FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 2 (2020), 815-823
https://doi.org/10.22190/FUMI2003815D

REFINEMENTS AND REVERSES OF HöLDER-MCCARTHY OPERATOR INEQUALITY VIA A CARTWRIGHT-FIELD RESULT

Silvestru Sever Dragomir

Abstract. By the use of a classical result of Cartwright and Field, in this paper we have obtained new refinements and reverses of Hölder-McCarthy operator inequality in the case of $p \in (0,1)$. A comparison for the two upper bounds obtained showing that neither of them is better in general, has also been performed.

Keywords: Hölder-McCarthy operator inequality; selfadjoint operator; Hilbert space; nonnegative operator.

1. Introduction

Let A be a nonnegative operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$, namely $\langle Ax, x\rangle \geq 0$ for any $x \in H$. We write this as $A \geq 0$.

By the use of the spectral resolution of A and the Hölder inequality, C. A. McCarthy [16] proved that

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \; p \in (1, \infty) \tag{1.1}$$

and

$$\langle A^p x, x\rangle \leq \langle Ax, x\rangle^p, \; p \in (0, 1) \tag{1.2}$$

for any $x \in H$ with $\|x\| = 1$.

Let A be a selfadjoint operator on H with

$$mI \leq A \leq MI, \tag{1.3}$$

where I is the *identity operator* and m, M are real numbers with $m < M$.

Received September 30, 2019; accepted February 25, 2020
2010 *Mathematics Subject Classification.* Primary 47A63, 47A30; Secondary 15A60, 26D15; 26D10

In [7, Theorem 3] Fujii et al. obtained the following interesting ratio inequality that provides a reverse of the *Hölder-McCarthy inequality* (1.1) for an operator A that satisfies the condition (1.3) with $m > 0$

$$\langle A^p x, x\rangle \le \left\{ \frac{1}{p^{1/p} q^{1/q}} \frac{M^p - m^p}{(M-m)^{1/p} (mM^p - Mm^p)^{1/q}} \right\}^p \langle Ax, x\rangle^p , \tag{1.4}$$

for any $x \in H$ with $\|x\| = 1$, where $q = p/(p-1)$, $p > 1$.

If A satisfies the condition (1.3) with $m \ge 0$, then we also have the additive reverse of (1.1) that has been obtained by the author in 2008, see [4]

$$\begin{aligned} \langle A^p x, x\rangle - \langle Ax, x\rangle^p &\le \tfrac{1}{2} p (M-m) \left[\left\|A^{p-1}x\right\|^2 - \left\langle A^{p-1}x, x\right\rangle^2 \right]^{1/2} \\ &\le \tfrac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1}\right) \end{aligned}$$

and

$$\begin{aligned} \langle A^p x, x\rangle - \langle Ax, x\rangle^p &\le \tfrac{1}{2} p \left(M^{p-1} - m^{p-1}\right) \left[\|Ax\|^2 - \langle Ax, x\rangle^2 \right]^{1/2} \\ &\le \tfrac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1}\right) \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

We also have the alternative upper bounds [4]

$$\begin{aligned} \langle A^p x, x\rangle - \langle Ax, x\rangle^p &\le \tfrac{1}{4} p \tfrac{(M-m)\left(M^{p-1} - m^{p-1}\right)}{M^{p/2} m^{p/2}} \langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle , \text{ (for } m > 0), \\ &\le p \tfrac{1}{4} (M-m) \left(M^{p-1} - m^{p-1}\right) \left(\tfrac{M}{m}\right)^{p/2} , \text{ (for } m > 0) \end{aligned}$$

and

$$\begin{aligned} \langle A^p x, x\rangle - \langle Ax, x\rangle^p &\le p \left(\sqrt{M} - \sqrt{m}\right) \left(M^{(p-1)/2} - m^{(p-1)/2}\right) \left[\langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle \right]^{\frac{1}{2}} \\ &\le p \left(\sqrt{M} - \sqrt{m}\right) \left(M^{(p-1)/2} - m^{(p-1)/2}\right) M^{p/2} \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

For various related inequalities, see [6]-[10] and [14]-[15].

