^3 s_i - 4e_{00}\mathcal{L}_{\hat{X}} s_i\beta^3 b_j\\
&-4e_{00}\mathcal{L}_{\hat{X}} s_j\beta^3 b_i + 4e_{00}\beta^3 b_i\eta_j + 4e_{00}\beta^3 b_j\eta_i - t_{00}\beta^3 b_i t_j\\
&-t_{00}\beta^3 b_j t_i + 8t_{00}\beta q_{j0} y_i + 24t_{00} b_i q_{00} y_j\\
&+24t_{00} b_j q_{00} y_i - 6e_{i0}\mathcal{L}_{\hat{X}} b_j\beta^3 s_0 - 6e_{i0}\mathcal{L}_{\hat{X}} s_0\beta^3 b_j\\
&+6e_{i0}\beta^3 b_j\eta - 6e_{j0}\mathcal{L}_{\hat{X}} b_i\beta^3 s_0 - 6e_{j0}\mathcal{L}_{\hat{X}} s_0\beta^3 b_i\\
&+6e_{j0}\beta^3 b_i eta - 4\mathcal{L}_{\hat{X}} e_{00}\beta^3 b_i s_j - 4\mathcal{L}_{\hat{X}} e_{00}\beta^3 b_j s_i\\
&-6\mathcal{L}_{\hat{X}} e_{i0}\beta^3 b_j s_0 - 6\mathcal{L}_{\hat{X}} e_{j0}\beta^3 b_i s_0 + 8a_{ij}\mathcal{L}_{\hat{X}} s_0\beta^3 s_0\\
&-4e_{00}\mathcal{L}_{\hat{X}} b_i\beta^3 s_j + 4\mathcal{L}_{\hat{X}} t_0\beta^3 s_i y_j + 4\mathcal{L}_{\hat{X}} t_0\beta^3 s_j y_i\\
&+4\mathcal{L}_{\hat{X}} y_i\beta^3 s_j t_0 - 2\mathcal{L}_{\hat{X}} y_i\beta^2 b_j s_0^2 - 2\mathcal{L}_{\hat{X}} b_i\beta^2 s_0^2 y_j\\
&-2\mathcal{L}_{\hat{X}} b_j\beta^2 s_0^2 y_i - 12e_{00}\mathcal{L}_{\hat{X}} y_i\beta^2 s_j - 12e_{00}\mathcal{L}_{\hat{X}} y_j\beta^2 s_i\\
&+12e_{00}\beta^2\eta_i y_j + 12e_{00}\beta^2\eta_j y_i + 2e_{00;0}\mathcal{L}_{\hat{X}} b_i\beta^2 b_j\\
&+2e_{00;0}\mathcal{L}_{\hat{X}} b_j\beta^2 b_i + 4\mathcal{L}_{\hat{X}} y_j\beta^3 s_i t_0 - 2\mathcal{L}_{\hat{X}} y_j\beta^2 b_i s_0^2\\
&-12a_{ij}e_{00}\mathcal{L}_{\hat{X}} s_0\beta^2 + 24a_{ij}e_{00}\beta^2\eta
\end{aligned}
$$

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\begin{aligned}
&-24a_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta^2 s_0 - 12e_{00}^2\mathcal{L}_{\hat{X}}b_i\beta b_j - 12e_{00}^2\mathcal{L}_{\hat{X}}b_j\beta b_i\\
&+8e_{00}e_{i0}\mathcal{L}_{\hat{X}}b_j\beta^2 + 4\mathcal{L}_{\hat{X}}s_i\beta^3 t_0 y_j\\
&+4\mathcal{L}_{\hat{X}}s_j\beta^3 t_0 y_i - 2\mathcal{L}_{\hat{X}}q_{00}\beta^2 b_i y_j - 2\mathcal{L}_{\hat{X}}q_{00}\beta^2 b_j y_i\\
&-16\mathcal{L}_{\hat{X}}t_0\beta^2 y_i y_j + 8e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_j - 18e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^2 y_j\\
&-18e_{i0}\mathcal{L}_{\hat{X}}y_j\beta^2 s_0 + 18e_{i0}\beta^2\eta y_j + 8e_{j0}\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_i\\
&-18e_{j0}\mathcal{L}_{\hat{X}}s_0\beta^2 y_i - 18e_{j0}\mathcal{L}_{\hat{X}}y_i\beta^2 s_0 + 18e_{j0}\beta^2\eta y_i\\
&+8e_{00}e_{j0}\mathcal{L}_{\hat{X}}b_i\beta^2 - 24e_{00}\mathcal{L}_{\hat{X}}a_{ij}\beta^2 s_0 + 8e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta^2 b_j\\
&+8e_{00}\mathcal{L}_{\hat{X}}e_{j0}\beta^2 b_i - 12e_{00}\mathcal{L}_{\hat{X}}s_i\beta^2 y_j\\
&-12e_{00}\mathcal{L}_{\hat{X}}s_j\beta^2 y_i + 16e_{00}\mathcal{L}_{\hat{X}}e_{j0}\beta y_i + 20e_{00}\mathcal{L}_{\hat{X}}s_0 y_i y_j\\
&+24e_{00}\mathcal{L}_{\hat{X}}y_i s_0 y_j + 24e_{00}\mathcal{L}_{\hat{X}}y_j s_0 y_i\\
&-24e_{00}\eta y_i y_j + 4e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta y_j + 4e_{00;0}\mathcal{L}_{\hat{X}}b_j\beta y_i\\
&+4e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta b_j + 4e_{00;0}\mathcal{L}_{\hat{X}}y_j\beta b_i - 2\mathcal{L}_{\hat{X}}y_i\beta^2 b_j q_{00}\\
&-16\mathcal{L}_{\hat{X}}y_i\beta^2 t_0 y_j - 16\mathcal{L}_{\hat{X}}y_i\beta s_0^2 y_j - 2\mathcal{L}_{\hat{X}}y_j\beta^2 b_i q_{00}\\
&-16\mathcal{L}_{\hat{X}}y_j\beta^2 t_0 y_i - 16\mathcal{L}_{\hat{X}}y_j\beta s_0^2 y_i + 16a_{ij}e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta\\
&-12\mathcal{L}_{\hat{X}}e_{00}\beta^2 s_i y_j - 12\mathcal{L}_{\hat{X}}e_{00}\beta^2 s_j y_i + 2\mathcal{L}_{\hat{X}}e_{00;0}\beta^2 b_i b_j\\
&-18\mathcal{L}_{\hat{X}}e_{i0}\beta^2 s_0 y_j - 18\mathcal{L}_{\hat{X}}e_{j0}\beta^2 s_0 y_i\\
&-2\mathcal{L}_{\hat{X}}b_i\beta^2 q_{00}y_j - 2\mathcal{L}_{\hat{X}}b_j\beta^2 q_{00}y_i + 16e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta y_j\\
&+16e_{j0}\mathcal{L}_{\hat{X}}e_{00}\beta y_i + 24\mathcal{L}_{\hat{X}}e_{00}s_0 y_i y_j + 4(\mathcal{L}_{\hat{X}}e_{00})_{;0}\beta b_i y_j\\
&+4(\mathcal{L}_{\hat{X}}e_{00})_{;0}\beta b_j y_i - 16\mathcal{L}_{\hat{X}}q_{00}\beta y_i y_j - 16\mathcal{L}_{\hat{X}}y_i\beta q_{00}y_j\\
&-16\mathcal{L}_{\hat{X}}y_j\beta q_{00}y_i + 16e_{00}e_{i0}\mathcal{L}_{\hat{X}}y_j\beta\\
&+16e_{00}e_{j0}\mathcal{L}_{\hat{X}}y_i\beta - 24e_{00}\mathcal{L}_{\hat{X}}e_{00}b_i y_j - 24e_{00}\mathcal{L}_{\hat{X}}e_{00}b_j y_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta y_j + (5/2a_{ij})e_{00;0}t_{00}\\
