FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 3 (2020), 775-788
https://doi.org/10.22190/FUMI2003775O

TAUBERIAN THEOREMS FOR THE WEIGHTED MEAN METHOD OF SUMMABILITY OF INTEGRALS

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Abstract. Let q be a positive weight function on $\mathbf{R}_+ := [0, \infty)$ which is integrable in Lebesgue's sense over every finite interval $(0, x)$ for $0 < x < \infty$, in symbol: $q \in L^1_{loc}(\mathbf{R}_+)$ such that $Q(x) := \int_0^x q(t)dt \neq 0$ for each $x > 0$, $Q(0) = 0$ and $Q(x) \to \infty$ as $x \to \infty$. Given a real or complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$, we define $s(x) := \int_0^x f(t)dt$ and

$$\tau_q^{(0)}(x) := s(x), \tau_q^{(m)}(x) := \frac{1}{Q(x)} \int_0^x \tau_q^{(m-1)}(t)q(t)dt \;\; (x > 0, m = 1, 2, ...),$$

provided that $Q(x) > 0$.

We say that $\int_0^\infty f(x)dx$ is summable to L by the m-th iteration of weighted mean method determined by the function $q(x)$, or for short, $(\overline{N}, q, m)$ integrable to a finite number L if

$$\lim_{x\to\infty} \tau_q^{(m)}(x) = L.$$

In this case, we write $s(x) \to L(\overline{N}, q, m)$.

It is known that if the limit $\lim_{x\to\infty} s(x) = L$ exists, then $\lim_{x\to\infty} \tau_q^{(m)}(x) = L$ also exists. However, the converse of this implication is not always true. Some suitable conditions together with the existence of the limit $\lim_{x\to\infty} \tau_q^{(m)}(x)$, which is so called Tauberian conditions, may imply convergence of $\lim_{x\to\infty} s(x)$.

In this paper, one- and two-sided Tauberian conditions in terms of the generating function and its generalizations for $(\overline{N}, q, m)$ summable integrals of real- or complex-valued functions have been obtained. Some classical type Tauberian theorems given for Cesàro summability $(C, 1)$ and weighted mean method of summability $(\overline{N}, q)$ have been extended and generalized.
Keywords: Tauberian conditions; weight function; summable integrals; finite interval.

1. Introduction

Let q be a positive weight function on $\mathbf{R}_+ := [0, \infty)$ which is integrable in Lebesgue's sense over every finite interval $(0, x)$ for $0 < x < \infty$, in symbol: $q \in$

Received September 17, 2019; accepted November 20, 2019.
2010 *Mathematics Subject Classification.* Primary 40E05; Secondary 40A10 , 40G05

$L^1_{loc}(\mathbf{R}_+)$ such that $Q(x) := \int_0^x q(t)dt \neq 0$ for each $x > 0$, $Q(0) = 0$ and $Q(x) \to \infty$ as $x \to \infty$. Given a real or complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$, we define $s(x) := \int_0^x f(t)dt$ and

$$\tau_q^{(0)}(x) := s(x), \tau_q^{(m)}(x) := \frac{1}{Q(x)} \int_0^x \tau_q^{(m-1)}(t)q(t)dt \ \ (x > 0, m = 1, 2, ...),$$

provided that $Q(x) > 0$.

For each integer $m \geq 0$, we define $v_q^{(m)}(x)$ by

$$v_q^{(m)}(x) = \begin{cases} \frac{Q(x)}{q(x)} f(x) & , m = 0 \\ \frac{1}{Q(x)} \int_0^x f(t)Q(t)dt & , m = 1 \\ \frac{1}{Q(x)} \int_0^x v_q^{(m-1)}(t)q(t)dt & , m \geq 2. \end{cases}$$

The identity

$$\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) = v_q^{(m)}(x) \tag{1.1}$$

is known as the weighted Kronecker identity for the weighted mean method of summability.

It is clear from (1.1) that

$$\frac{Q(x)}{q(x)} \frac{d}{dx} \tau_q^{(m)}(x) = v_q^{(m)}(x)$$

for each integer $m \geq 0$ (see [14]). Here, we call $v_q^{(m)}(x)$ the generator of $\tau_q^{(m-1)}(x)$ for each integer $m \geq 1$.