We have the following inequality that provides a refinement and a reverse for the celebrated *Young's scalar inequality*

$$\begin{aligned} \frac{1}{2} \nu (1-\nu) \frac{(b-a)^2}{\max\{a,b\}} &\le (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \\ &\le \frac{1}{2} \nu (1-\nu) \frac{(b-a)^2}{\min\{a,b\}} \end{aligned} \tag{1.5}$$

for any a, $b > 0$ and $\nu \in [0,1]$.

This result was obtained in 1978 by Cartwright and Field [1] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

For new recent reverses and refinements of Young's inequality see [2]-[3], [11]-[12], [13] and [19].

By the use of (1.5). we have obtained new refinements and reverses of Hölder-McCarthy operator inequality in the case when $p \in (0,1)$. A comparison for the two upper bounds obtained showing that neither of them is better in general, has also been performed.

2. Some Refinements and Reverse Results

We have:

Theorem 2.1. *Let m, M be real numbers so that $M > m > 0$. If A is a selfadjoint operator satisfying the condition (1.3) above, then for any $p \in (0,1)$ we have*

$$\begin{aligned} \frac{p(1-p)}{2}\frac{m}{M}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) &\le \frac{p(1-p)}{2M}\langle Ax,x\rangle\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) \\ &\le 1-\frac{\langle A^px,x\rangle}{\langle Ax,x\rangle^p} \\ &\le \frac{p(1-p)}{2m}\langle Ax,x\rangle\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) \\ &\le \frac{p(1-p)}{2}\frac{M}{m}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) \end{aligned} \tag{2.1}$$

for any $x \in H$ with $\|x\| = 1$.

In particular,

$$\begin{aligned} \frac{1}{8}\frac{m}{M}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) &\le \frac{\langle Ax,x\rangle}{8M}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) \\ \le 1-\frac{\langle A^{1/2}x,x\rangle}{\langle Ax,x\rangle^{1/2}} &\le \frac{\langle Ax,x\rangle}{8m}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) \\ &\le \frac{1}{8}\frac{M}{m}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right), \end{aligned} \tag{2.2}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. If $a,\ b \in [m, M]$, then by Cartwright-Field inequality (1.5) we have

$$\frac{1}{2M}p(1-p)(b-a)^2 \le (1-p)a+pb-a^{1-p}b^p \le \frac{1}{2m}p(1-p)(b-a)^2$$

or, equivalently

$$\begin{aligned} \frac{1}{2M}p(1-p)(b^2-2ab+a^2) &\le (1-p)a+pb-a^{1-p}b^p \\ &\le \frac{1}{2m}p(1-p)(b^2-2ab+a^2), \end{aligned} \tag{2.3}$$

for any $p \in (0,1)$.

Fix $a \in [m,M]$ and by using the operator functional calculus for A with $mI \le A \le MI$ we have

$$\begin{aligned} \frac{1}{2M}p(1-p)(A^2-2aA+a^2I) &\le (1-p)aI+pA-a^{1-p}A^p \\ &\le \frac{1}{2m}p(1-p)(A^2-2aA+a^2I). \end{aligned} \tag{2.4}$$

Then for any $x \in H$ with $\|x\|=1$ we have from (2.4) that

$$\begin{aligned} &\tfrac{1}{2M}p(1-p)(\langle A^2x,x\rangle - 2a\langle Ax,x\rangle + a^2) \\ &\le (1-p)a + p\langle Ax,x\rangle - a^{1-p}\langle A^px,x\rangle \\ &\le \tfrac{1}{2m}p(1-p)(\langle A^2x,x\rangle - 2a\langle Ax,x\rangle + a^2), \end{aligned} \tag{2.5}$$

for any $a \in [m,M]$.

If we choose in (2.5) $a = \langle Ax,x\rangle \in [m,M]$, then we get for any $x \in H$ with $\|x\|=1$ that

$$\begin{aligned} \frac{1}{2M}p(1-p)(\langle A^2x,x\rangle - \langle Ax,x\rangle^2) &\le \langle Ax,x\rangle - \langle Ax,x\rangle^{1-p}\langle A^px,x\rangle \\ &\le \tfrac{1}{2m}p(1-p)(\langle A^2x,x\rangle - \langle Ax,x\rangle^2), \end{aligned}$$

and by division with $\langle Ax,x\rangle > 0$ we obtain the second and third inequalities in (2.1).