&+2\mathcal{L}_{\hat{X}}e_{ij;0}\beta^4 - 10\mathcal{L}_{\hat{X}}t_0\beta^3 b_i y_j - 10\mathcal{L}_{\hat{X}}t_0\beta^3 b_j y_i\\
&-10\mathcal{L}_{\hat{X}}y_i\beta^3 b_j t_0 - 2\mathcal{L}_{\hat{X}}y_i\beta^3 s_0 s_j - 10\mathcal{L}_{\hat{X}}y_j\beta^3 b_i t_0\\
&-2\mathcal{L}_{\hat{X}}y_j\beta^3 s_0 s_i - 10\mathcal{L}_{\hat{X}}b_i\beta^3 t_0 y_j - 10\mathcal{L}_{\hat{X}}b_j\beta^3 t_0 y_i\\
&-6\mathcal{L}_{\hat{X}}s_0\beta^3 s_i y_j - 6\mathcal{L}_{\hat{X}}s_0\beta^3 s_j y_i - 2\mathcal{L}_{\hat{X}}s_i\beta^3 s_0 y_j\\
&-2\mathcal{L}_{\hat{X}}s_j\beta^3 s_0 y_i - 6\mathcal{L}_{\hat{X}}q_{i0}\beta^3 y_j - 6\mathcal{L}_{\hat{X}}q_{j0}\beta^3 y_i\\
&-6\mathcal{L}_{\hat{X}}y_i\beta^3 q_{j0} - 6\mathcal{L}_{\hat{X}}y_j\beta^3 q_{i0} - 112e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_i b_j s_0\\
&-21e_{i0}t_{00}\beta^2 s_j + 12e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta^2 b_j - 42e_{ij}t_{00}\beta^2 s_0\\
&-20a_{ij}\mathcal{L}_{\hat{X}}\beta\beta^3 t_0 - 24a_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2 s_0^2 - 10e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^3 s_j\\
&-20e_{ij}\mathcal{L}_{\hat{X}}\beta\beta^3 s_0 - 10e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^3 s_i - 10\mathcal{L}_{\hat{X}}\beta\beta^3 t_i y_j\\
&-10\mathcal{L}_{\hat{X}}\beta\beta^3 t_j y_i - 28\mathcal{L}_{\hat{X}}\beta\beta^2 q_{0i}y_j - 24\mathcal{L}_{\hat{X}}\beta\beta^2 q_{0j}y_i\\
&+18\mathcal{L}_{\hat{X}}\beta\beta^2 q_{i0}y_j + 18\mathcal{L}_{\hat{X}}\beta\beta^2 q_{j0}y_i + 56\mathcal{L}_{\hat{X}}\beta s_0^2 y_i y_j\\
&-12t_{00}\beta^2 b_i q_{0j} - 6t_{00}\beta^2 b_i q_{j0} - 20t_{00}\beta^2 b_j q_{0i}\\
&-6t_{00}\beta^2 b_j q_{i0} - 15t_{00}\beta^2 t_i y_j - 15t_{00}\beta^2 t_j y_i + 24t_{00}b_i s_0^2 y_j\\
&+24t_{00}b_j s_0^2 y_i + 10t_{00}\beta^3 s_i s_j
\end{aligned}
$$

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\begin{aligned}
& -30a_{ij}t_{00}\beta^2 t_0 - 30a_{ij}t_{00}\beta s_0^2 + 96e_{00}^2\mathcal{L}_{\hat{X}}\beta b_i b_j \\
& +12e_{00}e_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2 + 24e_{i0}e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^2 + 32a_{ij}e_{00}t_{00}s_0 \\
& +8a_{ij}e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta - 30a_{ij}t_{00}\beta q_{00} - 56e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta y_j \\
& -32e_{00}e_{i0}t_{00}b_j + 15e_{00}e_{ij}t_{00}\beta - 56e_{00}e_{j0}\mathcal{L}_{\hat{X}}\beta y_i \\
& -32e_{00}e_{j0}t_{00}b_i + 16e_{00}t_{00}s_i y_j + 16e_{00}t_{00}s_j y_i - 14e_{00;0}\mathcal{L}_{\hat{X}}\beta b_i y_j \\
& -14e_{00;0}\mathcal{L}_{\hat{X}}\beta b_j y_i - 8e_{00;0}t_{00}b_i b_j \\
& +30e_{i0}e_{j0}t_{00}\beta + 24e_{i0}t_{00}s_0 y_j + 24e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta y_j \\
& +15e_{i0;0}t_{00}\beta b_j + 24e_{j0}t_{00}s_0 y_i + 24e_{j0;0}\mathcal{L}_{\hat{X}}\beta\beta y_i \\
& +15e_{j0;0}t_{00}\beta b_i + 56\mathcal{L}_{\hat{X}}\beta q_{00}y_i y_j - 40t_{00}\beta q_{0i}y_j \\
& -30t_{00}\beta q_{0j}y_i + 8t_{00}\beta q_{i0}y_j - 21e_{j0}t_{00}\beta^2 s_i \\
& +12e_{j0;0}\mathcal{L}_{\hat{X}}\beta\beta^2 b_i - 24a_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2 q_{00}
\end{aligned}
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\begin{aligned}
A_6 \;=\;& 10\mathcal{L}_{\hat{X}}\beta\beta^3 q_{ij} + 10\mathcal{L}_{\hat{X}}\beta\beta^3 q_{ji} - 14a_{ij}t_{00}s_0^2 \\
& -2\mathcal{L}_{\hat{X}}t_i\beta^4 b_j + 2\mathcal{L}_{\hat{X}}t_j\beta^4 b_j + 12a_{ij}\mathcal{L}_{\hat{X}}t_0\beta^3 \\
& -30e_{ij;0}\mathcal{L}_{\hat{X}}\beta\beta^2 + 15t_{00}\beta^2 q_{ij} + 15t_{00}\beta^2 q_{ji} \\
& +4a_{ij}e_{00;0}\mathcal{L}_{\hat{X}}\beta - 14a_{ij}t_{00}q_{00} + 7e_{00}e_{ij}t_{00} + 14e_{i0}e_{j0}t_{00} \\
& +12e_{i0;0}\mathcal{L}_{\hat{X}}\beta y_j + 7e_{i0;0}t_{00}b_j - 19e_{ij;0}t_{00}\beta \\
& +12e_{j0;0}\mathcal{L}_{\hat{X}}\beta y_i + 7e_{j0;0}t_{00}b_i - 18t_{00}q_{0i}y_j - 14t_{00}q_{0j}y_i \\
& +12a_{ij}\beta^3 t_0 + 12a_{ij}\beta^2 s_0^2 - 4\mathcal{L}_{\hat{X}}s_i\beta^4 s_j - 4\mathcal{L}_{\hat{X}}s_j\beta^4 s_i \\
& -6e_{00}\mathcal{L}_{\hat{X}}e_{ij}\beta^2 - 12e_{i0}\mathcal{L}_{\hat{X}}e_{j0}\beta^2 - 6e_{i0;0}\mathcal{L}_{\hat{X}}b_j\beta^2 \\
& +4\mathcal{L}_{\hat{X}}b_i\beta^3 q_{0j} + 2\mathcal{L}_{\hat{X}}b_i\beta^3 q_{j0} + 4\mathcal{L}_{\hat{X}}b_j\beta^3 q_{0i} \\