We say that $\int_0^\infty f(x)dx$ is summable to L by the m-th iteration of weighted mean method determined by the function $q(x)$, or for short, $(\overline{N}, q, m)$ summable to a finite number L if

$$\lim_{x \to \infty} \tau_q^{(m)}(x) = L. \tag{1.2}$$

It is obvious that $(\overline{N}, q, m)$ summability reduces to the ordinary convergence for $m = 0$ and $(\overline{N}, q, 1)$ is the $(\overline{N}, q)$ method. If $q(x) = 1$ on $\mathbf{R}_+$, then $(\overline{N}, q, m)$ method is the Hölder method of order m and $(\overline{N}, q, 1)$ method is the Cesàro summability method $(C, 1)$.

It is well known that condition $Q(x) \to \infty$ as $x \to \infty$ is a necessary and sufficient condition that the existence of the integral

$$\int_0^\infty s(x)dx = L \tag{1.3}$$

implies (1.2). That is, the $(\overline{N}, q, m)$ summability method is regular, where m is a nonnegative integer. However, the converse of this implication is not always true. Notice that some suitable condition on $s(x)$ together with (1.2) may imply (1.3).

Such a condition is called a Tauberian condition and resulting theorem is said to be a Tauberian theorem.

Móricz [8] and Fekete and Móricz [6] obtained one-sided and two-sided Tauberian conditions for the weighted mean method $(\overline{N}, q)$ of integrals. Following these works, Totur and Okur [13] proved one-sided boundedness of $v_q^{(0)}(x)$ is a Tauberian condition for the weighted mean method of summability $(\overline{N}, q)$ of integrals. From the fact that condition $v_q^{(0)}(x) \geq -C$ implies slow decreasing of $s(x)$, Totur and Okur [13] generalized their first result and proved that slow decrease of $s(x)$ is also a Tauberian condition for $(\overline{N}, q)$ method. For a detailed study and some interesting results related to Tauberian theorems for the weighted mean method of summability, we refer the reader to Borwein and Kratz [1], Çanak and Totur [2], Çanak and Totur [3], Çanak and Totur [4], Özsaraç and Çanak [9], Sezer and Çanak [10], Tietz and Zeller [11] and Totur and Çanak [12], etc.

In this paper, one- and two-sided Tauberian conditions in terms of the generating function and its generalizations for summable integrals by m-th iteration of weighted means of real- or complex-valued functions have been obtained, respectively. Some classical type Tauberian theorems given for Cesàro summability $(C, 1)$ and weighted mean method of summability $(\overline{N}, q)$ have been extended and generalized.

2. Main results

For the main results of the paper, we need the following definitions and notations.

Definition 2.1. ([7]) A positive function Q is called regularly varying of index $\alpha > 0$ if

$$\lim_{x\to\infty} \frac{Q(\rho x)}{Q(x)} = \rho^{\alpha}, \quad \rho > 0. \tag{2.1}$$

It easily follows from Definition 2.1 that for all $\rho > 1$ and sufficiently large x,

$$\frac{\rho^{\alpha}}{2\left(\rho^{\alpha}-1\right)} \leq \frac{Q\left(\rho x\right)}{Q\left(\rho x\right)-Q\left(x\right)} \leq \frac{3\rho^{\alpha}}{2\left(\rho^{\alpha}-1\right)} \tag{2.2}$$

and for all $0 < \rho < 1$ and sufficiently large x,

$$\frac{\rho^{\alpha}}{2\left(1-\rho^{\alpha}\right)} \leq \frac{Q\left(\rho x\right)}{Q\left(x\right)-Q\left(\rho x\right)} \leq \frac{3\rho^{\alpha}}{2\left(1-\rho^{\alpha}\right)}. \tag{2.3}$$

We note that if (2.1) holds, then the following equivalent conditions are clearly satisfied (see [5]):

$$\liminf_{x\to\infty} \frac{Q(x)}{Q(\rho x)} < 1, \text{ for every } \rho > 1 \tag{2.4}$$

and

$$\liminf_{x\to\infty} \frac{Q(\rho x)}{Q(x)} < 1, \text{ for every } 0 < \rho < 1. \tag{2.5}$$

First, we consider real-valued functions and prove the following Tauberian theorems.