The rest is obvious. ∎

Remark 2.1. It is well known that, if $mI \le A \le MI$ with $M > 0$, then, see for instance [17, p. 27], we have

$$(1 \le)\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2} \le \frac{(m+M)^2}{4mM}$$

for any $x \in H$ with $\|x\|=1$, which implies that

$$(0 \le)\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2} - 1 \le \frac{(M-m)^2}{4mM}.$$

Using (2.1) and by denoting $h = \frac{M}{m}$ we get

$$(0 \le) 1 - \frac{\langle A^px,x\rangle}{\langle Ax,x\rangle^p} \le \frac{p(1-p)}{8}(h-1)^2 \tag{2.6}$$

and, in particular,

$$(0 \le)\, 1 - \frac{\left\langle A^{1/2}x, x\right\rangle}{\langle Ax, x\rangle^{1/2}} \le \frac{1}{32}(h-1)^2, \tag{2.7}$$

for any $x \in H$ with $\|x\| = 1$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \; h > 0. \tag{2.8}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

Observe that for any $h > 0$

$$K(h) - 1 = \frac{(h-1)^2}{4h} = K\left(\frac{1}{h}\right) - 1.$$

From (2.6) we then have

$$(0 \le)\, 1 - \frac{\langle A^p x, x\rangle}{\langle Ax, x\rangle^p} \le \frac{p(1-p)}{2} h\,[K(h) - 1] \tag{2.9}$$

and, in particular,

$$(0 \le)\, 1 - \frac{\left\langle A^{1/2}x, x\right\rangle}{\langle Ax, x\rangle^{1/2}} \le \frac{1}{8} h\,[K(h) - 1], \tag{2.10}$$

for any $x \in H$ with $\|x\| = 1$.

Also, if a, $b > 0$ then

$$K\left(\frac{b}{a}\right) - 1 = \frac{(b-a)^2}{4ab}.$$

Since $\min\{a,b\} \max\{a,b\} = ab$ if $a, b > 0$, then

$$\frac{(b-a)^2}{\max\{a,b\}} = \frac{\min\{a,b\}(b-a)^2}{ab} = 4\min\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right]$$

and

$$\frac{(b-a)^2}{\min\{a,b\}} = \frac{\max\{a,b\}(b-a)^2}{ab} = 4\max\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right]$$

and the inequality (1.5) can be written as

$$2\nu(1-\nu)\min\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right] \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le 2\nu(1-\nu)\max\{a,b\}\left[K\left(\tfrac{b}{a}\right) - 1\right]$$

for any a, $b > 0$ and $\nu \in [0,1]$.

Theorem 2.2. *Let m, M be real numbers so that $M > m > 0$. If A is a selfadjoint operator satisfying the condition (1.3) above, then for any $p \in (0,1)$ we have*

$$
\begin{aligned}
(0 \leq)\, 1 - \tfrac{\langle A^p x,x\rangle}{\langle Ax,x\rangle^p} &\leq p(1-p)[K(h)-1]\left(2 + \tfrac{\langle |A-\langle Ax,x\rangle I|x,x\rangle}{\langle Ax,x\rangle}\right) \\
&\leq p(1-p)[K(h)-1]\left[2 + \left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2} - 1\right)^{1/2}\right] \\
&\leq p(1-p)[K(h)-1]\left[2 + (K(h)-1)^{1/2}\right]
\end{aligned} \tag{2.11}
$$

for any $x \in H$ with $\|x\| = 1$.

In particular, we have

$$
\begin{aligned}
(0 \leq)\, 1 - \tfrac{\langle A^{1/2} x,x\rangle}{\langle Ax,x\rangle^{1/2}} &\leq \tfrac{1}{4}[K(h)-1]\left(2 + \tfrac{\langle |A-\langle Ax,x\rangle I|x,x\rangle}{\langle Ax,x\rangle}\right) \\
&\leq \tfrac{1}{4}[K(h)-1]\left[2 + \left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2} - 1\right)^{1/2}\right] \\
&\leq \tfrac{1}{4}[K(h)-1]\left[2 + (K(h)-1)^{1/2}\right]
\end{aligned} \tag{2.12}
$$

for any $x \in H$ with $\|x\| = 1$.