& +2\mathcal{L}_{\hat{X}}b_j\beta^3 q_{i0} + 4\mathcal{L}_{\hat{X}}q_{0i}\beta^3 b_j + 4\mathcal{L}_{\hat{X}}q_{0j}\beta^3 b_i + 2\mathcal{L}_{\hat{X}}q_{i0}\beta^3 b_j \\
& +2\mathcal{L}_{\hat{X}}q_{j0}\beta^3 b_i + 6\mathcal{L}_{\hat{X}}t_j\beta^3 y_i + 6\mathcal{L}_{\hat{X}}t_j\beta^3 y_j \\
& -6e_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta^2 - 12e_{j0}\mathcal{L}_{\hat{X}}e_{i0}\beta^2 - 6e_{j0;0}\mathcal{L}_{\hat{X}}b_i\beta^2 \\
& +12a_{ij}\beta^2 q_{00} - 6\mathcal{L}_{\hat{X}}e_{i0;0}\beta^2 b_j - 6\mathcal{L}_{\hat{X}}e_{j0;0}\beta^2 b_i \\
& +12\mathcal{L}_{\hat{X}}q_{0i}\beta^2 y_j + 12\mathcal{L}_{\hat{X}}q_{0j}\beta^2 y_i - 6\mathcal{L}_{\hat{X}}q_{i0}\beta^2 y_j \\
& -6\mathcal{L}_{\hat{X}}q_{j0}\beta^2 y_i + 6\mathcal{L}_{\hat{X}}y_i\beta^3 t_j + 6\mathcal{L}_{\hat{X}}y_j\beta^3 t_i \\
& +12a_{ij}\mathcal{L}_{\hat{X}}q_{00}\beta^2 - 12e_{00}^2\mathcal{L}_{\hat{X}}b_i b_j - 12e_{00}^2\mathcal{L}_{\hat{X}}b_j b_i \\
& -6e_{i0;0}\mathcal{L}_{\hat{X}}y_j\beta - 6e_{j0;0}\mathcal{L}_{\hat{X}}y_i\beta + 2\mathcal{L}_{\hat{X}}e_{00;0}b_i y_j \\
& +2\mathcal{L}_{\hat{X}}e_{00;0}b_j y_i - 6\mathcal{L}_{\hat{X}}e_{i0;0}\beta y_j - 6\mathcal{L}_{\hat{X}}e_{j0;0}\beta y_i \\
& -8\mathcal{L}_{\hat{X}}q_{00}y_i y_j - 8\mathcal{L}_{\hat{X}}y_i q_{00}y_j - 8\mathcal{L}_{\hat{X}}y_j q_{00}y_i \\
& +12\mathcal{L}_{\hat{X}}y_i\beta^2 q_{0j} - 6\mathcal{L}_{\hat{X}}y_i\beta^2 q_{j0} - 8\mathcal{L}_{\hat{X}}y_i s_0^2 y_j \\
& +12\mathcal{L}_{\hat{X}}y_j\beta^2 q_{0i} - 6\mathcal{L}_{\hat{X}}y_j\beta^2 q_{i0} - 8\mathcal{L}_{\hat{X}}y_j s_0^2 y_i \\
& +8a_{ij}e_{00}\mathcal{L}_{\hat{X}}e_{00} - 3a_{ij}\mathcal{L}_{\hat{X}}e_{00;0}\beta + 8e_{00}e_{i0}\mathcal{L}_{\hat{X}}y_j \\
& +8e_{00}e_{j0}\mathcal{L}_{\hat{X}}y_i + 8e_{00}\mathcal{L}_{\hat{X}}e_{i0}y_j + 8e_{00}\mathcal{L}_{\hat{X}}e_{j0}y_i \\
& -3e_{00;0}a_{ij}\beta + 2e_{00;0}\mathcal{L}_{\hat{X}}b_i y_j + 2e_{00;0}\mathcal{L}_{\hat{X}}b_j y_i \\
& +2e_{00;0}\mathcal{L}_{\hat{X}}y_i b_j + 2e_{00;0}\mathcal{L}_{\hat{X}}y_j b_i + 8e_{i0}\mathcal{L}_{\hat{X}}e_{00}y_j
\end{aligned}
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\begin{aligned}
&+8e_{j0}\mathcal{L}_{\hat{X}}e_{00}y_i+8\mathcal{L}_{\hat{X}}\beta\beta^2b_js_it_0-16\mathcal{L}_{\hat{X}}s_0\beta^2b_ib_js_0\\
&+32e_{00}\mathcal{L}_{\hat{X}}b_i\beta b_js_0+32e_{00}\mathcal{L}_{\hat{X}}b_j\beta b_is_0+32e_{00}\mathcal{L}_{\hat{X}}s_0\beta b_ib_j\\
&-32e_{00}\beta b_ib_j\eta+20t_{00}\beta b_is_jt_0+20t_{00}\beta b_js_it_0\\
&+32\mathcal{L}_{\hat{X}}e_{00}\beta b_ib_js_0+16\mathcal{L}_{\hat{X}}\beta\beta s_it_0y_j+16\mathcal{L}_{\hat{X}}\beta\beta s_jt_0y_i\\
&-20\mathcal{L}_{\hat{X}}s_0\beta b_is_0y_j-20\mathcal{L}_{\hat{X}}s_0\beta b_js_0y_i\\
&-16\mathcal{L}_{\hat{X}}\beta\beta^2b_ib_jt_0+8\mathcal{L}_{\hat{X}}\beta\beta^2b_is_jt_0+72e_{j0}\mathcal{L}_{\hat{X}}\beta\beta b_is_0\\
&+56\mathcal{L}_{\hat{X}}\beta\beta b_ib_jq_{00}-58\mathcal{L}_{\hat{X}}\beta\beta s_0s_iy_j\\
&-58\mathcal{L}_{\hat{X}}\beta\beta s_0s_jy_i-60t_{00}\beta b_ib_jt_0-56t_{00}\beta b_is_0s_j\\
&-56t_{00}\beta b_js_0s_i-44\mathcal{L}_{\hat{X}}\beta\beta^2b_is_0s_j\\
&-44\mathcal{L}_{\hat{X}}\beta\beta^2b_js_0s_i+56\mathcal{L}_{\hat{X}}\beta\beta b_ib_js_0^2+48e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_is_j\\
&+48e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_js_i-112e_{00}\mathcal{L}_{\hat{X}}\beta b_ib_js_0\\
&+72e_{i0}\mathcal{L}_{\hat{X}}\beta\beta b_js_0-12e_{00}\mathcal{L}_{\hat{X}}s_i\beta^2b_j-12e_{00}\mathcal{L}_{\hat{X}}s_j\beta^2b_i\\
&+12e_{00}\beta^2b_i\eta_j+12e_{00}\beta^2b_j\eta_i-18e_{i0}\mathcal{L}_{\hat{X}}b_j\beta^2s_0\\
&-18e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^2b_j+18e_{i0}\beta^2b_j\eta-18e_{j0}\mathcal{L}_{\hat{X}}b_i\beta^2s_0\\
&-18e_{j0}\mathcal{L}_{\hat{X}}s_0\beta^2b_i+18e_{j0}\beta^2b_ieta-12\mathcal{L}_{\hat{X}}e_{00}\beta^2b_is_j\\
&-12\mathcal{L}_{\hat{X}}e_{00}\beta^2b_js_i-18\mathcal{L}_{\hat{X}}e_{i0}\beta^2b_js_0-18\mathcal{L}_{\hat{X}}e_{j0}\beta^2b_is_0\\
&+6\mathcal{L}_{\hat{X}}s_j\beta^3b_is_0+4\mathcal{L}_{\hat{X}}s_j\beta^3b_it_0\\
&-8\mathcal{L}_{\hat{X}}t_0\beta^3b_ib_j+4\mathcal{L}_{\hat{X}}t_0\beta^3b_is_j+4\mathcal{L}_{\hat{X}}t_0\beta^3b_js_i\\
&+24a_{ij}\mathcal{L}_{\hat{X}}s_0\beta^2s_0-12e_{00}\mathcal{L}_{\hat{X}}b_i\beta^2s_j-12e_{00}\mathcal{L}_{\hat{X}}b_j\beta^2s_i\\