Theorem 2.1. *Let (2.1) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, m)$ summable to L and $v_q^{(m-1)}(x)$ is one-sided bounded, then $s(x)$ is $(\overline{N}, q, m-1)$ summable to L.*

Corollary 2.1. *([13]) Let (2.1) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, 1)$ summable to L and $v_q^{(0)}(x)$ is one-sided bounded, then $s(x)$ converges to L.*

Theorem 2.2. *Let (2.4) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, m)$ summable to L and $\tau_q^{(m-1)}(x)$ is slowly decreasing, then $s(x)$ is $(\overline{N}, q, m-1)$ summable to L.*

Corollary 2.2. *([13]) Let (2.4) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, 1)$ summable to L and slowly decreasing, then $s(x)$ converges to L.*

A real-valued function $s(x)$ defined on $\mathbf{R}_+$ is said to be slowly decreasing if

$$\lim_{\rho \to 1^+} \liminf_{x \to \infty} \min_{x \le t \le \rho x} (s(t) - s(x)) \ge 0. \tag{2.6}$$

Note that condition (2.6) can be equivalently reformulated as follows:

$$\lim_{\rho \to 1^-} \liminf_{x \to \infty} \min_{\rho x \le t \le x} (s(x) - s(t)) \ge 0. \tag{2.7}$$

Second, we consider complex-valued functions and prove the following Tauberian theorems.

Theorem 2.3. *Let (2.1) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, m)$ summable to L and $v_q^{(m-1)}(x)$ is bounded, then $s(x)$ is $(\overline{N}, q, m-1)$ summable to L.*

Corollary 2.3. *Let (2.1) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, 1)$ summable to L and $v_q^{(0)}(x)$ is bounded, then $s(x)$ converges to L.*

Theorem 2.4. *Let (2.4) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, m)$ summable to L and $\tau_q^{(m-1)}(x)$ is slowly oscillating, then $s(x)$ is $(\overline{N}, q, m-1)$ summable to L.*

Corollary 2.4. *Let (2.4) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, 1)$ summable to L and slowly oscillating, then $s(x)$ converges to L.*

A complex-valued function $s(x)$ defined on $\mathbf{R}_+$ is said to be slowly oscillating if

$$\lim_{\rho\to 1^+}\limsup_{x\to\infty}\max_{x\le t\le \rho x}\left|s\left(t\right)-s\left(x\right)\right|=0. \tag{2.8}$$

Note that condition (2.8) can be equivalently reformulated as follows:

$$\lim_{\rho\to 1^-}\limsup_{x\to\infty}\max_{\rho x\le t\le x}\left|s\left(x\right)-s\left(t\right)\right|=0. \tag{2.9}$$

3. An auxiliary result

The following two representations of $s\left(x\right)-\tau_q^{(1)}\left(x\right)$ will be needed in the proofs of our main results.

Lemma 3.1. *([13])*

(i) For $\rho>1$ and sufficiently large x,

$$\begin{aligned} s\left(x\right)-\tau_q^{(1)}\left(x\right) &= \frac{Q\left(\rho x\right)}{Q\left(\rho x\right)-Q\left(x\right)}\left(\tau_q^{(1)}\left(\rho x\right)-\tau_q^{(1)}\left(x\right)\right)\\ &- \frac{1}{Q\left(\rho x\right)-Q\left(x\right)}\int_x^{\rho x}\left(s\left(t\right)-s\left(x\right)\right)q\left(t\right)dt. \end{aligned}$$

(ii) For $0<\rho<1$ and sufficiently large x,

$$\begin{aligned} s\left(x\right)-\tau_q^{(1)}\left(x\right) &= \frac{Q\left(\rho x\right)}{Q\left(x\right)-Q\left(\rho x\right)}\left(\tau_q^{(1)}\left(x\right)-\tau_q^{(1)}\left(\rho x\right)\right)\\ &+ \frac{1}{Q\left(x\right)-Q\left(\rho x\right)}\int_{\rho x}^{x}\left(s\left(x\right)-s\left(t\right)\right)q\left(t\right)dt. \end{aligned}$$