Proof. From (2.11) we have for any a, $b > 0$ and $p \in [0,1]$ that

$$
(1-p)a + pb - a^{1-p}b^p \leq p(1-p)(a+b+|b-a|)\left[K\left(\frac{b}{a}\right) - 1\right] \tag{2.13}
$$

since

$$
\max\{a,b\} = \frac{1}{2}(a+b+|b-a|).
$$

If a, $b \in [m,M]$, then $\frac{b}{a} \in \left[\frac{m}{M}, \frac{M}{m}\right]$ and by the properties of Kantorovich's constant K, we have

$$
1 \leq K\left(\frac{b}{a}\right) \leq K\left(\frac{M}{m}\right) = K(h) \text{ for any } a, b \in [m,M].
$$

Therefore, by (2.13) we have

$$
(1-p)a + pb - a^{1-p}b^p \leq p(1-p)(a+b+|b-a|)[K(h)-1]
$$

for any a, $b \in [m,M]$ and $p \in [0,1]$.

Fix $a \in [m,M]$ and by using the operator functional calculus for A with $mI \leq A \leq MI$, we have

$$
(1-p)aI + pA - a^{1-p}A^p \leq p(1-p)[K(h)-1](aI + A + |A-aI|). \tag{2.14}
$$

Then for any $x \in H$ with $\|x\| = 1$ we get from (2.14) that

$$\begin{aligned} (2.15) \qquad & (1-p)\,a + p\,\langle Ax, x\rangle - a^{1-p}\,\langle A^p x, x\rangle \\ & \le p\,(1-p)\,[K\,(h) - 1]\,(a + \langle Ax, x\rangle + \langle |A - aI|\, x, x\rangle)\,, \end{aligned}$$

for any $a \in [m, M]$ and $p \in [0, 1]$.

Now, if we take $a = \langle Ax, x\rangle \in [m, M]$, where $x \in H$ with $\|x\| = 1$ in (2.15), then we obtain

$$\begin{aligned} & \langle Ax, x\rangle - \langle Ax, x\rangle^{1-p}\,\langle A^p x, x\rangle \\ & \le p\,(1-p)\,[K\,(h) - 1]\,(2\,\langle Ax, x\rangle + \langle |A - \langle Ax, x\rangle\, I|\, x, x\rangle)\,, \end{aligned}$$

which, by division with $\langle Ax, x\rangle > 0$ provides the first inequality in (2.11).

By Schwarz inequality, we have for $x \in H$ with $\|x\| = 1$ that

$$\begin{aligned} \langle |A - \langle Ax, x\rangle\, I|\, x, x\rangle & \le \left\langle (A - \langle Ax, x\rangle\, I)^2\, x, x\right\rangle^{1/2} \\ & = \left\langle \left(A^2 - 2\,\langle Ax, x\rangle\, A + \langle Ax, x\rangle^2\, I\right) x, x\right\rangle^{1/2} \\ & = \left(\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2\right)^{1/2}, \end{aligned}$$

which proves the second part of (2.11).

Since

$$\frac{\left\langle A^2 x, x\right\rangle}{\langle Ax, x\rangle^2} - 1 \le \frac{(M-m)^2}{4mM} = K\,(h) - 1$$

for $x \in H$ with $\|x\| = 1$, then the last part of (2.11) is thus proved. ∎

3. A Comparison for Upper Bounds

We observe that the inequality (2.9) provides for the quantity

$$(0 \le)\, 1 - \frac{\langle A^p x, x\rangle}{\langle Ax, x\rangle^p},\ x \in H \text{ with } \|x\| = 1,$$

the following upper bound

$$(3.1) \qquad B_1\,(p, h) := \frac{p\,(1-p)}{2} h\,[K\,(h) - 1]\,,$$

while the inequality (2.11) gives the upper bound

$$(3.2) \qquad B_2\,(p, h) := p\,(1-p)\,[K\,(h) - 1]\left[2 + (K\,(h) - 1)^{1/2}\right],$$

where $p \in (0, 1)$ and $h > 1$.