&-22\mathcal{L}_{\hat{X}}t_0\beta^2b_jy_i+12\mathcal{L}_{\hat{X}}t_0\beta^2s_iy_j+12\mathcal{L}_{\hat{X}}t_0\beta^2s_jy_i\\
&-22\mathcal{L}_{\hat{X}}y_i\beta^2b_jt_0+6\mathcal{L}_{\hat{X}}y_i\beta^2s_0s_j+12\mathcal{L}_{\hat{X}}y_i\beta^2s_jt_0\\
&-10\mathcal{L}_{\hat{X}}y_i\beta b_js_0^2-22\mathcal{L}_{\hat{X}}y_j\beta^2b_it_0+6\mathcal{L}_{\hat{X}}y_j\beta^2s_0s_i\\
&+12\mathcal{L}_{\hat{X}}y_j\beta^2s_it_0-10\mathcal{L}_{\hat{X}}y_j\beta b_is_0^2-8\mathcal{L}_{\hat{X}}b_i\beta^2b_jq_{00}\\
&-22\mathcal{L}_{\hat{X}}b_i\beta^2t_0y_j-10\mathcal{L}_{\hat{X}}b_i\beta s_0^2y_j-8\mathcal{L}_{\hat{X}}b_j\beta^2b_iq_{00}\\
&-22\mathcal{L}_{\hat{X}}b_j\beta^2t_0y_i-10\mathcal{L}_{\hat{X}}b_j\beta s_0^2y_i-8\mathcal{L}_{\hat{X}}q_{00}\beta^2b_ib_j\\
&-6\mathcal{L}_{\hat{X}}s_0\beta^2s_iy_j-6\mathcal{L}_{\hat{X}}s_0\beta^2s_jy_i-12e_{00}\mathcal{L}_{\hat{X}}y_j\beta s_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}y_jb_is_0+12e_{00}\beta\eta_iy_j+12e_{00}\beta\eta_jy_i\\
&-16e_{00}b_i\eta y_j-16e_{00}b_j\eta y_i+4e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta b_j\\
&+4e_{00;0}\mathcal{L}_{\hat{X}}b_j\beta b_i-12a_{ij}e_{00}\mathcal{L}_{\hat{X}}s_0\beta+24a_{ij}e_{00}\beta\eta\\
&-24a_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta s_0+16e_{00}e_{i0}\mathcal{L}_{\hat{X}}b_j\beta+6\mathcal{L}_{\hat{X}}s_i\beta^2s_0y_j\\
&+12\mathcal{L}_{\hat{X}}s_i\beta^2t_0y_j+6\mathcal{L}_{\hat{X}}s_j\beta^2s_0y_i+12\mathcal{L}_{\hat{X}}s_j\beta^2t_0y_i\\
&-22\mathcal{L}_{\hat{X}}t_0\beta^2b_iy_j-16\mathcal{L}_{\hat{X}}s_0s_0y_iy_j-8\mathcal{L}_{\hat{X}}t_0\beta y_iy_j\\
&-10\mathcal{L}_{\hat{X}}y_i\beta b_jq_{00}-8\mathcal{L}_{\hat{X}}y_i\beta t_0y_j+16e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta b_j\\
&-18e_{i0}\mathcal{L}_{\hat{X}}s_0\beta y_j-18e_{i0}\mathcal{L}_{\hat{X}}y_j\beta s_0+18e_{i0}\beta\eta y_j\\
&+16e_{j0}\mathcal{L}_{\hat{X}}e_{00}\beta b_i-18e_{j0}\mathcal{L}_{\hat{X}}s_0\beta y_i-18e_{j0}\mathcal{L}_{\hat{X}}y_i\beta s_0\\
&+18e_{j0}\beta\eta y_i-12\mathcal{L}_{\hat{X}}e_{00}\beta s_iy_j-12\mathcal{L}_{\hat{X}}e_{00}\beta s_jyi
\end{aligned}
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\begin{aligned}
&+16e_{00}e_{j0}\mathcal{L}_{\hat{X}}b_i\beta-24e_{00}a_{ij}\beta s_0-24e_{00}\mathcal{L}_{\hat{X}}e_{00}b_ib_j\\
&+16e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta b_j+16e_{00}\mathcal{L}_{\hat{X}}e_{j0}\beta b_i+16e_{00}\mathcal{L}_{\hat{X}}b_is_0y_j\\
&+16e_{00}\mathcal{L}_{\hat{X}}b_js_0y_i+16e_{00}\mathcal{L}_{\hat{X}}s_0b_iy_j\\
&+16e_{00}\mathcal{L}_{\hat{X}}s_0b_jy_i-12e_{00}\mathcal{L}_{\hat{X}}s_i\beta y_j-12e_{00}\mathcal{L}_{\hat{X}}s_j\beta y_i\\
&-12e_{00}\mathcal{L}_{\hat{X}}y_i\beta s_j+16e_{00}\mathcal{L}_{\hat{X}}y_ib_js_0-10\mathcal{L}_{\hat{X}}y_j\beta b_iq_{00}\\
&-8\mathcal{L}_{\hat{X}}y_j\beta t_0y_i+6t_{00}s_it_0y_j+6t_{00}s_jt_0y_i\\
&+16\mathcal{L}_{\hat{X}}e_{00}b_is_0y_j+16\mathcal{L}_{\hat{X}}e_{00}b_js_0y_i+4\mathcal{L}_{\hat{X}}e_{00;0}\beta b_ib_j\\
&-18\mathcal{L}_{\hat{X}}e_{i0}\beta s_0y_j-18\mathcal{L}_{\hat{X}}e_{j0}\beta s_0y_i-8\mathcal{L}_{\hat{X}}\beta t_0y_iy_j\\
&-10\mathcal{L}_{\hat{X}}b_i\beta q_{00}y_j-10\mathcal{L}_{\hat{X}}b_j\beta q_{00}y_i-10\mathcal{L}_{\hat{X}}q_{00}\beta b_iy_j\\
&-10\mathcal{L}_{\hat{X}}q_{00}\beta b_jy_i+2\mathcal{L}_{\hat{X}}s_0\beta^3b_is_j+2\mathcal{L}_{\hat{X}}s_0\beta^3b_js_i\\
&+6\mathcal{L}_{\hat{X}}s_i\beta^3b_js_0+4\mathcal{L}_{\hat{X}}s_i\beta^3b_jt_0-8\mathcal{L}_{\hat{X}}b_i\beta^3b_jt_0\\
&+6\mathcal{L}_{\hat{X}}b_i\beta^3s_0s_j+4\mathcal{L}_{\hat{X}}b_i\beta^3s_jt_0-8\mathcal{L}_{\hat{X}}b_i\beta^2b_js_0^2\\
&-8\mathcal{L}_{\hat{X}}b_j\beta^3b_it_0+6\mathcal{L}_{\hat{X}}b_j\beta^3s_0s_i+4\mathcal{L}_{\hat{X}}b_j\beta^3s_it_0\\
&-8\mathcal{L}_{\hat{X}}b_j\beta^2b_is_0^2-2\mathcal{L}_{\hat{X}}q_{ij}\beta^4-2\mathcal{L}_{\hat{X}}q_{ji}\beta^4\\
&+4e_{00}^2\mathcal{L}_{\hat{X}}a_{ij}+8e_{i0}\mathcal{L}_{\hat{X}}s_j\beta^3-8e_{i0}\beta^3\eta_j+16e_{ij}\mathcal{L}_{\hat{X}}s_0\beta^3\\
&-16e_{ij}\beta^3\eta+8e_{j0}\mathcal{L}_{\hat{X}}s_i\beta^3-8e_{j0}\beta^3\eta_i+8\mathcal{L}_{\hat{X}}e_{i0}\beta^3s_j\\
&+16\mathcal{L}_{\hat{X}}e_{ij}\beta^3s_0+8\mathcal{L}_{\hat{X}}e_{j0}\beta^3s_i+8\mathcal{L}_{\hat{X}}e_{ij;0}\beta^3\\
&+21e_{i0}t_{00}b_js_0+24e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta b_j-38e_{ij}t_{00}\beta s_0\\
&+36e_{j0}\mathcal{L}_{\hat{X}}\beta s_0y_i-19e_{j0}t_{00}\beta s_i+21e_{j0}t_{00}b_is_0\\