4. Proofs of main results

Proof of Theorem 2.1 Suppose that $s\left(x\right)$ is $\left(\overline{N},q,m\right)$ summable to L and $v_q^{(m-1)}\left(x\right)$ is one-sided bounded. By Lemma 3.1 (i), we have

$$\begin{aligned} \tau_q^{(m-1)}\left(x\right)-\tau_q^{(m)}\left(x\right) &= \frac{Q\left(\rho x\right)}{Q\left(\rho x\right)-Q\left(x\right)}\left(\tau_q^{(m)}\left(\rho x\right)-\tau_q^{(m)}\left(x\right)\right)\\ &- \frac{1}{Q\left(\rho x\right)-Q\left(x\right)}\int_x^{\rho x}\left(\tau_q^{(m-1)}\left(t\right)-\tau_q^{(m-1)}\left(x\right)\right)q\left(t\right)dt\\ &= \frac{Q\left(\rho x\right)}{Q\left(\rho x\right)-Q\left(x\right)}\left(\tau_q^{(m)}\left(\rho x\right)-\tau_q^{(m)}\left(x\right)\right)\\ &- \frac{1}{Q\left(\rho x\right)-Q\left(x\right)}\int_x^{\rho x}\left(\int_x^{t}\frac{d}{dz}\tau_q^{(m-1)}\left(z\right)dz\right)q\left(t\right)dt. \end{aligned}$$

Since $v_q^{(m-1)}(x)$ is one-sided bounded, we get

$$\begin{aligned}\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x) &\le \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right)\\ &+ \frac{C}{Q(\rho x)-Q(x)}\int_x^{\rho x}\left(\int_x^t \frac{q(z)}{Q(z)}dz\right)q(t)\,dt\\ &= \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right)\\ &+ \frac{C}{Q(\rho x)-Q(x)}\int_x^{\rho x} q(t)\log\frac{Q(t)}{Q(x)}dt\\ &= \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right)+C\log\frac{Q(\rho x)}{Q(x)}. \qquad (4.1)\end{aligned}$$

By (2.2) and $\left(\overline{N},q,m\right)$ summability of $s(x)$, we have

$$\lim_{x\to\infty}\frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right)=0. \qquad (4.2)$$

Taking (4.2) into account in (4.1), we obtain

$$\limsup_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right)\le\limsup_{x\to\infty}\left(C\log\frac{Q(\rho x)}{Q(x)}\right)=C\log\rho^{\alpha}.$$

Letting $\rho\to 1^+$ in the last inequality, we have

$$\limsup_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right)\le 0. \qquad (4.3)$$

Similarly, from Lemma 3.1 (ii), we have

$$\begin{aligned}\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x) &= \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\ &+ \frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right)q(t)\,dt\\ &= \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\ &+ \frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}\left(\int_t^x \frac{d}{dz}\tau_q^{(m-1)}(z)\,dz\right)q(t)\,dt.\end{aligned}$$

Since $v_q^{(m-1)}(x)$ is one-sided bounded, we get

$$\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x) \ge \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)$$

$$
\begin{aligned}
&- \frac{C}{Q(x)-Q(\rho x)} \int_{\rho x}^{x} \left(\int_{t}^{x} \frac{q(z)}{Q(z)} dz \right) q(t)\, dt \\
&= \frac{Q(\rho x)}{Q(x)-Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\
&- \frac{C}{Q(x)-Q(\rho x)} \int_{\rho x}^{x} q(t) \log \frac{Q(x)}{Q(t)} dt \\
&= \frac{Q(\rho x)}{Q(x)-Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) - C \log \frac{Q(x)}{Q(\rho x)}. \qquad (4.4)
\end{aligned}
$$

By (2.3) and $\left(\overline{N}, q, m\right)$ summability of $s(x)$, we obtain

$$
\lim_{x\to\infty} \frac{Q(\rho x)}{Q(x)-Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) = 0. \qquad (4.5)
$$

Taking (4.5) into account in (4.4), we obtain

$$
\limsup_{x\to\infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \geq - \liminf_{x\to\infty} \left(C \log \frac{Q(x)}{Q(\rho x)} \right) = -C \log \rho^{\alpha}.
$$

Letting $\rho \to 1^-$ in the last inequality, we have

$$
\limsup_{x\to\infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \geq 0. \qquad (4.6)
$$

Combining (4.3) and (4.6), we obtain $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L. $\square$