Now, if we depict the 3D plot for the difference of the bounds B_1 and B_2, namely

$$D(x,y) := B_1(y,x) - B_2(y,x)$$

on the box $[1,8] \times [0,1]$, then we observe that it takes both positive and negative values, showing that the bounds $B_1(p,h)$ and $B_2(p,h)$ can not be compared in general, namely neither of them is better for any $p \in (0,1)$ and $h > 1$.

REFERENCES

1. D. I. Cartwright, M. J. Field: *A refinement of the arithmetic mean-geometric mean inequality*, Proc. Amer. Math. Soc., **71** (1978), 36-38.
2. S. S. Dragomir: *A note on Young's inequality*, Preprint RGMIA Res. Rep. Coll. **18** (2015), Art. 126. [http://rgmia.org/papers/v18/v18a126.pdf].
3. S. S. Dragomir: *A note on new refinements and reverses of Young's inequality*, Preprint RGMIA Res. Rep. Coll. **18** (2015), Art. . [http://rgmia.org/papers/v1].
4. S. S. Dragomir: *Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces*, J. Inequal. & Appl., Volume **2010**, Article ID 496821, 15 pages doi:10.1155/2010/496821. Preprint RGMIA *Res. Rep. Coll.*, **11** (2008), Supliment. Art. 15. [Online http://rgmia.org/papers/v11e/RevJensenOp.pdf].
5. S. S. Dragomir: *Operator Inequalities of the Jensen, Čebyšev and Grüss Type*, Springer Briefs in Mathematics, Springer, 2012.
6. M. Fujii, S. Izumino, R. Nakamoto: *Classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality*. Nihonkai Math. J. **5** (1994), no. 1, 61–67.
7. M. Fujii, S. Izumino , R. Nakamoto, Y. Seo: *Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities*, Nihonkai Math. J. **8** (1997), 117-122.
8. T. Furuta: *Extensions of Hölder-McCarthy and Kantorovich inequalities and their applications*. Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 3, 38–41.
9. T. Furuta: *Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities*. J. Inequal. Appl. **2** (1998), no. 2, 137–148.
10. T. Furuta: *The Hölder-McCarthy and the Young inequalities are equivalent for Hilbert space operators*. Amer. Math. Monthly **108** (2001), no. 1, 68–69.
11. F. Kittaneh, Y. Manasrah: *Improved Young and Heinz inequalities for matrix*, J. Math. Anal. Appl. **361** (2010), 262-269.
12. F. Kittaneh, Y. Manasrah: *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra, **59** (2011), 1031-1037.
13. W. Liao, J. Wu, J. Zhao: *New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant*, Taiwanese J. Math. **19** (2015), No. 2, pp. 467-479.
14. C.-S. Lin, Y. J. Cho: *On Hölder-McCarthy-type inequalities with powers*. J. Korean Math. Soc. **39** (2002), no. 3, 351–361.

15. C.-S. Lin a, Y. J. Cho: *On Kantorovich inequality and Hölder-McCarthy inequalities*. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **11** (2004), no. 4, 481–490.

16. C. A. McCarthy: c_p, Israel J. Math. **5** (1967), 249–271.

17. J. Pečarić, T. Furuta, J. Mićić Hot, Y. Seo: *Mond-Pečarić Method in Operator inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space.* Monographs in Inequalities, 1. ELEMENT, Zagreb, 2005. xiv+262 pp.+loose errata. ISBN: 953-197-572-8

18. W. Specht: *Zer Theorie der elementaren Mittel*, Math. Z., **74** (1960), pp. 91-98.

19. M. Tominaga: *Specht's ratio in the Young inequality*, Sci. Math. Japon., **55** (2002), 583-588.

Silvestru Sever Dragomir
College of Engineering and of Science
Department of Mathematics
P. O. Box 8000
Melbourne, Australia
sever.dragomir@vu.edu.au