&+20\mathcal{L}_{\hat{X}}\beta\beta^3s_is_j-36a_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2t_0-48a_{ij}\mathcal{L}_{\hat{X}}\beta\beta s_0^2\\
&-30e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2s_j-60e_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2s_0-30e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^2s_i\\
-\;&24\mathcal{L}_{\hat{X}}\beta\beta^2b_iq_{0j}-12\mathcal{L}_{\hat{X}}\beta\beta^2b_iq_{j0}-28\mathcal{L}_{\hat{X}}\beta\beta^2b_jq_{0i}\\
&-12\mathcal{L}_{\hat{X}}\beta\beta^2b_jq_{i0}-18\mathcal{L}_{\hat{X}}\beta\beta^2t_iy_j-18\mathcal{L}_{\hat{X}}\beta\beta^2t_jy_i\\
&+44\mathcal{L}_{\hat{X}}\beta b_is_0^2y_j+44\mathcal{L}_{\hat{X}}\beta b_js_0^2y_i+44\mathcal{L}_{\hat{X}}\beta b_iq_{00}y_j\\
&+44\mathcal{L}_{\hat{X}}\beta b_jq_{00}y_i-24t_{00}\beta b_iq_{0j}-12t_{00}\beta b_iq_{j0}\\
&-34t_{00}\beta b_jq_{0i}-12t_{00}\beta b_jq_{i0}-7t_{00}\beta t_iy_j-7t_{00}\beta t_jy_i\\
&+28t_{00}b_ib_jq_{00}-13t_{00}b_it_0y_j-13t_{00}b_jt_0y_i\\
-\;&22t_{00}s_0s_iy_j-22t_{00}s_0s_jy_i+3t_{00}\beta^2b_it_j+3t_{00}\beta^2b_jt_i\\
&+30t_{00}\beta^2s_is_j+28t_{00}b_ib_js_0^2+48a_{ij}e_{00}\mathcal{L}_{\hat{X}}\beta s_0\\
-\;&48a_{ij}\mathcal{L}_{\hat{X}}\beta\beta q_{00}-14a_{ij}t_{00}\beta t_0-56e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta b_j\\
&+24e_{00}e_{ij}\mathcal{L}_{\hat{X}}\beta\beta-56e_{00}e_{j0}\mathcal{L}_{\hat{X}}\beta b_i+24e_{00}\mathcal{L}_{\hat{X}}\beta s_iy_j\\
&+24e_{00}\mathcal{L}_{\hat{X}}\beta s_jy_i+14e_{00}t_{00}b_is_j+14e_{00}t_{00}b_js_i\\
&-14e_{00;0}\mathcal{L}_{\hat{X}}\beta b_ib_j+48e_{i0}e_{j0}\mathcal{L}_{\hat{X}}\beta\beta+36e_{i0}\mathcal{L}_{\hat{X}}\beta s_0y_j\\
&-19e_{i0}t_{00}\beta s_j+24e_{j0;0}\mathcal{L}_{\hat{X}}\beta\beta b_i-56\mathcal{L}_{\hat{X}}\beta\beta q_{0i}y_j\\
&-48\mathcal{L}_{\hat{X}}\beta\beta q_{0j}y_i+6\mathcal{L}_{\hat{X}}\beta\beta q_{i0}y_j+6\mathcal{L}_{\hat{X}}\beta\beta q_{j0}y_i\\
A_7\;=\;&30\mathcal{L}_{\hat{X}}\beta\beta^2q_{ij}+30\mathcal{L}_{\hat{X}}\beta\beta^2q_{ji}-24a_{ij}\mathcal{L}_{\hat{X}}\beta q_{00}
\end{aligned}
$$

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\begin{aligned}
&+12e_{00}e_{ij}\mathcal{L}_{\hat{X}}\beta+24e_{i0}e_{j0}\mathcal{L}_{\hat{X}}\beta+6\mathcal{L}_{\hat{X}}t_j\beta^3b_j\\
&+12a_{ij}\mathcal{L}_{\hat{X}}t_0\beta^2+12e_{i0}\mathcal{L}_{\hat{X}}s_j\beta^2-6e_{i0}t_{00}s_j\\
&+12e_{i0;0}\mathcal{L}_{\hat{X}}\beta b_j-12e_{ij}t_{00}s_0-30e_{ij;0}\mathcal{L}_{\hat{X}}\beta\beta-6e_{j0}t_{00}s_i\\
&+12e_{j0;0}\mathcal{L}_{\hat{X}}\beta b_i-28\mathcal{L}_{\hat{X}}\beta q_{0i}y_j-24\mathcal{L}_{\hat{X}}\beta q_{0j}y_i\\
&+15t_{00}\beta q_{ij}+15t_{00}\beta q_{ji}-12t_{00}b_iq_{0j}-6t_{00}b_iq_{j0}\\
&-16t_{00}b_jq_{0i}-6t_{00}b_jq_{i0}-24a_{ij}\mathcal{L}_{\hat{X}}\beta s_0^2-12e_{i0}\beta^2\eta_j\\
&+24e_{ij}\mathcal{L}_{\hat{X}}s_0\beta^2-24e_{ij}\beta^2\eta+12e_{j0}\mathcal{L}_{\hat{X}}s_i\beta^2\\
&-12e_{j0}\beta^2\eta_i+12\mathcal{L}_{\hat{X}}a_{ij}\beta^2t_0+12\mathcal{L}_{\hat{X}}a_{ij}\beta s_0^2\\
&+12\mathcal{L}_{\hat{X}}e_{i0}\beta^2s_j+24\mathcal{L}_{\hat{X}}e_{ij}\beta^2s_0+12\mathcal{L}_{\hat{X}}e_{j0}\beta^2s_i\\
&+8e_{00}\mathcal{L}_{\hat{X}}e_{j0}b_i-4e_{00}\mathcal{L}_{\hat{X}}s_iy_j-4e_{00}\mathcal{L}_{\hat{X}}s_jy_i\\
&-4e_{00}\mathcal{L}_{\hat{X}}y_is_j-4e_{00}\mathcal{L}_{\hat{X}}y_js_i+4e_{00}\eta_iy_j+4e_{00}\eta_jy_i\\
&+2e_{00;0}\mathcal{L}_{\hat{X}}b_ib_j+2e_{00;0}\mathcal{L}_{\hat{X}}b_jb_i+8e_{i0}\mathcal{L}_{\hat{X}}e_{00}b_j\\
&-12e_{i0}\mathcal{L}_{\hat{X}}e_{j0}\beta-6e_{i0}\mathcal{L}_{\hat{X}}s_0y_j-6e_{i0}\mathcal{L}_{\hat{X}}y_js_0\\
&+6e_{i0}\eta y_j-6e_{i0;0}\mathcal{L}_{\hat{X}}b_j\beta+12\mathcal{L}_{\hat{X}}b_i\beta^2q_{0j}+6\mathcal{L}_{\hat{X}}b_i\beta^2q_{j0}\\
&-6\mathcal{L}_{\hat{X}}b_is_0^2y_j+12\mathcal{L}_{\hat{X}}b_j\beta^2q_{0i}+6\mathcal{L}_{\hat{X}}b_j\beta^2q_{i0}\\
&-6\mathcal{L}_{\hat{X}}b_js_0^2y_i+12\mathcal{L}_{\hat{X}}q_{0i}\beta^2b_j+12\mathcal{L}_{\hat{X}}q_{0j}\beta^2b_i\\
&+6\mathcal{L}_{\hat{X}}q_{i0}\beta^2b_j+6\mathcal{L}_{\hat{X}}q_{j0}\beta^2b_i+6\mathcal{L}_{\hat{X}}t_j\beta^2y_i+6\mathcal{L}_{\hat{X}}t_j\beta^2y_j\\
&+8e_{j0}\mathcal{L}_{\hat{X}}e_{00}b_i-12e_{j0}\mathcal{L}_{\hat{X}}e_{i0}\beta-6e_{j0}\mathcal{L}_{\hat{X}}s_0y_i-6e_{j0}\mathcal{L}_{\hat{X}}y_is_0\\
&+6e_{j0}\eta y_i-6e_{j0;0}\mathcal{L}_{\hat{X}}b_i\beta+12\mathcal{L}_{\hat{X}}a_{ij}\beta q_{00}\\