Proof of Theorem 2.2 Suppose that $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L and $\tau_q^{(m-1)}(x)$ is slowly decreasing. By Lemma 3.1 (i), we have

$$
\begin{aligned}
\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) &= \frac{Q(\rho x)}{Q(\rho x)-Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
&- \frac{1}{Q(\rho x)-Q(x)} \int_{x}^{\rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right) q(t)\, dt \\
&\leq \frac{Q(\rho x)}{Q(\rho x)-Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
&- \frac{1}{Q(\rho x)-Q(x)} \int_{x}^{\rho x} q(t) \min_{x\leq t\leq \rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right) dt \\
&= \frac{Q(\rho x)}{Q(\rho x)-Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
&- \min_{x\leq t\leq \rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right). \qquad (4.7)
\end{aligned}
$$

Taking the lim sup of both sides of (4.7), we get

$$\limsup_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right) \le \limsup_{x\to\infty}\frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right) - \liminf_{x\to\infty}\min_{x\le t\le\rho x}\left(\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right). \tag{4.8}$$

By (2.4), we have

$$0<\limsup_{x\to\infty}\frac{Q(\rho x)}{Q(\rho x)-Q(x)}=1+\left(\liminf_{x\to\infty}\frac{Q(\rho x)}{Q(x)}-1\right)^{-1}<\infty.$$

Since $s(x)$ is $(\overline{N},q,m)$ summable to L, the first term on the right-hand side vanishes in (4.8). From this, we obtain

$$\limsup_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right)\le-\liminf_{x\to\infty}\min_{x\le t\le\rho x}\left(\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right).$$

Taking the limit of (4.8) as $\rho\to 1^+$, we have

$$\limsup_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right)\le 0. \tag{4.9}$$

Similarly, by Lemma 3.1 (ii), we have

$$\begin{aligned}
\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x) &= \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\
&+ \frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right)q(t)\,dt\\
&\ge \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\
&+ \frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}q(t)\min_{\rho x\le t\le x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right)dt\\
&= \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\
&+ \min_{\rho x\le t\le x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right).
\end{aligned} \tag{4.10}$$

From (4.10), we get

$$\liminf_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right) \ge \liminf_{x\to\infty}\frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right) + \liminf_{x\to\infty}\min_{\rho x\le t\le x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right). \tag{4.11}$$

By (2.4), we have

$$0 < \liminf_{x\to\infty} \frac{Q(\rho x)}{Q(x) - Q(\rho x)} = \left(\limsup_{x\to\infty} \frac{Q(x)}{Q(\rho x)} - 1\right)^{-1} < \infty.$$

Since $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L, the first term on the right-hand side vanishes in (4.11). From this, we obtain

$$\liminf_{x\to\infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right) \geq \liminf_{x\to\infty} \min_{\rho x \leq t \leq x} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right).$$

Taking the limit of (4.11) as $\rho \to 1^-$, we have

$$\liminf_{x\to\infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right) \geq 0. \tag{4.12}$$

Combining (4.9) and (4.12), we obtain $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L. □

Proof of Theorem 2.3 Suppose that $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L and $v_q^{(m-1)}(x)$ is bounded. By Lemma 3.1 (i), we have

$$\begin{aligned}
\left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| &\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left|\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x)\right| \\
&+ \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left|\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x)\right| q(t)\, dt \\
&= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left|\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x)\right| \\
&+ \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left|\int_x^t \frac{d}{dz}\tau_q^{(m-1)}(z)\, dz\right| q(t)\, dt.
\end{aligned}$$

Since $v_q^{(m-1)}(x)$ is bounded, we get

$$\begin{aligned}
\left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| &\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left|\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x)\right| \\
&+ \frac{C}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left|\int_x^t \frac{q(z)}{Q(z)} dz\right| q(t)\, dt \\
&= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left|\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x)\right| \\
&+ \frac{C}{Q(\rho x) - Q(x)} \int_x^{\rho x} q(t) \log \frac{Q(t)}{Q(x)} dt \\
&\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left|\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x)\right| + C \log \frac{Q(\rho x)}{Q(x)}. \qquad (4.13)
\end{aligned}$$