&-4\mathcal{L}_{\hat{X}}e_{00}s_iy_j-4\mathcal{L}_{\hat{X}}e_{00}s_jy_i+2\mathcal{L}_{\hat{X}}e_{00;0}b_ib_j\\
&-6\mathcal{L}_{\hat{X}}e_{i0}s_0y_j-6\mathcal{L}_{\hat{X}}e_{i0;0}\beta b_j-6\mathcal{L}_{\hat{X}}e_{j0}s_0y_i\\
&-6\mathcal{L}_{\hat{X}}e_{j0;0}\beta b_i-2\mathcal{L}_{\hat{X}}\beta q_{i0}y_j-2\mathcal{L}_{\hat{X}}\beta q_{j0}y_i-6\mathcal{L}_{\hat{X}}b_iq_{00}y_j\\
&-6\mathcal{L}_{\hat{X}}b_jq_{00}y_i-6\mathcal{L}_{\hat{X}}q_{00}b_iy_j-6\mathcal{L}_{\hat{X}}q_{00}b_jy_i\\
&+12\mathcal{L}_{\hat{X}}q_{0i}\beta y_j+12\mathcal{L}_{\hat{X}}q_{0j}\beta y_i-2\mathcal{L}_{\hat{X}}q_{i0}\beta y_j-2\mathcal{L}_{\hat{X}}q_{j0}\beta y_i\\
&+6\mathcal{L}_{\hat{X}}y_i\beta^2t_j-6\mathcal{L}_{\hat{X}}y_ib_js_0^2+6\mathcal{L}_{\hat{X}}y_j\beta^2t_i\\
&-6\mathcal{L}_{\hat{X}}y_jb_is_0^2-4a_{ij}e_{00}\mathcal{L}_{\hat{X}}s_0+8a_{ij}e_{00}\eta-8a_{ij}\mathcal{L}_{\hat{X}}e_{00}s_0\\
&+12a_{ij}\mathcal{L}_{\hat{X}}q_{00}\beta+8e_{00}e_{i0}\mathcal{L}_{\hat{X}}b_j+8e_{00}e_{j0}\mathcal{L}_{\hat{X}}b_i\\
&-8e_{00}\mathcal{L}_{\hat{X}}a_{ij}s_0+8e_{00}\mathcal{L}_{\hat{X}}e_{i0}b_j-6e_{00}\mathcal{L}_{\hat{X}}e_{ij}\beta\\
&+12\mathcal{L}_{\hat{X}}y_i\beta q_{0j}-2\mathcal{L}_{\hat{X}}y_i\beta q_{j0}-6\mathcal{L}_{\hat{X}}y_ib_jq_{00}+12\mathcal{L}_{\hat{X}}y_j\beta q_{0i}\\
&-2\mathcal{L}_{\hat{X}}y_j\beta q_{i0}-6\mathcal{L}_{\hat{X}}y_jb_iq_{00}-6e_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta\\
&-32\mathcal{L}_{\hat{X}}s_0\beta b_ib_js_0+16\mathcal{L}_{\hat{X}}\beta\beta b_is_jt_0+16\mathcal{L}_{\hat{X}}\beta\beta b_js_it_0\\
&-88\mathcal{L}_{\hat{X}}\beta\beta b_ib_jt_0-88\mathcal{L}_{\hat{X}}\beta\beta b_is_0s_j-88\mathcal{L}_{\hat{X}}\beta\beta b_js_0s_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}s_0b_ib_j-12e_{00}\mathcal{L}_{\hat{X}}s_i\beta b_j\\
&-12e_{00}\mathcal{L}_{\hat{X}}s_j\beta b_i+12e_{00}\beta b_i\eta_j+12e_{00}\beta b_j\eta_i-16e_{00}b_ib_j\eta\\
&+10\mathcal{L}_{\hat{X}}s_i\beta s_0y_j-18e_{i0}\mathcal{L}_{\hat{X}}b_j\beta s_0\\
&-18e_{i0}\mathcal{L}_{\hat{X}}s_0\beta b_j+18e_{i0}\beta b_j\eta-18e_{j0}\mathcal{L}_{\hat{X}}b_i\beta s_0
\end{aligned}
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\begin{aligned}
& -18e_{j0}\mathcal{L}_{\hat{X}}s_0\beta b_i + 18e_{j0}\beta b_i\eta - 12\mathcal{L}_{\hat{X}}e_{00}\beta b_i s_j \\
& -12\mathcal{L}_{\hat{X}}e_{00}\beta b_j s_i + 16\mathcal{L}_{\hat{X}}e_{00}b_i b_j s_0 - 18\mathcal{L}_{\hat{X}}e_{i0}\beta b_j s_0 \\
& -18\mathcal{L}_{\hat{X}}e_{j0}\beta b_i s_0 + 12\mathcal{L}_{\hat{X}}s_j\beta^2 b_i t_0 - 16\mathcal{L}_{\hat{X}}t_0\beta^2 b_i b_j \\
& +12\mathcal{L}_{\hat{X}}t_0\beta^2 b_i s_j + 12\mathcal{L}_{\hat{X}}t_0\beta^2 b_j s_i + 24a_{ij}\mathcal{L}_{\hat{X}}s_0\beta s_0 \\
& -12e_{00}\mathcal{L}_{\hat{X}}b_i\beta s_j + 16e_{00}\mathcal{L}_{\hat{X}}b_i b_j s_0 - 12e_{00}\mathcal{L}_{\hat{X}}b_j\beta s_i \\
& +16e_{00}\mathcal{L}_{\hat{X}}b_j b_i s_0 - 14\mathcal{L}_{\hat{X}}y_i\beta b_j t_0 + 10\mathcal{L}_{\hat{X}}y_i\beta s_0 s_j \\
& +12\mathcal{L}_{\hat{X}}y_i\beta s_j t_0 - 14\mathcal{L}_{\hat{X}}y_j\beta b_i t_0 + 10\mathcal{L}_{\hat{X}}y_j\beta s_0 s_i \\
& +12\mathcal{L}_{\hat{X}}y_j\beta s_i t_0 + 8\mathcal{L}_{\hat{X}}\beta s_i t_0 y_j + 8\mathcal{L}_{\hat{X}}\beta s_j t_0 y_i \\
& -16\mathcal{L}_{\hat{X}}b_i\beta b_j q_{00} - 14\mathcal{L}_{\hat{X}}b_i\beta t_0 y_j - 16\mathcal{L}_{\hat{X}}b_j\beta b_i q_{00} \\
& -14\mathcal{L}_{\hat{X}}b_j\beta t_0 y_i - 16\mathcal{L}_{\hat{X}}q_{00}\beta b_i b_j \\
& -2\mathcal{L}_{\hat{X}}s_0\beta s_i y_j - 2\mathcal{L}_{\hat{X}}s_0\beta s_j y_i - 12\mathcal{L}_{\hat{X}}s_0 b_i s_0 y_j - 12\mathcal{L}_{\hat{X}}s_0 b_j s_0 y_i \\
& +8t_{00}b_i s_j t_0 + 8t_{00}b_j s_i t_0 + 12\mathcal{L}_{\hat{X}}s_i\beta t_0 y_j \\
& +10\mathcal{L}_{\hat{X}}s_j\beta s_0 y_i + 12\mathcal{L}_{\hat{X}}s_j\beta t_0 y_i - 14\mathcal{L}_{\hat{X}}t_0\beta(b_i y_j + b_j y_i) \\
& +12\mathcal{L}_{\hat{X}}t_0\beta s_i y_j + 12\mathcal{L}_{\hat{X}}t_0\beta s_j y_i + 6\mathcal{L}_{\hat{X}}s_0\beta^2 b_i s_j \\
& +6\mathcal{L}_{\hat{X}}s_0\beta^2 b_j s_i + 18\mathcal{L}_{\hat{X}}s_i\beta^2 b_j s_0 + 12\mathcal{L}_{\hat{X}}s_i\beta^2 b_j t_0 \\
& +18\mathcal{L}_{\hat{X}}s_j\beta^2 b_i s_0 - 16\mathcal{L}_{\hat{X}}b_i\beta^2 b_j t_0 + 18\mathcal{L}_{\hat{X}}b_i\beta^2 s_0 s_j \\
& +12\mathcal{L}_{\hat{X}}b_i\beta^2 s_j t_0 - 16\mathcal{L}_{\hat{X}}b_i\beta b_j s_0^2 - 16\mathcal{L}_{\hat{X}}b_j\beta^2 b_i t_0 \\
& +18\mathcal{L}_{\hat{X}}b_j\beta^2 s_0 s_i + 12\mathcal{L}_{\hat{X}}b_j\beta^2 s_i t_0 - 16\mathcal{L}_{\hat{X}}b_j\beta b_i s_0^2 \\
& -6e_{ij;0}t_{00} + 12(\mathcal{L}_{\hat{X}}e_{ij})_{;0}\beta^2 - a_{ij}(\mathcal{L}_{\hat{X}}e_{00})_{;0} \\
& -e_{00;0}\mathcal{L}_{\hat{X}}a_{ij} - 2e_{i0;0}\mathcal{L}_{\hat{X}}y_j - 2e_{j0;0}\mathcal{L}_{\hat{X}}y_i - 2(\mathcal{L}_{\hat{X}}e_{i0})_{;0}y_j \\
& -2(\mathcal{L}_{\hat{X}}e_{j0})_{;0}y_i - 2\mathcal{L}_{\hat{X}}b_i\beta^3 t_j - 2\mathcal{L}_{\hat{X}}b_j\beta^3 t_i \\
& -16\mathcal{L}_{\hat{X}}s_i\beta^3 s_j - 16\mathcal{L}_{\hat{X}}s_j\beta^3 s_i - 8\mathcal{L}_{\hat{X}}t_i\beta^3 b_j - 2\mathcal{L}_{\hat{X}}t_j\beta^3 b_i \\
& -8\mathcal{L}_{\hat{X}}q_{ij}\beta^3 - 8\mathcal{L}_{\hat{X}}q_{ji}\beta^3 - 26t_{00}b_i s_0 s_j \\
& +12\mathcal{L}_{\hat{X}}\beta\beta^2 b_i t_j + 12\mathcal{L}_{\hat{X}}\beta\beta^2 b_j t_i + 60\mathcal{L}_{\hat{X}}\beta\beta^2 s_i s_j \\
& +56\mathcal{L}_{\hat{X}}\beta b_i b_j s_0^2 - 12a_{ij}\mathcal{L}_{\hat{X}}\beta\beta t_0 + 24e_{00}\mathcal{L}_{\hat{X}}\beta b_i s_j \\
& +24e_{00}\mathcal{L}_{\hat{X}}\beta b_j s_i - 30e_{i0}\mathcal{L}_{\hat{X}}\beta\beta s_j + 36e_{i0}\mathcal{L}_{\hat{X}}\beta b_j s_0 \\
& -60e_{ij}\mathcal{L}_{\hat{X}}\beta\beta s_0 - 30e_{j0}\mathcal{L}_{\hat{X}}\beta\beta s_i + 36e_{j0}\mathcal{L}_{\hat{X}}\beta b_i s_0 \\
& -48\mathcal{L}_{\hat{X}}\beta\beta b_i q_{0j} - 24\mathcal{L}_{\hat{X}}\beta\beta b_i q_{j0} - 56\mathcal{L}_{\hat{X}}\beta\beta b_j q_{0i} \\
& -24\mathcal{L}_{\hat{X}}\beta\beta b_j q_{i0} - 6\mathcal{L}_{\hat{X}}\beta\beta t_i y_j - 6\mathcal{L}_{\hat{X}}\beta\beta t_j y_i \\
& +56\mathcal{L}_{\hat{X}}\beta b_i b_j q_{00} - 28\mathcal{L}_{\hat{X}}\beta b_i t_0 y_j - 28\mathcal{L}_{\hat{X}}\beta b_j t_0 y_i - 34\mathcal{L}_{\hat{X}}\beta s_0 s_i y_j \\
& -34\mathcal{L}_{\hat{X}}\beta s_0 s_j y_i + 9t_{00}\beta b_i t_j \\
& +9t_{00}\beta b_j t_i + 30t_{00}\beta s_i s_j - 40t_{00}b_i b_j t_0 - 26t_{00}b_j s_0 s_i \\
A_8 = & -20e_{ij}\mathcal{L}_{\hat{X}}\beta s_0 - 10e_{j0}\mathcal{L}_{\hat{X}}\beta s_i + 30\mathcal{L}_{\hat{X}}\beta\beta q_{ij} \\
& +30\mathcal{L}_{\hat{X}}\beta\beta q_{ji} - 24\mathcal{L}_{\hat{X}}\beta b_i q_{0j} - 12\mathcal{L}_{\hat{X}}\beta b_i q_{j0} \\
& -28\mathcal{L}_{\hat{X}}\beta b_j q_{0i} - 12\mathcal{L}_{\hat{X}}\beta b_j q_{i0} + 5t_{00}b_i t_j + 5t_{00}b_j t_i \\
& +10t_{00}s_i s_j - 12\mathcal{L}_{\hat{X}}t_i\beta^2 b_j - 6\mathcal{L}_{\hat{X}}t_j\beta^2 b_i
\end{aligned}
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\begin{aligned}
&+6\mathcal{L}_{\hat{X}}t_j\beta^2 b_j+4a_{ij}\mathcal{L}_{\hat{X}}\beta t_0+8a_{ij}\mathcal{L}_{\hat{X}}s_0s_0+4a_{ij}\mathcal{L}_{\hat{X}}t_0\beta-4e_{00;0}\mathcal{L}_{\hat{X}}b_is_j\\
&-4e_{00;0}\mathcal{L}_{\hat{X}}b_js_i-4e_{00;0}\mathcal{L}_{\hat{X}}s_ib_j\\
&-4e_{00;0}\mathcal{L}_{\hat{X}}s_jb_i+4e_{00;0}b_i\eta_j+4e_{00;0}b_j\eta_i-6e_{i0}\mathcal{L}_{\hat{X}}b_js_0\\
&-6e_{i0}\mathcal{L}_{\hat{X}}s_0b_j+8e_{i0}\mathcal{L}_{\hat{X}}s_j\beta-10e_{i0}\mathcal{L}_{\hat{X}}\beta s_j\\
&-8e_{i0}\beta\eta_j+6e_{i0}b_j\eta+16e_{ij}\mathcal{L}_{\hat{X}}s_0\beta-16e_{ij}\beta\eta\\
&-6e_{j0}\mathcal{L}_{\hat{X}}b_is_0-6e_{j0}\mathcal{L}_{\hat{X}}s_0b_i+8e_{j0}\mathcal{L}_{\hat{X}}s_i\beta-8e_{j0}\beta\eta_i\\
&+6e_{j0}b_i\eta+4\mathcal{L}_{\hat{X}}a_{ij}\beta t_0-4\mathcal{L}_{\hat{X}}e_{00;0}b_is_j\\
&-4\mathcal{L}_{\hat{X}}e_{00;0}b_js_i+8\mathcal{L}_{\hat{X}}e_{i0}\beta s_j-6\mathcal{L}_{\hat{X}}e_{i0}b_js_0+16\mathcal{L}_{\hat{X}}e_{ij}\beta s_0\\
&+8\mathcal{L}_{\hat{X}}e_{j0}\beta s_i-6\mathcal{L}_{\hat{X}}e_{j0}b_is_0-6\mathcal{L}_{\hat{X}}b_i\beta^2t_j-8\mathcal{L}_{\hat{X}}b_ib_js_0^2\\
&-6\mathcal{L}_{\hat{X}}b_j\beta^2t_i-8\mathcal{L}_{\hat{X}}b_jb_is_0^2-24\mathcal{L}_{\hat{X}}s_i\beta^2s_j\\
&-24\mathcal{L}_{\hat{X}}s_j\beta^2s_i+2\mathcal{L}_{\hat{X}}\beta t_iy_j+2\mathcal{L}_{\hat{X}}\beta t_jy_i+12\mathcal{L}_{\hat{X}}b_i\beta q_{0j}\\
&+6\mathcal{L}_{\hat{X}}b_i\beta q_{j0}-8\mathcal{L}_{\hat{X}}b_ib_jq_{00}-2\mathcal{L}_{\hat{X}}b_it_0y_j\\
&+12\mathcal{L}_{\hat{X}}b_j\beta q_{0i}+6\mathcal{L}_{\hat{X}}b_j\beta q_{i0}-8\mathcal{L}_{\hat{X}}b_jb_iq_{00}-2\mathcal{L}_{\hat{X}}b_jt_0y_i\\
&-8\mathcal{L}_{\hat{X}}q_{00}b_ib_j+12\mathcal{L}_{\hat{X}}q_{0i}\beta b_j+12\mathcal{L}_{\hat{X}}q_{0j}\beta b_i\\