By (2.2) and $\left(\overline{N},q,m\right)$ summability of $s(x)$, we have

$$\lim_{x\to\infty}\frac{Q\left(\rho x\right)}{Q\left(\rho x\right)-Q\left(x\right)}\left|\tau_q^{(m)}\left(\rho x\right)-\tau_q^{(m)}\left(x\right)\right|=0.$$

Taking the lim sup of both sides of (4.13) gives

$$\limsup_{x\to\infty}\left|\tau_q^{(m-1)}\left(x\right)-\tau_q^{(m)}\left(x\right)\right|\leq\limsup_{x\to\infty}\left(C\log\frac{P\left(\rho x\right)}{Q\left(x\right)}\right)=C\log\rho^{\alpha}.$$

Letting $\rho\to 1^{+}$ in last inequality, we have

$$\limsup_{x\to\infty}\left|\tau_q^{(m-1)}\left(x\right)-\tau_q^{(m)}\left(x\right)\right|\leq 0. \tag{4.14}$$

Similarly, from Lemma 3.1 (ii), we have

$$\begin{aligned}\left|\tau_q^{(m-1)}\left(x\right)-\tau_q^{(m)}\left(x\right)\right| &\leq \frac{P\left(\rho x\right)}{Q\left(x\right)-Q\left(\rho x\right)}\left|\tau_q^{(m)}\left(x\right)-\tau_q^{(m)}\left(\rho x\right)\right|\\ &+\frac{1}{Q\left(x\right)-Q\left(\rho x\right)}\int_{\rho x}^{x}\left|\tau_q^{(m-1)}\left(x\right)-\tau_q^{(m-1)}\left(t\right)\right|q\left(t\right)dt\\ &=\frac{Q\left(\rho x\right)}{Q\left(x\right)-Q\left(\rho x\right)}\left|\tau_q^{(m)}\left(x\right)-\tau_q^{(m)}\left(\rho x\right)\right|\\ &+\frac{1}{Q\left(x\right)-Q\left(\rho x\right)}\int_{\rho x}^{x}\left|\int_{t}^{x}\frac{d}{dz}\tau_q^{(m-1)}\left(z\right)dz\right|q\left(t\right)dt.\end{aligned}$$

Since $v_q^{(m-1)}\left(x\right)$ is bounded, we get

$$\begin{aligned}\left|\tau_q^{(m-1)}\left(x\right)-\tau_q^{(m)}\left(x\right)\right| &\leq \frac{P\left(\rho x\right)}{Q\left(x\right)-Q\left(\rho x\right)}\left|\tau_q^{(m)}\left(x\right)-\tau_q^{(m)}\left(\rho x\right)\right|\\ &+\frac{C}{Q\left(x\right)-Q\left(\rho x\right)}\int_{\rho x}^{x}\left|\int_{t}^{x}\frac{p\left(z\right)}{P\left(z\right)}dz\right|q\left(t\right)dt\\ &=\frac{Q\left(\rho x\right)}{Q\left(x\right)-Q\left(\rho x\right)}\left|\tau_q^{(m)}\left(x\right)-\tau_q^{(m)}\left(\rho x\right)\right|\\ &+\frac{C}{Q\left(x\right)-Q\left(\rho x\right)}\int_{\rho x}^{x}q\left(t\right)\log\frac{Q\left(x\right)}{Q\left(t\right)}dt\\ &\leq\frac{Q\left(\rho x\right)}{Q\left(x\right)-Q\left(\rho x\right)}\left|\tau_q^{(m)}\left(x\right)-\tau_q^{(m)}\left(\rho x\right)\right|+C\log\frac{Q\left(x\right)}{Q\left(\rho x\right)}. \qquad (4.15)\end{aligned}$$

By (2.3) and $\left(\overline{N},p,m\right)$ summability of $s(x)$, we have

$$\lim_{x\to\infty}\frac{Q\left(\rho x\right)}{P\left(x\right)-Q\left(\rho x\right)}\left|\tau_q^{(m)}\left(x\right)-\tau_q^{(m)}\left(\rho x\right)\right|=0.$$

From (4.15), we get

$$\limsup_{x\to\infty}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right|\le\limsup_{x\to\infty}\left(C\log\frac{Q(x)}{Q(\rho x)}\right)=C\log\rho^{\alpha}.$$