&+6\mathcal{L}_{\hat{X}}q_{i0}\beta b_j+6\mathcal{L}_{\hat{X}}q_{j0}\beta b_i+4\mathcal{L}_{\hat{X}}s_is_0y_j+4\mathcal{L}_{\hat{X}}s_it_0y_j\\
&+4\mathcal{L}_{\hat{X}}s_js_0y_i+4\mathcal{L}_{\hat{X}}s_jt_0y_i-2\mathcal{L}_{\hat{X}}t_0b_iy_j\\
&-2\mathcal{L}_{\hat{X}}t_0b_jy_i+4\mathcal{L}_{\hat{X}}t_0s_iy_j+4\mathcal{L}_{\hat{X}}t_0s_jy_i+2\mathcal{L}_{\hat{X}}t_j\beta y_i\\
&+2\mathcal{L}_{\hat{X}}t_j\beta y_j+2\mathcal{L}_{\hat{X}}y_i\beta t_j-2\mathcal{L}_{\hat{X}}y_ib_jt_0\\
&+4\mathcal{L}_{\hat{X}}y_is_0s_j+4\mathcal{L}_{\hat{X}}y_is_jt_0+2\mathcal{L}_{\hat{X}}y_j\beta t_i-2\mathcal{L}_{\hat{X}}y_jb_it_0\\
&+4\mathcal{L}_{\hat{X}}y_js_0s_i+4\mathcal{L}_{\hat{X}}y_js_it_0-8\mathcal{L}_{\hat{X}}t_0\beta b_ib_j\\
&+12\mathcal{L}_{\hat{X}}t_0\beta b_is_j+12\mathcal{L}_{\hat{X}}t_0\beta b_js_i+6\mathcal{L}_{\hat{X}}s_0\beta b_is_j\\
&+6\mathcal{L}_{\hat{X}}s_0\beta b_js_i-16\mathcal{L}_{\hat{X}}s_0b_ib_js_0+18\mathcal{L}_{\hat{X}}s_i\beta b_js_0\\
&+12\mathcal{L}_{\hat{X}}s_i\beta b_jt_0+18\mathcal{L}_{\hat{X}}s_j\beta b_is_0+12\mathcal{L}_{\hat{X}}s_j\beta b_it_0\\
&+8\mathcal{L}_{\hat{X}}\beta b_is_jt_0+8\mathcal{L}_{\hat{X}}\beta b_js_it_0-8\mathcal{L}_{\hat{X}}b_i\beta b_jt_0\\
&+18\mathcal{L}_{\hat{X}}b_i\beta s_0s_j+12\mathcal{L}_{\hat{X}}b_i\beta s_jt_0-8\mathcal{L}_{\hat{X}}b_j\beta b_it_0\\
&+18\mathcal{L}_{\hat{X}}b_j\beta s_0s_i+12\mathcal{L}_{\hat{X}}b_j\beta s_it_0+4\mathcal{L}_{\hat{X}}a_{ij}s_0^2\\
&-12\mathcal{L}_{\hat{X}}q_{ij}\beta^2-12\mathcal{L}_{\hat{X}}q_{ji}\beta^2+4a_{ij}\mathcal{L}_{\hat{X}}q_{00}-2e_{00;0}\mathcal{L}_{\hat{X}}e_{ij}\\
&-2e_{i0;0}\mathcal{L}_{\hat{X}}b_j-2e_{ij}\mathcal{L}_{\hat{X}}e_{00;0}-2e_{j0;0}\mathcal{L}_{\hat{X}}b_i\\
&+4\mathcal{L}_{\hat{X}}a_{ij}q_{00}-2\mathcal{L}_{\hat{X}}e_{i0;0}b_j-10e_{ij;0}\mathcal{L}_{\hat{X}}\beta+5t_{00}q_{ij}+5t_{00}q_{ji}\\
&+8\mathcal{L}_{\hat{X}}e_{ij;0}\beta-2\mathcal{L}_{\hat{X}}e_{j0;0}b_i+4\mathcal{L}_{\hat{X}}q_{0i}y_j\\
&+4\mathcal{L}_{\hat{X}}q_{0j}y_i+4\mathcal{L}_{\hat{X}}y_iq_{0j}+4\mathcal{L}_{\hat{X}}y_jq_{0i}\\
&+24\mathcal{L}_{\hat{X}}\beta\beta b_it_j+24\mathcal{L}_{\hat{X}}\beta\beta b_jt_i+60\mathcal{L}_{\hat{X}}\beta\beta s_is_j\\
&-72\mathcal{L}_{\hat{X}}\beta b_ib_jt_0-44\mathcal{L}_{\hat{X}}\beta b_is_0s_j-44\mathcal{L}_{\hat{X}}\beta b_js_0s_i\\
&-4\mathcal{L}_{\hat{X}}e_{i0}e_{j0}-4\mathcal{L}_{\hat{X}}e_{j0}e_{i0}
\end{aligned}
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\begin{aligned}
A_9 \;=\;& 4\mathcal{L}_{\hat{X}}t_0b_js_i-8\mathcal{L}_{\hat{X}}t_i\beta b_j-6\mathcal{L}_{\hat{X}}t_j\beta b_i+2\mathcal{L}_{\hat{X}}t_j\beta b_j\\
&-6\mathcal{L}_{\hat{X}}b_i\beta t_j+12\mathcal{L}_{\hat{X}}\beta b_it_j+12\mathcal{L}_{\hat{X}}\beta b_jt_i
\end{aligned}
$$

$$
\begin{aligned}
&+20\mathcal{L}_{\hat{X}}\beta s_i s_j + 6\mathcal{L}_{\hat{X}} b_i s_0 s_j + 4\mathcal{L}_{\hat{X}} b_i s_j t_0 - 6\mathcal{L}_{\hat{X}} b_j \beta t_i \\
&+6\mathcal{L}_{\hat{X}} b_j s_0 s_i + 4\mathcal{L}_{\hat{X}} b_j s_i t_0 + 2\mathcal{L}_{\hat{X}} s_0 b_i s_j \\
&+2\mathcal{L}_{\hat{X}} s_0 b_j s_i - 16\mathcal{L}_{\hat{X}} s_i \beta s_j + 6\mathcal{L}_{\hat{X}} s_i b_j s_0 + 4\mathcal{L}_{\hat{X}} s_i b_j t_0 \\
&-16\mathcal{L}_{\hat{X}} s_j \beta s_i + 6\mathcal{L}_{\hat{X}} s_j b_i s_0 + 4\mathcal{L}_{\hat{X}} s_j b_i t_0 \\
&+4\mathcal{L}_{\hat{X}} t_0 b_i s_j + 2\mathcal{L}_{\hat{X}} e_{j0} s_i + 2e_{j0}\mathcal{L}_{\hat{X}} s_i + 4\mathcal{L}_{\hat{X}} e_{ij} s_0 \\
&+4e_{ij}\mathcal{L}_{\hat{X}} s_0 + 2\mathcal{L}_{\hat{X}} e_{i0} s_j + 2e_{i0}\mathcal{L}_{\hat{X}} s_j - 4e_{ij}\eta \\
&-2e_{j0}\eta_i + 4\mathcal{L}_{\hat{X}} b_i q_{0j} + 2\mathcal{L}_{\hat{X}} b_i q_{j0} + 4\mathcal{L}_{\hat{X}} b_j q_{0i} \\
&+2\mathcal{L}_{\hat{X}} b_j q_{i0} + 4\mathcal{L}_{\hat{X}} q_{0i} b_j + 4\mathcal{L}_{\hat{X}} q_{0j} b_i + 2\mathcal{L}_{\hat{X}} q_{i0} b_j \\
&-8\mathcal{L}_{\hat{X}} q_{ij}\beta + 2\mathcal{L}_{\hat{X}} q_{j0} b_i - 8\mathcal{L}_{\hat{X}} qji\beta + 10\mathcal{L}_{\hat{X}}\beta q_{ij} \\
&+10\mathcal{L}_{\hat{X}}\beta q_{ji} - 2e_{i0}\eta_j + 2\mathcal{L}_{\hat{X}} e_{ij;0}
\end{aligned}
$$

$$
\begin{aligned}
A_{10} &= -2(\mathcal{L}_{\hat{X}} b_i t_j + \mathcal{L}_{\hat{X}} b_j t_i + 2\mathcal{L}_{\hat{X}} s_i s_j + 2\mathcal{L}_{\hat{X}} s_j s_i \\
&\quad +\mathcal{L}_{\hat{X}} t_i b_j + \mathcal{L}_{\hat{X}} t_j b_i + \mathcal{L}_{\hat{X}} q_{ij} + \mathcal{L}_{\hat{X}} q_{ji}).
\end{aligned}
$$

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