Letting $\rho\to 1^-$ in last inequality, we have

$$\limsup_{x\to\infty}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right|\le 0. \tag{4.16}$$

From either (4.14) or (4.16), we conclude $s(x)$ is $\left(\overline{N},q,m-1\right)$ summable to L. □

Proof of Theorem 2.4 Suppose that $s(x)$ is $\left(\overline{N},q,m\right)$ summable to L and $\tau_q^{(m-1)}(x)$ is slowly oscillating. By Lemma 3.1 (i), we have

$$\begin{aligned}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right| &= \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+\frac{1}{Q(\rho x)-Q(x)}\int_x^{\rho x}\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|q(t)\,dt\\ &\le \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+\frac{1}{Q(\rho x)-Q(x)}\int_x^{\rho x}q(t)\max_{x\le t\le\rho x}\left(\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|\right)dt\\ &\le \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+\max_{x\le t\le\rho x}\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|. \end{aligned}\tag{4.17}$$

From (4.17), we get

$$\begin{aligned}\limsup_{x\to\infty}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right| &\le\limsup_{x\to\infty}\tfrac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+\limsup_{x\to\infty}\max\nolimits_{x\le t\le\rho x}\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|.\end{aligned}\tag{4.18}$$

By (2.4), we have

$$0<\limsup_{x\to\infty}\frac{Q(\rho x)}{Q(\rho x)-Q(x)}=1+\left(\liminf_{x\to\infty}\frac{Q(\rho x)}{Q(x)}-1\right)^{-1}<\infty.$$

Since $s(x)$ is $\left(\overline{N},q,m\right)$ summable to L, the first term on the right side vanishes in (4.18). From this, we obtain

$$\limsup_{x\to\infty}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right|\le\limsup_{x\to\infty}\max_{x\le t\le\rho x}\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|.$$

Taking the limit of (4.18) as $\rho \to 1^+$, we have

$$\limsup_{x\to\infty} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| \leq 0. \tag{4.19}$$

Similarly, by Lemma 3.1 (ii), we have

$$\begin{aligned}
\left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| &= \frac{Q(\rho x)}{Q(x)-Q(\rho x)} \left|\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x)\right| \\
&+ \frac{1}{Q(x)-Q(\rho x)} \int_{\rho x}^{x} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right| q(t)\, dt \\
&\leq \frac{Q(\rho x)}{Q(x)-Q(\rho x)} \left|\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x)\right| \\
&+ \frac{1}{Q(x)-Q(\rho x)} \int_{\rho x}^{x} q(t) \max_{\rho x\leq t\leq x} \left(\left|\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right|\right) dt \\
&\leq \frac{Q(\rho x)}{Q(x)-Q(\rho x)} \left|\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x)\right| \\
&+ \max_{\rho x\leq t\leq x} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right|. \qquad (4.20)
\end{aligned}$$

From (4.20), we get

$$\begin{aligned}
\limsup_{x\to\infty} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| &\leq \limsup_{x\to\infty} \tfrac{Q(\rho x)}{Q(x)-Q(\rho x)} \left|\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x)\right| \\
&+ \limsup_{x\to\infty} \max_{\rho x\leq t\leq x} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right|. \qquad (4.21)
\end{aligned}$$

By (2.4), we have

$$0 < \liminf_{x\to\infty} \frac{Q(\rho x)}{Q(x)-Q(\rho x)} = \left(\limsup_{x\to\infty} \frac{Q(x)}{Q(\rho x)} - 1\right)^{-1} < \infty.$$

Since $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L, the first term on the right-hand side vanishes in (4.21). From this, we obtain

$$\limsup_{x\to\infty} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| \leq \limsup_{x\to\infty} \max_{\rho x\leq t\leq x} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right|.$$

Taking the limit of (4.21) as $\rho \to 1^-$, we have

$$\limsup_{x\to\infty} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| \leq 0. \tag{4.22}$$

From either (4.19) or (4.22), we conclude $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L. $\square$

5. Conclusion

In this paper, we introduce Tauberian conditions in terms of the generator and its generalizations for summable integrals by m-th iteration of weighted means of real- or complex-valued functions, respectively. Tauberian conditions for summable double integrals by m-th iteration of weighted means of real- or complex-valued functions will be illustrated in a forthcoming work.

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